SINGULARITY FORMATION FOR A NONLINEAR
VARIATIONAL SINE-GORDON EQUATION IN A
MULTIDIMENSIONAL SPACE

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ABSTRACT. We study a multidimensional nonlinear variational sine-Gordon equation, which can be used to describe long waves on a dipole chain in the continuum limit. By using the method of characteristics, we show that a solution of a nonlinear variational sine-Gordon equation with certain initial data in a multidimensional space has a singularity in finite time.

1. Introduction

The multidimensional nonlinear variational sine-Gordon equation is
\begin{equation}
 u_{tt} - c(u)\nabla \cdot (c(u)\nabla u) + \frac{\omega^2}{2} \sin(2u) = 0,
\end{equation}
where $\omega$ is a constant. Here the wave speed $c(u) > 0$ satisfies
\begin{equation}
 c^2(u) = a \cos^2 u + b \sin^2 u
\end{equation}
for some constants $a > 0$ and $b > 0$. If $a = b$, then the equation (1.1) reduces to the multidimensional nonlinear Klein-Gordon equation. In fact, this equation originates from the study of long waves on a dipole chain in the continuum limit which occurs in an anisotropic system [16]. The wave in a massive liquid crystal director field is the example of the system as well. The equation (1.1) can be regarded as the sine-Gordon version of the nonlinear variational wave equation
\begin{equation}
 u_{tt} - c(u)\nabla \cdot (c(u)\nabla u) = 0
\end{equation}
which is used in the theory of nematic liquid crystals. Refer to [1–5,8–11,13–15] for more information and mathematical results on the nonlinear variational
wave equation and its sine-Gordon version. In particular, singularity of solutions were studied in [6, 7, 12].

In this paper we are concerned with the singularity formation of smooth solutions for the multidimensional nonlinear variational sine-Gordon equations. Let \( u = u(t, r) \) and \( r = |x|, \) where \( x = (x_1, \ldots, x_l) \). Then (1.1) transforms into

\[
(1.3) \quad u_{tt} - c(u)(c(u)u_r)_r - \frac{(l - 1)c^2(u)u_r}{r} + \frac{\omega^2}{2} \sin(2u) = 0.
\]

The wave speed \( c(\cdot) \in C^2 \) is assumed to satisfy

\[
(1.4) \quad 0 < c_0 \leq c(\cdot) \leq c_1, \quad |c'(\cdot)| \leq c_1
\]

for some constants \( c_0 > 0 \) and \( c_1 > 0 \).

We notice that a constant solution of (1.3) becomes a critical point of \( c(\cdot) \) which causes some difficulty in using the characteristic method employed in [6, 7] for the singularity formation. Thus we have to find another type of solutions to make use of the characteristic method. In this paper, we overcome the difficulty by finding a proper function that is not a critical point of the wave speed. In fact, it is known that there exists a unique solution \( y = Y(t) \), which is a spatial independent solution of (1.3), for a second order differential equation

\[
\frac{d^2y}{dt^2} + \frac{1}{2} \omega^2 \sin(2y(t)) = 0
\]

with the data

\[
y(0) = k_0 \quad \text{and} \quad \frac{dy}{dt}(0) = 0.
\]

Then \( Y(t) \approx k_0 \cos(\omega t) \) for sufficiently small \( 0 < k_0 \ll 1, 0 < Y(t) \leq k_0 \) and \( c'(k_0) > 0 \) for a certain time period. Let such a time period be \([0, t_0)\). Let us define

\[
\frac{c'(Y(0))}{4} = \frac{c'(k_0)}{4} := c_2.
\]

Let us state the singularity formation which is the main result of the paper.

**Theorem 1.1** (Main theorem). Let us assume that (1.3) with initial data

\[
(1.5) \quad \left\{ \begin{array}{l}
          u(0, r) = Y(0) + \varepsilon \psi \left( \frac{r - r_0}{\varepsilon} \right), \\
          u_t(0, r) = (-c(u(0, r))) + \varepsilon u_r(0, r)
        \end{array} \right.
\]

has a smooth solution \( u(t, r) \in C^1([0, T) \times \mathbb{R}^+) \). Here \( \varepsilon \) and \( r_0 \) are positive constants to be determined. If \( c'(k_0) > 0 \), and \( \psi \) satisfies

\[
(1.6) \quad \psi(\cdot) \in C^4((-1, 1)), \quad \psi \neq 0, \quad \text{and} \quad \psi'(0) < -2 \max \left\{ \frac{3}{2c_0 r_0^2}, \frac{16c_1^2 d}{r_0 c_0 c_2} \right\},
\]

where \( d = \frac{l - 1}{2} > 0 \), then \( T < \infty \).
2. The proof of main theorem

The purpose of this section is to establish the singularity formation of (1.1). We prove Theorem 1.1 by using the method of [6,7]. More precisely, by deriving the energy equation of the Riemann variables, we show that the blow-up result for the equation (1.1) occurs at a certain finite time by the characteristic method.

Let us define
\[ d = \frac{t_0 - 1}{2} > 0 \] and take \( r_0 \) small enough so that \( \frac{r_0}{c_1} \ll t_0 \). Let us introduce new variables
\[ U := (u_t + c(u)u_r)r^d, \quad S := (u_t - c(u)u_r)r^d, \]
which yields that
\[ u_t = \frac{U + S}{2r^d}, \quad u_r = \frac{U - S}{2c(u)r^d}. \]

Equation (1.3) transforms into the system of equations for \( U \) and \( S \):
\[
\begin{align*}
U_t - c(u)U_r &= \frac{\partial}{\partial r} \left( U^2 - S^2 \right) - \frac{dc}{dr} S - \frac{r^d \omega^2 \sin(2u)}{2}, \\
S_t + c(u)S_r &= \frac{\partial}{\partial r} \left( S^2 - U^2 \right) + \frac{dc}{dr} U - \frac{r^d \omega^2 \sin(2u)}{2}.
\end{align*}
\] (2.2)

From (2.2) we can obtain the following equation of conservative form
\[
\frac{\partial}{\partial t} \left( U^2 + S^2 + 2r^2 \omega^2 \sin^2 u \right) + \frac{\partial}{\partial r} \left[ c(u)(S^2 - U^2) \right] = 0.
\] (2.3)

According to (1.5), the initial data for the Riemann variables are given by
\[
U(0, r) = 0 = S(0, r), \quad U_t(0, r) = c(u(0, r))u_r(0, r).
\] (2.4)

As a consequence, the blow-up of the smooth solutions to Cauchy problem (1.3) with (1.5) is transformed into the blow-up of the smooth solution to (2.2) with (2.4).

Next, let us introduce an energy function which is uniformly bounded by its initial energy. For the time being, let \( \varepsilon < \frac{1}{4} r_0 \). From (1.6) and (2.4) one has
\[
U(0, r) = 0 = S(0, r)
\]
for all \( r \in [0, \infty) \setminus [r_0 - \varepsilon, r_0 + \varepsilon] \). Let us define the energy function \( E(t) \) as follows:
\[
E(t) = \int_0^{r_0 + \varepsilon} \left[ U^2(t, r) + S^2(t, r) + 2r^2 \omega^2 \sin^2 u(t, r) \right] dr.
\] (2.5)

From integration (2.3) on \( t \), we notice that \( E \) is time-independent, which implies \( E(t) = E(0) \). Furthermore, we can obtain
\[
E(0) = \int_0^{r_0 + \varepsilon} \left( U^2(0, r) + S^2(0, r) + 2r^2 \omega^2 \sin^2 u(0, r) \right) dr
\]
\[
= \int_{r_0 - \varepsilon}^{r_0 + \varepsilon} \left\{ \left[ \varepsilon^2 + (\varepsilon - 2c(u(0, r)))^2 \right] \left( \psi' \left( \frac{r - r_0}{\varepsilon} \right) \right)^2 + 2\omega^2 \sin^2 u(0, r) \right\} r^d dr.
\]
Then
\[ E(0) \leq (2r_0)^{2d} \left\{ 4\omega^2 \varepsilon + (\varepsilon^2 + (2c_1 + \varepsilon)^2) \int_{r_0 - \varepsilon}^{r_0 + \varepsilon} \left( \psi' \left( \frac{r - r_0}{\varepsilon} \right) \right)^2 dr \right\} \]
(2.6)
\[ \leq \left\{ 4\omega^2 + \left[ 1 + (2c_1 + 1) \right] \int_{-1}^{1} (\psi'(z))^2 dz \right\} (2r_0)^{2d} \varepsilon := Mr_0^{2d} \varepsilon \]
for some positive constant \( M \).

Let \((r_1, 0)\) and \((r_2, 0)\) be two points with \(0 < r_1 < r_2\). We define two characteristic curves passing through them, respectively. One is a positive characteristic curve \( r^+(t) \) emitting from \((r_1, 0)\) and the other is a negative characteristic curve \( r^-(t) \) emitting from \((r_2, 0)\) as follows:
\[ \frac{dr^+(t)}{dt} = c_0 u(t, r^+(t)), \quad r^+(0) = r_1, \]
and
\[ \frac{dr^-(t)}{dt} = -c_0 u(t, r^-(t)), \quad r^-(0) = r_2. \]

From (1.4), let us choose \( r_1 \) and \( r_2 \) so that \( r_2 - r_1 \leq \frac{2c_0 (r_0 - \varepsilon)}{c_1} \).

Then it follows that two characteristic curves \( r^+(t) \) and \( r^-(t) \) will intersect at the point \((r_m, t_m)\) with \( r_1 < r_m < r_2 \) and \( t_m < \frac{r_0 - \varepsilon}{c_1} \). Applying the Green’s formula for equation (2.3) to a region enclosed by the characteristic curves \( r = r^\pm(t) \) and \( r \)-axis which is depicted in Figure 1, we have
\[ \int_{r_1}^{r_m} \left[ U^2(t^+(r), r) + r^{2d} \omega^2 \sin^2 u(t^+(r), r) \right] dr
+ \int_{r_m}^{r_2} \left[ S^2(t^-(r), r) + r^{2d} \omega^2 \sin^2 u(t^-(r), r) \right] dr
= \frac{1}{2} \int_{r_1}^{r_2} \left[ U^2(0, r) + S^2(0, r) + 2r^{2d} \omega^2 \sin^2 u(0, r) \right] dr \leq Mr_0^{2d} \varepsilon \]
(2.7)
by the energy estimate (2.6).

Let \( r = \hat{r}(t) \) with
\[ \frac{d\hat{r}(t)}{dt} = c(u(t, \hat{r}(t))), \quad \hat{r}(0) = r_0 \quad (r_1 < r_0 < r_2) \]
be another positive characteristic curve starting from \((r_0, 0)\). We will show that \( c'(u) \) is always positive on the curve \( r = \hat{r}(t) \) if \( \varepsilon \) is sufficiently small. Let us define \( \partial_1 := \partial_t + c(u) \partial_r \). From (2.1), one obtain
\[ \partial_1 u(t, \hat{r}(t)) = \frac{U(t, \hat{r}(t))}{\hat{r}^d(t)}. \]
(2.8)
Figure 1. The region bounded by two characteristics

If we integrate (2.8) along the characteristic from 0 to \( t \) with \( t < \frac{r_0 - \varepsilon}{c_1} \) and use (2.7), then it follows that

\[
\left| u(t, \hat{r}(t)) - u(0, r_0) \right| \leq \int_0^t \frac{|U(v, r)|}{\hat{r}'(v)} dv \leq \sqrt{t} \left( \int_0^t \left( \frac{U(v, \hat{r}(v))}{\hat{r}'(v)} \right)^2 dv \right)^{\frac{1}{2}} \\
\leq \frac{\sqrt{r_0 - \varepsilon}}{r_0^2 c_1 c_0} \left( \int_{r_0}^{r_1} U^2(\hat{r}(s), s) ds \right)^{\frac{1}{2}} \leq \sqrt{\frac{M(r_0 - \varepsilon)}{c_1 c_0}} \sqrt{\varepsilon}.
\]

From the smoothness of \( c(\cdot) \), there exists \( \varepsilon_0 \) small enough such that for any \( \varepsilon \in (0, \varepsilon_0) \) it follows that on the curve \( r = \hat{r}(t) \)

\[
0 < c_2 = \frac{c'(k_0)}{4} \leq \frac{c'(u(0, r(0)))}{2} \leq c'(u(t, \hat{r}(t))).
\]

We now prove that the solutions of (2.2) can blow up for the given initial condition (2.4), namely, \( S(t, \hat{r}(t)) \) becomes infinite at finite time before \( t = \frac{r_0 - \varepsilon}{c_1} \). First, let us take small enough \( r_0 > 0 \) so that

\[
\left( \frac{c_1 M}{4c_0^2} + \frac{3d}{2c_1} \omega^2 \right) r_0^{1+d} < \frac{1}{3} \quad r_0^{2d} \leq \frac{c_2 r_0^2}{16c_1 3^d(c_1^2 M + 2c_0^2 3^d \omega^2)}.
\]

Furthermore, let us choose

\[
\varepsilon < \min \left\{ \varepsilon_0, c_0, \frac{3}{4} r_0, \frac{\sqrt{c_0}}{4d} \sqrt{c_1 M}, \frac{r_0}{c_0} \frac{\sqrt{c_1 M}}{4d} \right\}
\]
and
\[
S(0, r_0) = (-2c(u(0, r_0)) + \varepsilon)r_0^d \psi'(0) \\
> 2c_0 r_0^d \max \left\{ \frac{3}{2c_0 r_0^d}, \frac{16c_0^2 3d}{r_0 c_0 c_2} \right\} \\
> \max \left\{ 3, \frac{32c_0^2 (3r_0)^d}{(r_0 - \varepsilon)c_2} \right\}.
\]
(2.12)

**Lemma 2.1.** Let \( S(0, r_0) \) satisfy (2.12). Then \( S(t, \hat{r}(t)) \) goes to infinity as \( t\), which is less than \( \frac{\alpha - \varepsilon}{\varepsilon_1} \), for sufficiently small \( \varepsilon \) in (2.11).

**Proof.** First, let us show that \( S(t, \hat{r}(t)) > 1 \) when \( S(t, \hat{r}(t)) \) is smooth in \([0, \frac{\alpha - \varepsilon}{\varepsilon_1}]\). Let us assume on the contrary. That is, there exists \( \tau \in (0, \frac{\alpha - \varepsilon}{\varepsilon_1}) \) such that \( S(t, \hat{r}(t)) \) belongs to \( C^1([0, \tau]) \). Let \( S(t, \hat{r}(t)) > 1 \) for all \( t \in [0, \tau] \), and \( S(\tau, \hat{r}(\tau)) = 1 \). From (1.4), (2.1) and (2.9), one has
\[
\partial_1 \left( \frac{1}{S} \right) = -\frac{1}{S^2} \left[ \frac{c'}{4c_0 r_0^d} (S^2 - U^2) + \frac{dc}{r} U - \frac{r^d}{2} \omega^2 \sin(2u) \right] \\
\leq -\frac{c'}{4c_0 r_0^d} + \frac{1}{S^2} \left( \frac{c'}{4c_0 r_0^d} U^2 + \frac{dc}{r} |U| + \frac{r^d}{2} \omega^2 \sin(2u) \right) \\
\leq -\frac{c_2}{4c_1 (3r_0)^d} + \frac{1}{S^2} \left( \frac{c_2}{4c_0 r_0^d} U^2 + \frac{dc_1}{r_0} |U| + \frac{(3r_0)^d \omega^2}{2} \right).
\]
(2.13)

If we integrate (2.13) from 0 to \( \tau \) along the positive characteristic \( r = \hat{r}(t) \) with \( \tau < \frac{\alpha - \varepsilon}{\varepsilon_1} \), then it follows that
\[
\frac{1}{S(\tau, \hat{r}(\tau))} \leq \frac{1}{S(0, r_0)} + \int_0^\tau \frac{1}{S^2} \left( \frac{c_2}{4c_0 r_0^d} U^2 + \frac{dc_1}{r_0} |U| + \frac{(3r_0)^d \omega^2}{2} \right) dt \\
\leq \frac{1}{S(0, r_0)} + \frac{c_1}{4c_0^2 r_0^d} \int_{r_0}^{\hat{r}(t)} U^2 dr + \frac{c_1 d^2}{c_0 r_0} \left( \int_{r_0}^{\hat{r}(t)} U^2 dr \right)^{\frac{1}{2}} + \frac{(3r_0)^d \omega^2}{2} \cdot \frac{r_0}{c_1} \\
\leq \frac{1}{S(0, r_0)} + \frac{c_1}{4c_0^2 r_0^d} M t_0^d \varepsilon + d \cdot \frac{c_1}{c_0 r_0} \sqrt{M t_0^d \varepsilon} + \frac{3d}{2c_1} \omega^2 r_0^{1+d} \\
\leq \frac{1}{S(0, r_0)} + \frac{c_1}{4c_0^2 r_0^d} M t_0^{1+d} + \frac{c_1 M}{c_0 r_0} d t_0^d \varepsilon + \frac{3d}{2c_1} \omega^2 r_0^{1+d} \\
\leq \frac{2}{3} + \sqrt{\frac{c_1 M}{c_0 r_0} d t_0^d \varepsilon} < 1,
\]
which contradicts to \( S(\tau, \hat{r}(\tau)) = 1 \). In fact, the last inequality comes from the choice of \( r_0 \) in (2.10) and \( \varepsilon \) in (2.11). Therefore, \( S(t, \hat{r}(t)) > 1 \) holds for \( 0 \leq t < \frac{\alpha - \varepsilon}{\varepsilon_1} \).

Finally, let us prove that \( S(t, \hat{r}(t)) \) becomes infinite at finite time before \( t = \frac{\alpha - \varepsilon}{\varepsilon_1} \). Integration of (2.13) from 0 to \( t \) with \( t < \frac{\alpha - \varepsilon}{\varepsilon_1} \), which together with
the choice of $S(0, r_0)$ in (2.12) yields
\[
\frac{1}{S(t, \hat{r}(t))} \leq \frac{1}{S(0, r_0)} - \int_0^t \frac{c_2}{4c_1(3r_0)^d} dt
\]
\[
+ \int_0^t \frac{1}{S^2} \left( \frac{c_2}{4c_0r_0} U^2 + \frac{d_2}{r_0} |U| + \left( \frac{3r_0}{4c_1} \right) \omega^2 \right) dt
\]
\[
\leq \frac{1}{S(0, r_0)} - \frac{c_2}{4c_1(3r_0)^d} t + \left( \frac{c_1 M}{4c_0} + \frac{3d\omega^2}{2c_1} \right) r_0^{1+d} + d r_0^{d-\frac{1}{2}} \sqrt{\frac{c_1 M}{c_0}} \varepsilon
\]
(2.14)
\[
\leq \frac{c_2}{4c_1(3r_0)^d} \left( \frac{r_0 - \varepsilon}{2c_1} + r_0 \frac{8c_1}{3r_0} \right)
\]
We find that there exists $t^*$ satisfying
\[
t^* = \frac{r_0 - \varepsilon}{2c_1} + r_0 \frac{8c_1}{3r_0} = \frac{5r_0 - 4\varepsilon}{8c_1} < \frac{r_0 - \varepsilon}{c_1}
\]
and
\[
\frac{1}{S(t, \hat{r}(t))} \to 0
\]
as $t \to t^*$ by taking $r_0 > 4\varepsilon/3$. Hence $S(t, \hat{r}(t)) \to \infty$ as $t \to t^*$. □

According to the above argument, we show that the singularity of the smooth solution $u$ occurs at some point, that is $\nabla u \to \infty$ as $t \to t^*$ for some $t^* < \frac{c_0 - \varepsilon}{c_2}$. Therefore we complete the proof of the Main theorem.

References


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