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RESULTS ON THE ALGEBRAIC DIFFERENTIAL INDEPENDENCE OF THE RIEMANN ZETA FUNCTION AND THE EULER GAMMA FUNCTION

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Abstract. In 2010, Li-Ye [13, Theorem 0.1] proved that

$$P\left(\zeta(z), \zeta'(z), \ldots, \zeta^{(m)}(z), \Gamma(z), \Gamma'(z), \Gamma^{''}(z)\right) \not\equiv 0 \quad \text{in} \ \ \mathbb{C},$$

where m is a non-negative integer, and $P(u_0, u_1, \ldots, u_m, v_0, v_1, v_2)$ is any non-trivial polynomial in its arguments with coefficients in the field $\mathbb{C}$. Later on, Li-Ye [15, Theorem 1] proved that

$$P\left(z, \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z), \zeta(z)\right) \not\equiv 0$$

in $z \in \mathbb{C}$ for any non-trivial distinguished polynomial $P(z, u_0, u_1, \ldots, u_n, v)$ with coefficients in a set L_δ of the zero function and a class of non-zero functions f from $\mathbb{C}$ to $\mathbb{C} \cup \{\infty\}$ (cf. [15, Definition 1]). In this paper, we prove that $P\left(z, \zeta(z), \zeta'(z), \ldots, \zeta^{(m)}(z), \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z)\right) \not\equiv 0$ in $z \in \mathbb{C}$, where m and n are two non-negative integers, and

$$P(z, u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$$

is any non-trivial polynomial in the $m + n + 2$ variables

$$u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n$$

with coefficients being meromorphic functions of order less than one, and the polynomial $P(z, u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$ is a distinguished polynomial in the $n + 1$ variables $v_0, v_1, \ldots, v_n$. The question studied in this paper is concerning the conjecture of Markus from [16]. The main results obtained in this paper also extend the corresponding results from Li-Ye [12] and improve the corresponding results from Chen-Wang [5] and Wang-Li-Liu-Li [23], respectively.

1. Introduction and main results

Throughout this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. To prove the main results in

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the present paper, we will apply Nevanlinna's theory and adopt the standard notations of the Nevanlinna's theory. We assume that the readers are familiar with the standard notations which are used in the Nevanlinna's theory such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$, the counting function $N(r, f)$ and the reduced counting function $\overline{N}(r, f)$ of a meromorphic function f that are explained in [8, 11, 24, 25], their detail notions are defined as follows: for a non-constant meromorphic function f in the complex plane, the proximity function $m(r, f)$, the counting function $N(r, f)$, the reduced counting function $\overline{N}(r, f)$ and the characteristic function $T(r, f)$, are defined as

$$\begin{aligned}
m(r, f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \\
N(r, f) &= \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f)\log r, \\
\overline{N}(r, f) &= \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t} dt + \overline{n}(0, f)\log r \quad \text{and} \\
T(r, f) &= m(r, f) + N(r, f), \quad \text{respectively.}
\end{aligned}$$

Here $\log^+ x = \max\{0, \log x\}$ for any non-negative real number x, $n(t, f)$ denotes the numbers of the poles of f in $\{z : |z| \le t\}$, where each pole of f in $\{z : |z| \le t\}$ is counted according to its multiplicity as a pole of f, while $\overline{n}(t, f)$ denotes the reduced form of $n(t, f)$. The properties of $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$ and $T(r, f)$ can be found in [8, 11, 24, 25].

In 1886, Hölder [10] began to study the question of the algebraic differential independence of the Euler gamma function Γ and proved that the Euler gamma function Γ does not satisfy any non-trivial algebraic differential equations with polynomial coefficients. Later on, Bank-Kaufman [2] improved the corresponding result from Hölder [10] and proved that the Euler gamma function Γ does not satisfy any non-trivial algebraic differential equation whose coefficients are meromophic functions ϕ with their Nevanlinna's characteristics satisfying $T(r, \phi) = o(r)$ as $r \to +\infty$. On the other hand, as one of the well-known list of 23 problems introduced by Hilbert [9], Problem 18 wrote: whether or not the Riemann zeta function ζ and allied functions satisfy any non-trivial algebraic differential equation? The question was solved in [18, 19]. We recall that the Riemann zeta function ζ is associated with the Euler gamma function Γ by the Riemann functional equation (cf. [1, p. 217])

$$\zeta(1 - z) = 2^{1-z}\pi^{-z}\Gamma(z)\zeta(z)\cos\frac{\pi z}{2}.$$

One may ask, whether or not the Euler gamma function Γ and the Riemann zeta function ζ are related by any non-trivial algebraic differential equation? In this direction, Markus [16] deduced that the Euler gamma function Γ and the composition function $\zeta(\sin(2\pi z))$ are differential independent over $\mathbb{C}$. Moreover, Markus [16] conjectured that the Euler gamma function Γ is not a solution of any non-trivial algebraic differential equation, even allowing coefficients that are differential polynomials in ζ over $\mathbb{C}$. In other words, this conjecture can be expressed as follows:

Conjecture 1.1 (The conjecture of Markus, [16]). *Let*

$$P(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$$

be a polynomial in $u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n$ in the field $\mathbb{C}$, where m and n are two positive integers. If

$$P\left(\zeta(z), \zeta'(z), \ldots, \zeta^{(m)}(z), \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z)\right) = 0$$

identically in $z \in \mathbb{C}$, then P is identically zero.

In recent ten years, many mathematicians studied Conjecture 1.1 and obtained many interesting results, which can be found, for example, in Li-Ye [12–15] and [23]. For the case of the following narrations, we recall the following notations from Li-Ye [12]: let $P(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$ be any polynomial in the $m+n+2$ variables $u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n$ with coefficients in $\mathbb{C}$, where and in what follows, m and n are two positive integers. For a non-negative integer μ, we let

$$\begin{aligned}(1)\quad \Lambda &= \Lambda(\mu)\\ &= \{(\lambda_0, \lambda_1, \ldots, \lambda_\mu) : \lambda_j \text{ is a non-negative integer with } 0 \le j \le \mu < \infty\}\end{aligned}$$

be a finite multi-index set, and let

$$\Lambda_k = \{\lambda \in \Lambda : |\lambda| = k\} \quad \text{with} \quad |\lambda| = \sum_{j=0}^{\mu} \lambda_j \tag{2}$$

and

$$\Lambda_k^* = \{\lambda \in \Lambda : |\lambda|_* = k\} \quad \text{with} \quad |\lambda|_* = \sum_{j=0}^{\mu} j\lambda_j. \tag{3}$$

Then, it follows that there is a non-negative integer N such that

$$P(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n) = \sum_{j=0}^{N} \sum_{\lambda \in \Lambda_j} a_\lambda(u_0, u_1, \ldots, u_m) v_0^{\lambda_0} v_1^{\lambda_1} \cdots v_n^{\lambda_n}, \tag{4}$$

where $a_\lambda(u_0, u_1, \ldots, u_m)$ is a polynomial in the $m+1$ variables $u_0, u_1, \ldots, u_m$ with coefficients in $\mathbb{C}$. We set

$$P_j(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n) = \sum_{\lambda \in \Lambda_j} a_\lambda(u_0, u_1, \ldots, u_m) v_0^{\lambda_0} v_1^{\lambda_1} \cdots v_n^{\lambda_n}. \tag{5}$$

Moreover, we write

$$P_j(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n) = \sum_{p=0}^{N_j} P_{j,p}(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n), \tag{6}$$

with

$$P_{j,p}(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n) = \sum_{\lambda \in \Lambda_j \bigcap \Lambda_p^*} a_\lambda(u_0, u_1, \ldots, u_m) v_0^{\lambda_0} v_1^{\lambda_1} \cdots v_n^{\lambda_n}, \tag{7}$$

where N_j is a non-negative integer. Consequently, we have by (1)-(7) that

$$\begin{aligned} &P(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n) \\ &= \sum_{j=0}^{N} P_j(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n) \\ &= \sum_{j=0}^{N} \sum_{p=0}^{N_j} P_{j,p}(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n). \end{aligned} \tag{8}$$

We recall the following results from Li-Ye [12] and Li-Ye [13], respectively:

Theorem 1.2 ([12, Theorem1]). *Let $P(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$ defined as in (8) be any non-trivial polynomial in the $m + n + 2$ variables*

$$u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n$$

with coefficients in $\mathbb{C}$, where m and n are two non-negative integers. If

$$\sum_{\lambda \in \Lambda_j \bigcap \Lambda_p^*} a_\lambda(u_0, u_1, \ldots, u_m) \not\equiv 0 \tag{9}$$

whenever $P_{j,p}(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n) \not\equiv 0$ for all possible j and p such that $0 \le p \le N_j$ with $0 \le j \le N$ and $j \in \mathbb{Z}$, then

$$P\left(\zeta(z), \zeta'(z), \ldots, \zeta^{(m)}(z), \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z)\right) \not\equiv 0$$

in $z \in \mathbb{C}$.

Theorem 1.3 ([13, Theorem 0.1]). *Let m be a non-negative integer, and let $P(u_0, u_1, \ldots, u_m, v_0, v_1, v_2)$ be a polynomial in its arguments with coefficients in the field $\mathbb{C}$. If*

$$P\left(\zeta(z), \zeta'(z), \ldots, \zeta^{(m)}(z), \Gamma(z), \Gamma'(z), \Gamma''(z)\right) = 0$$

identically in $z \in \mathbb{C}$, then the polynomial P is identically zero.

The following definition is from Li-Ye [15]:

Definition 1 ([15, Definition 2]). Let $I = (i_0, i_1, \ldots, i_n)$ be a multi-index with $|I| = \sum_{k=0}^{n} i_k$. A polynomial in the variable $u_0, u_1, \ldots, u_n$ with functional coefficients a_I in a set $\mathcal{S}$ consisting of the zero function and a class of non-zero functions can be always written into

$$P(u_0, u_1, \ldots, u_n) = \sum_{I \in \Lambda} a_I {u_0}^{i_0} {u_1}^{i_1} \cdots {u_n}^{i_n},$$

where Λ is a multi-index set. We call $P(u_0, u_1, \ldots, u_n)$ an $\mathcal{S}$-distinguished polynomial in $u_0, u_1, \ldots, u_n$ or simply an $\mathcal{S}$-distinguished polynomial, if the multi-index set Λ satisfies $|I_i| \neq |I_j|$ for distinct multi-indices I_i, I_j in Λ.

Remark 1.4 ([15, p. 1458]). By Definition 1 we can see that any polynomial $P(z, u)$ in one argument u with functional coefficients is a distinguished polynomial in u. In addition, any polynomial $P(z, u, v)$ in two arguments u and v with functional coefficients can be written into

$$P(z, u, v) = \sum_{k=0}^{m} P_k(z, u) v^k,$$

where $P_1(z, u), P_2(z, u), \ldots, P_m(z, u)$, not all identically zero, are distinguished polynomials with functional coefficients.

The question of the algebraic independence for the special case of Γ and ζ is solved by Li-Ye [14]. Indeed, they proved that Γ and ζ cannot satisfy non-zero polynomial equation $P(z, u, v) = 0$. Later on, Li-Ye [15] showed that

$$P\left(z, \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z), \zeta(z)\right) \not\equiv 0$$

in $z \in \mathbb{C}$ for any non-trivial distinguished polynomial $P(z, u_0, u_1, \ldots, u_n, v)$ in v with coefficients being allowed to be any polynomial of ζ over $\mathbb{C}$, over the ring of polynomials or more generally over a set L_δ of the zero function and a class of non-zero functions f from $\mathbb{C}$ to $\mathbb{C} \cup \{\infty\}$ (cf. [15, Definition 1]).

Recently, Wang-Li-Liu-Li [23] and Chen-Wang [5] respectively proved the following results of which Theorem 1.5 improved Theorem 1.3:

Theorem 1.5 ([5, Theorem 1.5]). *Let m, n, l, α be non-negative integers and $l > n > \alpha \geq 0$, and let $P(u_0, u_1, \ldots, u_m, v_\alpha, v_n, v_l)$ be a polynomial in the $m+4$ variables $u_0, u_1, \ldots, u_m, v_\alpha, v_n, v_l$ with coefficients being polynomials in $z \in \mathbb{C}$. If*

$$P\left(\zeta(z), \zeta'(z), \ldots, \zeta^{(m)}(z), \Gamma^{(\alpha)}(z), \Gamma^{(n)}(z), \Gamma^{(l)}(z)\right) = 0$$

identically in $z \in \mathbb{C}$, then P is identically zero.

Theorem 1.6 ([5, Theorem 1.5]). *Let m and n be two positive integers, let N be a non-negative integer, and let $P(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$ be a polynomial in the $m + n + 2$ variables $u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n$ with coefficients in the field $\mathbb{C}$ such that*

$$P(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n) = \sum_{j=0}^{N} P_j(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n),$$

where

$$P_j(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n) = \sum_{I \in \Lambda_j} a_I(u_0, u_1, \ldots, u_m) {v_0}^{\lambda_0} {v_1}^{\lambda_1} \cdots {v_n}^{\lambda_n}$$

is a distinguished polynomial in $v_0, v_1, \ldots, v_n$ *for* $0 \leq j \leq N$ *and* $j \in \mathbb{Z}$, *and* $a_I(u_0, u_1, \ldots, u_m)$ *for* $I \in \Lambda_j$, $0 \leq j \leq N$ *and* $j \in \mathbb{Z}$ *is a polynomial in the* $m+1$ *variables* $u_0, u_1, \ldots, u_m$ *with coefficients in the field* $\mathbb{C}$. *If*

$$P\left(\zeta(z), \zeta'(z), \ldots, \zeta^{(m)}(z), \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z)\right) = 0$$

identically in $z \in \mathbb{C}$, *then* P *is identically zero.*

Next we give the definition of the order of a non-constant meromorphic function:

Definition 2 ([8,11,24,25])**.** For a non-constant meromorphic function f, the order of f is defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Remark 1.7. For a non-constant entire function f, the order of f is defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max\limits_{|z|\leq r}\{|f(z)|\}$.

Regarding Theorems 1.2-1.5, one may ask the following question that is much more general than Conjecture 1.1:

Question 1.8. Whether or not

$$P\left(\zeta(z), \zeta'(z), \ldots, \zeta^{(m)}(z), \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z)\right) \not\equiv 0$$

in $z \in \mathbb{C}$, where $P(u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$ is any non-trivial polynomial in the $m+n+2$ variables $u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n$ with coefficients being meromorphic functions of order less than one, while m and n are two non-negative integers?

We will study Question 1.8, and prove the following result:

Theorem 1.9. *Let* $P(z, u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$ *be a polynomial in the* $m+n+2$ *variables* $u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n$ *with coefficients being meromorphic functions of order less than one, where* m *and* n *are two non-negative integers. Suppose that* $P(z, u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$ *is a distinguished polynomial in the* $n+1$ *variables* $v_0, v_1, \ldots, v_n$ *such that*

$$\begin{aligned}
(10) \qquad & P(z, u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n) \\
&= \sum_{j=0}^{t} a_{I_j}(z, u_0, u_1, \ldots, u_m) v_0^{i_{j,0}} v_1^{i_{j,1}} \cdots v_n^{i_{j,n}} \\
&= a_{I_0}(z, u_0, u_1, \ldots, u_m) + a_{I_1}(z, u_0, u_1, \ldots, u_m) v_0^{i_{1,0}} v_1^{i_{1,1}} \cdots v_n^{i_{1,n}} + \cdots \\
&\quad + a_{I_t}(z, u_0, u_1, \ldots, u_m) v_0^{i_{t,0}} v_1^{i_{t,1}} \cdots v_n^{i_{t,n}},
\end{aligned}$$

where t is some non-negative integer, and $a_{I_j}(z, u_0, u_1, \ldots, u_m)$ with $0 \le j \le t$ is a polynomial in the $m+1$ variables $u_0, u_1, \ldots, u_m$ with coefficients being meromorphic functions of order less than one. If

$$P\left(z, \zeta(z), \zeta'(z), \ldots, \zeta^{(m)}(z), \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z)\right) = 0 \tag{11}$$

identically in $z \in \mathbb{C}$, then the polynomial P is identically zero.

By Theorem 1.9 we deduce the following result that improves Theorem 1.6:

Corollary 1.10. *Let $P(z, u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$ be a polynomial in the $m+n+2$ variables $u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n$ with coefficients being polynomials over $\mathbb{C}$, where m and n are two non-negative integers. Suppose that $P(z, u_0, u_1, \ldots, u_m, v_0, v_1, \ldots, v_n)$ is a distinguished polynomial in the $n+1$ variables $v_0, v_1, \ldots, v_n$ such that* (10) *holds, where t is some non-negative integer, and $a_{I_j}(z, u_0, u_1, \ldots, u_m)$ with $0 \le j \le t$ is a polynomial in the $m+1$ variables $u_0, u_1, \ldots, u_m$ with coefficients being polynomials in the field $\mathbb{C}$. If* (11) *holds identically in $z \in \mathbb{C}$, then the polynomial P is identically zero.*

2. Preliminaries

In this section, we will introduce some lemmas that play an important role in proving the main results of this paper. First of all, we introduce the following result that is due to Miles [17]:

Lemma 2.1 ([17, Theorem])**.** *There exist absolute constants A and B such that if f is any meromorphic function in the plane, then there exist entire functions f_1 and f_2 such that $f = f_1/f_2$ and such that $T(r, f_j) < AT(Br, f)$ for $1 \le j \le 2$ and $r > 0$.*

We also need the following result due to Voronin [22]:

Lemma 2.2 ([22] or [20, p. 11, Theorem 1.6])**.** *For any fixed complex numbers z with $Re(z) = \sigma$ satisfying $\frac{1}{2} < \sigma < 1$, the set*

$$\left\{(\zeta(z+i\tau), \zeta'(z+i\tau), \ldots, \zeta^{(n-1)}(z+i\tau)) : \tau \in \mathbb{R}\right\}$$

is dense in $\mathbb{C}^n$.

The following result was originally proved in [15]:

Lemma 2.3 ([15, p. 1462])**.** *For the Euler gamma function Γ and any given positive integer $q \ge 1$ we have*

$$\frac{\Gamma^{(q)}(z)}{\Gamma(z)} = (\log z)^q(1 + o(1)),$$

uniformly for any small $\varepsilon > 0$ and for all $z \in \mathbb{C} \setminus \{z : |\arg z - \pi| \le \varepsilon\}$ such that $z \to \infty$.

The following result is called minimum modulus theorem, which can be found, for example, in Berenstein-Gay [3, p. 362, 4.5.14]:

Lemma 2.4 (Minimum Modulus Theorem, [3, p. 362, 4.5.14])**.** *Let f be holomorphic in the disk $\{z : |z| < 2eR\} =: B(0, 2eR)$ and continuous in the closure of this disk. Assume $f(0) = 1$ and let $\varepsilon > 0$ be such that $0 < \varepsilon < \frac{3e}{2}$. Then in the disk $|z| \leq R$, and outside a collection of closed disks the sum of whose radii does not exceed $4\varepsilon R$, we have*

$$\log|f(z)| > -\left(2 + \log\frac{3e}{2\varepsilon}\right)\log M\left(2eR, f\right).$$

The proof of Lemma 2.4 can be found, for example, in Berenstein-Gay [3, pp. 362–363, 4.5.14], which is mainly based upon the following famous Boutroux-Cartan theorem (cf. [4]):

Lemma 2.5 (Boutroux-Cartan Theorem, [4])**.** *Let $z_1, z_2, \ldots, z_n$ be n arbitrary points in the finite complex plane $\mathbb{C}$. Then, for every $H > 0$, the set of the points z satisfying the inequality*

$$\prod_{j=1}^{n}|z - z_j| \leq \left(\frac{H}{e}\right)^n$$

can be covered by a collection of disks whose number does not exceed n, and the sum of whose radii does not exceed $2H$.

Remark 2.6. All the disks as mentioned in Lemma 2.5 are called the Boutroux-Cartan exceptional disks about the positive constant H and the n points $z_1, z_2, \ldots, z_n$ as mentioned in Lemma 2.5 (cf. [24, p. 58]). Moreover, each such point z_j with $1 \leq j \leq n$ as mentioned in Lemma 2.5 can be repeated as many time as its multiplicity implies.

Remark 2.7. From the lines of the proof of Lemma 2.4 (cf. [3, pp. 362–363, 4.5.14]), we can see that the Boutroux-Cartan exceptional disks about the positive constant $2\varepsilon R$ and the zeros, say $z_1, z_2, \ldots, z_n$ of f in the closed disk $|z| \leq R$ are the closed disks mentioned in Lemma 2.4, whose number does not exceed n, and the sum of whose radii does not exceed $4\varepsilon R$, where each such zero z_j with $1 \leq j \leq n$ of f in the closed disk $|z| \leq R$ is repeated as many times as its multiplicity as a zero of f.

Following [21, p. 289], we introduce the definition of a Dirichlet series: by a Dirichlet series we mean, in this paper, a series of the form $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$, where the coefficients a_n are any given numbers, and z is a complex variable. The more general type of series $\sum_{n=1}^{\infty}a_n e^{-\lambda_n s}$ is also known as a Dirichlet series. The special type is obtained by putting $\lambda_n = \log n$. For the theory of the general type we must refer to Hardy-Riesz [7]. Throughout this paper we shall write $z = \sigma + it$, where σ and t are real numbers. We have already had one important example of a Dirichlet series, the Riemann zeta function $\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z}$. We recall the following result due to Chiang-Feng [6]:

Lemma 2.8 ([6, Lemma 4])**.** *Suppose that the Dirichlet series*

$$F_j(z) = \sum_{n=1}^{\infty} \frac{a_j(n)}{n^z}$$

is convergent in the region $\{\sigma + it : \sigma_0 < \sigma < +\infty\}$ with σ_0 being some positive number, where j is a positive integer such $1 \leq j \leq N$ with N being a positive integer, and suppose that ϕ_j with $1 \leq j \leq N$ is a meromorphic function in the complex plane such that its Nevanlinna's characteristic satisfies $T(r, \phi_j) = o(r)$ as $r \to +\infty$. If

$$\sum_{j=1}^{N} \phi_j(z) F_j(z) = 0$$

holds identically in $z \in \{\sigma + it : \sigma_0 < \sigma < +\infty\}$, then

$$\sum_{j=1}^{N} a_j(n) \phi_j(z) = 0$$

holds identically in the complex plane for each positive integer n.

By Lemma 2.8 we can get the following result that is the differentiation analogue of Theorem 1 from [6]:

Lemma 2.9. *The Riemann zeta function ζ does not satisfy any algebraic differential equation with coefficients being meromorphic functions ϕ such that their Nevanlinna's characteristics satisfy $T(r, \phi) = o(r)$ as $r \to +\infty$. That is, if*

$$f\left(z, \zeta(z), \zeta'(z), \ldots, \zeta^{(m)}(z)\right) = 0$$

identically in $z \in \mathbb{C}$, where m is a non-negative integer, and $f(z, u_0, u_1, \ldots, u_m)$ is a polynomial in $m+1$ variables $u_0, u_1, \ldots, u_m$ with coefficients being meromorphic functions ϕ such that their Nevanlinna's characteristics satisfy $T(r, \phi) = o(r)$ as $r \to +\infty$, then $f(z, u_0, u_1, \ldots, u_m)$ is identically zero.

Proof. Suppose that

$$f(z, u_0, u_1, \ldots, u_m) = \sum_{j=1}^{N} \phi_j u_0^{i_{j,0}} u_1^{i_{j,1}} \cdots u_m^{i_{j,m}}, \tag{12}$$

where $i_{j,0}, i_{j,1}, \ldots, i_{j,m}$ with $1 \leq j \leq N$ are non-negative integers and $I_j = (i_{j,0}, i_{j,1}, \ldots, i_{j,m})$ with $1 \leq j \leq N$ is the corresponding multi-indice such that $I_{j_1} \neq I_{j_2}$ for any two positive integers j_1 and j_2 satisfying $1 \leq j_1 < j_2 \leq N$, while $\phi_1, \phi_2, \ldots, \phi_N$ are meromorphic functions not all identically zero in the complex plane such that

$$T(r, \phi_j) = o(r) \quad \text{with} \quad 1 \leq j \leq N \tag{13}$$

as $r \to +\infty$.

Since the Riemann zeta function ζ satisfies the algebraic differential equation $f\left(z,\zeta(z),\zeta'(z),\ldots,\zeta^{(m)}(z)\right)=0$ in $z\in\mathbb{C}$, we have by (12) that

$$f\Big(z,\zeta(z),\zeta'(z),\ldots,\zeta^{(m)}(z)\Big) = \sum_{j=1}^{N}\phi_j(z)\zeta^{i_{j,0}}(z)(\zeta'(z))^{i_{j,1}}\cdots(\zeta^{(m)}(z))^{i_{j,m}}=0 \tag{14}$$

identically in $z\in\mathbb{C}$.

By the definition of the Riemann zeta function $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$, we deduce that the differential monomial $\zeta^{i_{j,0}}(z)(\zeta'(z))^{i_{j,1}}\cdots(\zeta^{(m)}(z))^{i_{j,m}}$ with $1\le j\le N$ is a Dirichlet series, say

$$\sum_{n=1}^{\infty}\frac{A_j(n)}{n^z}=:\zeta^{i_{j,0}}(z)(\zeta'(z))^{i_{j,1}}\cdots(\zeta^{(m)}(z))^{i_{j,m}} \quad \text{with} \quad 1\le j\le N, \tag{15}$$

where $A_j(n)$ with $1\le j\le N$ and $n\in\mathbb{N}$ is some complex number depending only upon j and n, and the Dirichlet series $\sum_{n=1}^{\infty}\frac{A_j(n)}{n^z}$ is convergent in the region $\{z\in\mathbb{C}:\operatorname{Re}(z)>1\}$. By (14), (15) and Lemma 2.8 we have for each $n\in\mathbb{N}$ that

$$\sum_{j=1}^{N}A_j(n)\phi_j(z)=0 \tag{16}$$

identically in $z\in\mathbb{C}$.

Next we let $z_0\in\mathbb{C}$ be a fixed point such that $\phi_j(z_0)\neq\infty$ for each $1\le j\le N$, and such that $\sum_{j=1}^{N}|\phi_j(z_0)|\neq 0$. Then, it follows by (15) and (16) that

$$\sum_{j=1}^{N}\phi_j(z_0)\zeta^{i_{j,0}}(z)(\zeta'(z))^{i_{j,1}}\cdots(\zeta^{(m)}(z))^{i_{j,m}}=0. \tag{17}$$

However, Ostrowski [19] proved that the Riemann zeta function ζ is not a solution of any non-trivial algebraic differential equation with coefficients being polynomials in $\mathbb{C}$. This implies that (17) is impossible. This proves Lemma 2.9. □

3. Proof of Theorem 1.9

Since $P(u_0,u_1,\ldots,u_m,v_0,v_1,\ldots,v_n)$ is a non-trivial polynomial in the $m+n+2$ variables $u_0,u_1,\ldots,u_m,v_0,v_1,\ldots,v_n$ with coefficients being meromorphic functions of order less than one, where m and n are two positive integers, we may rewrite $P(z,u_0,u_1,\ldots,u_m,v_0,v_1,\ldots,v_n)$ into the following form:

$$P(z,u_0,u_1,\ldots,u_m,v_0,v_1,\ldots,v_n) = \sum_{j=0}^{t}a_{I_j}(z,u_0,u_1,\ldots,u_m)v_0^{i_{j,0}}v_1^{i_{j,1}}\cdots v_n^{i_{j,n}} \tag{18}$$

$$= a_{I_0}(z,u_0,u_1,\ldots,u_m) + a_{I_1}(z,u_0,u_1,\ldots,u_m)v_0^{i_{1,0}}v_1^{i_{1,1}}\cdots v_n^{i_{1,n}} + \cdots$$
$$+ a_{I_t}(z,u_0,u_1,\ldots,u_m)v_0^{i_{t,0}}v_1^{i_{t,1}}\cdots v_n^{i_{t,n}},$$

where $I_j = (i_{j,0}, i_{j,1}, \ldots, i_{j,n})$ and $|I_j| = \sum_{k=0}^{n} i_{j,k}$ with $0 \le j \le t$ and $j \in \mathbb{Z}^+ \cup \{0\}$ such that

$$0 = |I_0| < |I_1| < \cdots < |I_t|,$$

and $a_{I_j}(z,u_0,u_1,\ldots,u_m)$ with $0 \le j \le t$ and $j \in \mathbb{Z}^+ \cup \{0\}$ is a polynomial in $u_0,u_1,\ldots,u_m$ with coefficients being meromorphic functions of order less than one. Since $\zeta,\zeta',\ldots,\zeta^{(m)}$ and $\Gamma,\Gamma',\ldots,\Gamma^{(n)}$ satisfy

$$P(z,u_0,u_1,\ldots,u_m,v_0,v_1,\ldots,v_n) = 0 \tag{19}$$

in $z \in \mathbb{C}$, we have by (18) and (19) that

$$\begin{aligned} &P\left(z,\zeta,\zeta',\ldots,\zeta^{(m)},\Gamma,\Gamma',\ldots,\Gamma^{(n)}\right) \\ &= a_{I_0}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right) + a_{I_1}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=0}^{n}\left(\Gamma^{(k)}\right)^{i_{1,k}} \\ &\quad + \cdots + a_{I_t}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=0}^{n}\left(\Gamma^{(k)}\right)^{i_{t,k}} = 0 \end{aligned} \tag{20}$$

identically in $z \in \mathbb{C}$. The second equality of (20) can be rewritten into

$$\begin{aligned} &- a_{I_0}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right) \\ &= a_{I_1}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{1,k}}\Gamma^{|I_1|} \\ &\quad + a_{I_2}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{2,k}}\Gamma^{|I_2|} + \cdots \\ &\quad + a_{I_t}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{t,k}}\Gamma^{|I_t|} \end{aligned} \tag{21}$$

identically in $z \in \mathbb{C}$. Next we prove that the first coefficient $a_{I_0}(z,\zeta,\zeta',\ldots,\zeta^{(m)})$ is identically zero in $\mathbb{C}$. On the contrary, supposing that $a_{I_0}(z,\zeta,\zeta',\ldots,\zeta^{(m)}) \not\equiv 0$ in $\mathbb{C}$, we will derive a contradiction. Indeed, since $a_{I_j}(z,\zeta,\zeta',\ldots,\zeta^{(m)})$ with $0 \le j \le t$ and $j \in \mathbb{Z}^+ \cup \{0\}$ is a polynomial in $\zeta,\zeta',\ldots,\zeta^{(m)}$ with coefficients being meromorphic functions of order less than one, we have

$$a_{I_j}(z,\zeta,\zeta',\ldots,\zeta^{(m)}) = \sum_{p=1}^{N_j} f_{j,p}\zeta^{i_{j,p,0}}(\zeta')^{i_{j,p,1}}\cdots(\zeta^{(m)})^{i_{j,p,m}}, \tag{22}$$

where and in what follows, N_j with $0 \le j \le t$ and $j \in \mathbb{Z}^+ \cup \{0\}$ is some positive integer, $f_{j,p}$ with $0 \le j \le t$, $1 \le p \le N_j$ and $j,p \in \mathbb{Z}^+ \cup \{0\}$ is some meromorphic function of order less than one, and $i_{j,p,0}, i_{j,p,1}, \ldots, i_{j,p,m}$ with

$0 \le j \le t$, $1 \le p \le N_j$ and $j, p \in \mathbb{Z}^+ \cup \{0\}$ are some non-negative integers. By Lemma 2.1 we can see that there exist entire functions $H_{j,p,1}$ and $H_{j,p,2}$ with $0 \le j \le t$, $1 \le p \le N_j$ and $j, p \in \mathbb{Z}^+ \cup \{0\}$, and there exist absolute constants $A_{j,p}$ and $B_{j,p}$ with $0 \le j \le t$, $1 \le p \le N_j$ and $j, p \in \mathbb{Z}^+ \cup \{0\}$ such that

$$f_{j,p} = \frac{H_{j,p,1}}{H_{j,p,2}}, \tag{23}$$

and such that

$$T(r, H_{j,p,l}) < A_{j,p} T(B_{j,p} r, f_{j,p}) \text{ with } l \in \{1,2\}, \text{ when } \ r > 0. \tag{24}$$

By Definition 2 and Remark 1.7 we deduce

$$\rho(H_{j,p,l}) \le \rho(f_{j,p}) < 1 \tag{25}$$

with $0 \le j \le t$, $1 \le p \le N_j$ for $j, p \in \mathbb{Z}^+ \cup \{0\}$ and $l \in \{1,2\}$. By substituting (23) into (22) we have

$$a_{I_j}(z, \zeta, \zeta', \ldots, \zeta^{(m)}) = \sum_{p=1}^{N_j} \frac{H_{j,p,1}}{H_{j,p,2}} \zeta^{i_{j,p,0}} (\zeta')^{i_{j,p,1}} \cdots (\zeta^{(m)})^{i_{j,p,m}} \tag{26}$$

with $0 \le j \le t$, $1 \le p \le N_j$ and $j, p \in \mathbb{Z}^+ \cup \{0\}$. By substituting (26) into (21) we have

$$\begin{aligned} & -\sum_{p=1}^{N_0} \frac{H_{0,p,1}}{H_{0,p,2}} \zeta^{i_{0,p,0}} (\zeta')^{i_{0,p,1}} \cdots (\zeta^{(m)})^{i_{0,p,m}} \\ = & \left(\sum_{p=1}^{N_1} \frac{H_{1,p,1}}{H_{1,p,2}} \zeta^{i_{1,p,0}} (\zeta')^{i_{1,p,1}} \cdots (\zeta^{(m)})^{i_{1,p,m}} \right) \prod_{k=1}^{n} \left(\frac{\Gamma^{(k)}}{\Gamma} \right)^{i_{1,k}} \Gamma^{|I_1|} \\ & + \left(\sum_{p=1}^{N_2} \frac{H_{2,p,1}}{H_{2,p,2}} \zeta^{i_{2,p,0}} (\zeta')^{i_{2,p,1}} \cdots (\zeta^{(m)})^{i_{2,p,m}} \right) \prod_{k=1}^{n} \left(\frac{\Gamma^{(k)}}{\Gamma} \right)^{i_{2,k}} \Gamma^{|I_2|} \\ & + \cdots \\ & + \left(\sum_{p=1}^{N_t} \frac{H_{t,p,1}}{H_{t,p,2}} \zeta^{i_{t,p,0}} (\zeta')^{i_{t,p,1}} \cdots (\zeta^{(m)})^{i_{t,p,m}} \right) \prod_{k=1}^{n} \left(\frac{\Gamma^{(k)}}{\Gamma} \right)^{i_{t,k}} \Gamma^{|I_t|} \end{aligned} \tag{27}$$

in $z \in \mathbb{C}$. Next we set

$$H = \prod_{j=0}^{t} \prod_{p=1}^{N_j} H_{j,p,2} \quad \text{and} \quad H_{j,p} = \frac{H H_{j,p,1}}{H_{j,p,2}} \tag{28}$$

with $1 \le j \le t$, $1 \le p \le N_j$ and $j, p \in \mathbb{Z}^+$.

Multiplying both sides of (27) by the entire function H defined in (28), and then using the right equality in (28), we have

$$-\sum_{p=1}^{N_0} H_{0,p} \zeta^{i_{0,p,0}} (\zeta')^{i_{0,p,1}} \cdots (\zeta^{(m)})^{i_{0,p,m}} \tag{29}$$

$$= \left(\sum_{p=1}^{N_1} H_{1,p}\zeta^{i_{1,p,0}}(\zeta')^{i_{1,p,1}}\cdots(\zeta^{(m)})^{i_{1,p,m}}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{1,k}}\Gamma^{|I_1|}$$
$$+ \left(\sum_{p=1}^{N_2} H_{2,p}\zeta^{i_{2,p,0}}(\zeta')^{i_{2,p,1}}\cdots(\zeta^{(m)})^{i_{2,p,m}}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{2,k}}\Gamma^{|I_2|} + \cdots$$
$$+ \left(\sum_{p=1}^{N_t} H_{t,p}\zeta^{i_{t,p,0}}(\zeta')^{i_{t,p,1}}\cdots(\zeta^{(m)})^{i_{t,p,m}}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{t,k}}\Gamma^{|I_t|}$$

in $z \in \mathbb{C}$. Noting that $H \not\equiv 0$, where H is the entire function defined in (28), we deduce by (26) with $j = 0$, the supposition $a_{I_0}(z, \zeta, \zeta', \ldots, \zeta^{(m)}) \not\equiv 0$ in $\mathbb{C}$ and the right equality in (28) that

$$Ha_{I_0}(z, \zeta, \zeta', \ldots, \zeta^{(m)}) = \sum_{p=1}^{N_0} H_{0,p}\zeta^{i_{0,p,0}}(\zeta')^{i_{0,p,1}}\cdots(\zeta^{(m)})^{i_{0,p,m}} \not\equiv 0. \tag{30}$$

By (25) and (28) we deduce that H and $H_{j,p}$ with $0 \le j \le t$ and $1 \le p \le N_j$ are entire functions such that

$$\rho(H) < 1 \quad \text{and} \quad \rho(H_{j,p}) < 1 \quad \text{with} \quad 0 \le j \le t \quad \text{and} \quad 1 \le p \le N_j. \tag{31}$$

By Lemma 2.2 we can see that for any fixed point $w_* = (w_0, w_1, \ldots, w_m) \in \mathbb{C}^{m+1}$ and its some neighborhood $\Omega \subset \mathbb{C}^{m+1}$, there exist infinitely many points $z_l = \frac{3}{4} + iy_l$ such that when $l \to +\infty$, we have

$$z_l = \frac{3}{4} + iy_l \to \infty, \tag{32}$$

$$\Big(\zeta(z_l), \zeta'(z_l), \ldots, \zeta^{(m)}(z_l)\Big) \to (w_0, w_1, \ldots, w_m) \tag{33}$$

and

$$\zeta^{i_{j,p,0}}(z_l)(\zeta'(z_l))^{i_{j,p,1}}\cdots(\zeta^{(m)}(z_l))^{i_{j,p,m}} \to w_0^{i_{j,p,0}}w_1^{i_{j,p,1}}\cdots w_m^{i_{j,p,m}} =: c_{j,p} \tag{34}$$

with $0 \le j \le t$ and $1 \le p \le N_j$, where $c_{j,p}$ with $0 \le j \le t$ and $1 \le p \le N_j$ is a finite complex number.

By (31) and Remark 1.7 we can see that there exists some positive number δ_0 satisfying $0 < \delta_0 < 1$, such that

$$|H(z_l)| \le M(|z_l|, H) \le e^{|z_l|^{\delta_0}} \quad \text{and} \quad |H_{j,p}(z_l)| \le M(|z_l|, H_{j,p}) \le e^{|z_l|^{\delta_0}} \tag{35}$$

for any positive integer l.

Using the reasoning of the lines of [21, p. 151] we have for a fixed value of $x = 3/4$ that

$$\left|\Gamma\left(\frac{3}{4} + iy\right)\right| \sim e^{-\frac{\pi}{2}|y|}|y|^{\frac{1}{4}}\sqrt{2\pi} \tag{36}$$

as $y \to \pm\infty$. On the other hand, by Lemma 2.3 we have for any given positive integer $q \geq 1$ that

$$\frac{\Gamma^{(q)}(z)}{\Gamma(z)} = (\log z)^q (1 + o(1)), \tag{37}$$

uniformly for any small $\varepsilon > 0$ and for all $z \in \mathbb{C} \setminus \{z : |\arg z - \pi| \leq \varepsilon\}$ such that $z \to \infty$. According to (30), we consider the following two cases:

Case 1. Suppose that there exists an infinite subsequence of the infinite sequence $\{z_l\}$, say itself, and there exists some positive number ε_0, such that

$$\begin{aligned} &\left|H(z_l)a_{I_0}(z_l, \zeta(z_l), \zeta'(z_l), \dots, \zeta^{(m)}(z_l))\right| \\ &\quad = \left|\sum_{p=1}^{N_0} H_{0,p}(z_l)\zeta^{i_{0,p,0}}(z_l)(\zeta'(z_l))^{i_{0,p,1}} \cdots (\zeta^{(m)}(z_l))^{i_{0,p,m}}\right| \geq \varepsilon_0 \end{aligned} \tag{38}$$

as $z_l = \frac{3}{4} + y_l \to \infty$.

Since $0 < \delta_0 < 1$ and

$$\lim_{l \to +\infty} \frac{|z_l|}{|y_l|} = \lim_{l \to +\infty} \frac{\left|\frac{3}{4} + iy_l\right|}{|y_l|} = \lim_{l \to +\infty} \frac{\sqrt{\left(\frac{3}{4}\right)^2 + y_l^2}}{|y_l|} = 1,$$

we can find a sufficiently small positive number ε_1 satisfying $0 < \varepsilon_1 < 1$, such that

$$|z_l| \leq (1 + \varepsilon_1)|y_l| \quad \text{and} \quad 1 < (1 + \varepsilon_1)^{\delta_0} < \frac{\pi}{2} \tag{39}$$

as $l \to +\infty$. By (29), (34), (35), (36), (37), (38) and (39) we deduce

$$\begin{aligned} \varepsilon_0 &\leq \left|\sum_{p=1}^{N_0} H_{0,p}(z_l)\zeta^{i_{0,p,0}}(z_l)(\zeta'(z_l))^{i_{0,p,1}} \cdots (\zeta^{(m)}(z_l))^{i_{0,p,m}}\right| \\ &\leq \left(\sum_{p=1}^{N_1} |H_{1,p}(z_l)| \left|\zeta^{i_{1,p,0}}(z_l)(\zeta'(z_l))^{i_{1,p,1}} \cdots (\zeta^{(m)}(z_l))^{i_{1,p,m}}\right|\right) \\ &\quad \times \prod_{k=1}^{n} \left|\frac{\Gamma^{(k)}(z_l)}{\Gamma(z_l)}\right|^{i_{1,k}} |\Gamma(z_l)|^{|I_1|} \\ &\quad + \left(\sum_{p=1}^{N_2} |H_{2,p}(z_l)| \left|\zeta^{i_{2,p,0}}(z_l)(\zeta'(z_l))^{i_{2,p,1}} \cdots (\zeta^{(m)}(z_l))^{i_{2,p,m}}\right|\right) \\ &\quad \times \prod_{k=1}^{n} \left|\frac{\Gamma^{(k)}(z_l)}{\Gamma(z_l)}\right|^{i_{2,k}} |\Gamma(z_l)|^{|I_2|} + \cdots \\ &\quad + \left(\left|\sum_{p=1}^{N_t} H_{t,p}(z_l)\zeta^{i_{t,p,0}}(z_l)(\zeta'(z_l))^{i_{t,p,1}} \cdots (\zeta^{(m)}(z_l))^{i_{t,p,m}}\right|\right) \end{aligned} \tag{40}$$

$$
\begin{aligned}
&\times \prod_{k=1}^{n} \left| \frac{\Gamma^{(k)}(z_l)}{\Gamma(z_l)} \right|^{i_{t,k}} |\Gamma(z_l)|^{|I_t|} \\
&\leq e^{|z_l|^{\delta_0}} \left(\sum_{p=1}^{N_1} |c_{1,p}| \right) \prod_{k=1}^{n} \left| (\log z_l)^k \,(1+o(1)) \right|^{i_{1,k}} e^{|I_1|\left(\varepsilon - \frac{\pi}{2}\right)|y_l|} (\sqrt{2\pi})^{|I_1|} \\
&\quad + e^{|z_l|^{\delta_0}} \left(\sum_{p=1}^{N_2} |c_{2,p}| \right) \prod_{k=1}^{n} \left| (\log z_l)^k \,(1+o(1)) \right|^{i_{2,k}} e^{|I_2|\left(\varepsilon - \frac{\pi}{2}\right)|y_l|} (\sqrt{2\pi})^{|I_2|} \\
&\quad + \cdots \\
&\quad + e^{|z_l|^{\delta_0}} \left(\sum_{p=1}^{N_t} |c_{t,p}| \right) \prod_{k=1}^{n} \left| (\log z_l)^k \,(1+o(1)) \right|^{i_{t,k}} e^{|I_t|\left(\varepsilon - \frac{\pi}{2}\right)|y_l|} (\sqrt{2\pi})^{|I_t|} \\
&\leq e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}} \left(\sum_{p=1}^{N_1} |c_{1,p}| \right) \prod_{k=1}^{n} \left| (\log z_l)^k \,(1+o(1)) \right|^{i_{1,k}} e^{|I_1|\left(\varepsilon - \frac{\pi}{2}\right)|y_l|} (\sqrt{2\pi})^{|I_1|} \\
&\quad + e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}} \left(\sum_{p=1}^{N_2} |c_{2,p}| \right) \prod_{k=1}^{n} \left| (\log z_l)^k \,(1+o(1)) \right|^{i_{2,k}} e^{|I_2|\left(\varepsilon - \frac{\pi}{2}\right)|y_l|} (\sqrt{2\pi})^{|I_2|} \\
&\quad + \cdots \\
&\quad + e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}} \left(\sum_{p=1}^{N_t} |c_{t,p}| \right) \prod_{k=1}^{n} \left| (\log z_l)^k \,(1+o(1)) \right|^{i_{t,k}} e^{|I_t|\left(\varepsilon - \frac{\pi}{2}\right)|y_l|} (\sqrt{2\pi})^{|I_t|} \\
&\to 0 \quad \text{as} \quad l \to +\infty.
\end{aligned}
$$

This is a contradiction.

Case 2. Suppose that

$$
\begin{aligned}
(41) \qquad & H(z_l)a_{I_0}(z_l, \zeta(z_l), \zeta'(z_l), \ldots, \zeta^{(m)}(z_l)) \\
&= \sum_{p=1}^{N_0} H_{0,p}(z_l)\zeta^{i_{0,p,0}}(z_l)(\zeta'(z_l))^{i_{0,p,1}} \cdots (\zeta^{(m)}(z_l))^{i_{0,p,m}} \to 0
\end{aligned}
$$

as $z_l = \frac{3}{4} + y_l \to \infty$. We consider the following two subcases:

Subcase 2.1. Suppose that (41) holds and there exists an infinite subsequence of the infinite sequence $\{z_l\}$, say itself, such that

$$
\begin{aligned}
(42) \qquad & H(z_l)a_{I_0}(z_l, \zeta(z_l), \zeta'(z_l), \ldots, \zeta^{(m)}(z_l)) \\
&= \sum_{p=1}^{N_0} H_{0,p}(z_l)\zeta^{i_{0,p,0}}(z_l)(\zeta'(z_l))^{i_{0,p,1}} \cdots (\zeta^{(m)}(z_l))^{i_{0,p,m}} \neq 0
\end{aligned}
$$

for each positive integer l.

First of all, we use the lines of Case 1 to get (40). By (40) and (42) we have

$$
(43) \quad 0 < \left| \sum_{p=1}^{N_0} H_{0,p}(z_l)\zeta^{i_{0,p,0}}(z_l)(\zeta'(z_l))^{i_{0,p,1}} \cdots (\zeta^{(m)}(z_l))^{i_{0,p,m}} \right|
$$

$$
\begin{aligned}
&\leq e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}} \left(\sum_{p=1}^{N_1}|c_{1,p}|\right) \prod_{k=1}^{n} \left|(\log z_l)^k (1+o(1))\right|^{i_{1,k}} e^{|I_1|\left(\varepsilon-\frac{\pi}{2}\right)|y_l|} (\sqrt{2\pi})^{|I_1|} \\
&\quad + e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}} \left(\sum_{p=1}^{N_2}|c_{2,p}|\right) \prod_{k=1}^{n} \left|(\log z_l)^k (1+o(1))\right|^{i_{2,k}} e^{|I_2|\left(\varepsilon-\frac{\pi}{2}\right)|y_l|} (\sqrt{2\pi})^{|I_2|} \\
&\quad + \cdots \\
&\quad + e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}} \left(\sum_{p=1}^{N_t}|c_{t,p}|\right) \prod_{k=1}^{n} \left|(\log z_l)^k (1+o(1))\right|^{i_{t,k}} e^{|I_t|\left(\varepsilon-\frac{\pi}{2}\right)|y_l|} (\sqrt{2\pi})^{|I_t|}
\end{aligned}
$$

as $l \to +\infty$. By (34) and the equality in (42) we deduce

$$
\begin{aligned}
&H(z_l)a_{I_0}(z_l, \zeta(z_l), \zeta'(z_l), \ldots, \zeta^{(m)}(z_l)) \\
&= \sum_{p=1}^{N_0} H_{0,p}(z_l)\zeta^{i_{0,p,0}}(z_l)(\zeta'(z_l))^{i_{0,p,1}} \cdots (\zeta^{(m)}(z_l))^{i_{0,p,m}} \\
&= \sum_{p=1}^{N_0} H_{0,p}(z_l) b_{0,p}(z_l) \\
&= \sum_{p=1}^{N_0} H_{0,p}(z_l) c_{0,p}(1+o(1))
\end{aligned} \tag{44}
$$

with

$$
b_{0,p}(z_l) = \zeta^{i_{0,p,0}}(z_l)(\zeta'(z_l))^{i_{0,p,1}} \cdots (\zeta^{(m)}(z_l))^{i_{0,p,m}} = c_{0,p}(1+o(1)) \tag{45}
$$

as $l \to +\infty$. For a large fixed positive integer l, by (28) and (44) we can see that $\sum_{p=1}^{N_0} H_{0,p}(z) b_{0,p}(z_l)$ is an entire function in $z \in \mathbb{C}$. Moreover, it follows by (42), (44) and (45) that

$$
\sum_{p=1}^{N_0} H_{0,p}(z_l) b_{0,p}(z_l) \neq 0, \tag{46}
$$

which implies that

$$
\sum_{p=1}^{N_0} H_{0,p}(z) b_{0,p}(z_l) \not\equiv 0 \quad \text{in} \quad \mathbb{C}. \tag{47}
$$

By (47) we can see that there exists an integer m_0 and there exists a finite non-zero complex number A_0 such that $A_0 z^{m_0}\left(\sum_{p=1}^{N_0} b_{0,p}(z_l) H_{0,p}(z)\right)$ is an entire function in $z \in \mathbb{C}$ and such that

$$
\lim_{z \to 0} A_0 z^{m_0} \left(\sum_{p=1}^{N_0} b_{0,p}(z_l) H_{0,p}(z)\right) = 1 \tag{48}
$$

and

$$A_0 z_l^{m_0}\left(\sum_{p=1}^{N_0} b_{0,p}(z_l)H_{0,p}(z_l)\right)\neq 0,\infty \tag{49}$$

for the large positive integer l. By (48), (49), Lemma 2.4 and the obtained result that $A_0 z^{m_0}\left(\sum_{p=1}^{N_0} b_{0,p}(z_l)H_{0,p}(z)\right)$ is an entire function in $z\in\mathbb{C}$, we can see that for the large positive integer l and a small positive number ε satisfying $0<\varepsilon<\frac{3e}{2}$, we have for $R=2|z_l|$ that

$$\begin{aligned}
&\log\left|A_0 z_l^{m_0}\left(\sum_{p=1}^{N_0} b_{0,p}(z_l)H_{0,p}(z_l)\right)\right| \\
&\quad > -\left(2+\log\frac{3e}{2\varepsilon}\right)\log M\left(2eR, A_0 z^{m_0}\left(\sum_{p=1}^{N_0} b_{0,p}(z_l)H_{0,p}(z)\right)\right)
\end{aligned} \tag{50}$$

for the large positive integer l. By (35), (45), (49) and (50) we deduce

$$\begin{aligned}
&\log\frac{1}{\left|\sum\limits_{p=1}^{N_0} b_{0,p}(z_l)H_{0,p}(z_l)\right|}+\log\left|\frac{1}{A_0}\right|-m_0\log|z_l| \\
&= \log\frac{1}{\left|A_0 z_l^{m_0}\left(\sum\limits_{p=1}^{N_0} b_{0,p}(z_l)H_{0,p}(z_l)\right)\right|} \\
&< \left(2+\log\frac{3e}{2\varepsilon}\right)\log M\left(2eR, A_0 z^{m_0}\left(\sum_{p=1}^{N_0} b_{0,p}(z_l)H_{0,p}(z)\right)\right) \\
&= \left(2+\log\frac{3e}{2\varepsilon}\right)\log\left\{\max_{|z|=2eR}\left\{\left|A_0 z^{m_0}\left(\sum_{p=1}^{N_0} b_{0,p}(z_l)H_{0,p}(z)\right)\right|\right\}\right\} \\
&\le \left(2+\log\frac{3e}{2\varepsilon}\right)\log\left(|A_0||z_l|^{m_0}\max_{|z|=2eR}\left\{\left|\sum_{p=1}^{N_0} b_{0,p}(z_l)H_{0,p}(z)\right|\right\}\right) \\
&\le \left(2+\log\frac{3e}{2\varepsilon}\right)\log\left(|A_0||z_l|^{m_0}\left(\sum_{p=1}^{N_0} |b_{0,p}(z_l)|\,M(2eR,H_{0,p})\right)\right) \\
&\le \left(2+\log\frac{3e}{2\varepsilon}\right)\log\left(|A_0||z_l|^{m_0}\left(\sum_{p=1}^{N_0} |2c_{0,p}|\,M(2eR,H_{0,p})\right)\right) \\
&\le \left(2+\log\frac{3e}{2\varepsilon}\right)\log\left(|A_0||z_l|^{m_0}\left(e^{(2eR)^{\delta_0}}\sum_{p=1}^{N_0} |2c_{0,p}|\right)\right)
\end{aligned} \tag{51}$$

$$= \left(2+\log\frac{3e}{2\varepsilon}\right)\log\left(|A_0||z_l|^{m_0}\left(e^{(4e|z_l|)^{\delta_0}}\sum_{p=1}^{N_0}|2c_{0,p}|\right)\right)$$
$$= \left(2+\log\frac{3e}{2\varepsilon}\right)(4e|z_l|)^{\delta_0}+|m_0|\left(2+\log\frac{3e}{2\varepsilon}\right)\log|z_l|+O(1)$$

for the large positive integer l. By (39) and (51) we have

$$\log\frac{1}{\left|\sum_{p=1}^{N_0}b_{0,p}(z_l)H_{0,p}(z_l)\right|} \tag{52}$$
$$\le \left(2+\log\frac{3e}{2\varepsilon}\right)(4e|z_l|)^{\delta_0}+|m_0|\left(3+\log\frac{3e}{2\varepsilon}\right)\log|z_l|+O(1)$$
$$\le \left(2+\log\frac{3e}{2\varepsilon}\right)(4e)^{\delta_0}(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}+|m_0|\left(3+\log\frac{3e}{2\varepsilon}\right)\log|z_l|+O(1)$$
$$= B_1|y_l|^{\delta_0}+B_2\log|z_l|+B_3$$

for the large positive integer l. Here B_3 is a sufficiently large positive constant, B_1 and B_2 are also positive constants that satisfy

$$B_1=\left(2+\log\frac{3e}{2\varepsilon}\right)(4e)^{\delta_0}(1+\varepsilon_1)^{\delta_0}\quad\text{and}\quad B_2=|m_0|\left(3+\log\frac{3e}{2\varepsilon}\right),$$

respectively. By (52) we have

$$\frac{1}{\left|\sum_{p=1}^{N_0}b_{0,p}(z_l)H_{0,p}(z_l)\right|}\le e^{B_3}|z_l|^{B_2}e^{B_1|y_l|^{\delta_0}} \tag{53}$$

for the large positive integer l.

By dividing two sides of the second inequality of (43) by

$$\left|\sum_{p=1}^{N_0}b_{0,p}(z_l)H_{0,p}(z_l)\right|,$$

we have by (53), the second equality of (44) and the result $0<\delta_0<1$ that

$$1\le\frac{e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}}\left(\sum_{p=1}^{N_1}|c_{1,p}|\right)\prod_{k=1}^{n}\left|(\log z_l)^k(1+o(1))\right|^{i_{1,k}}e^{|I_1|\left(\varepsilon-\frac{\pi}{2}\right)|y_l|}(\sqrt{2\pi})^{|I_1|}}{\sum_{p=1}^{N_0}b_{0,p}(z_l)H_{0,p}(z_l)}$$
$$+\frac{e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}}\left(\sum_{p=1}^{N_2}|c_{2,p}|\right)\prod_{k=1}^{n}\left|(\log z_l)^k(1+o(1))\right|^{i_{2,k}}e^{|I_2|\left(\varepsilon-\frac{\pi}{2}\right)|y_l|}(\sqrt{2\pi})^{|I_2|}}{\sum_{p=1}^{N_0}b_{0,p}(z_l)H_{0,p}(z_l)}$$
$$+\cdots$$

$$+\frac{e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}}\left(\sum_{p=1}^{N_t}|c_{t,p}|\right)\prod_{k=1}^{n}\left|(\log z_l)^k(1+o(1))\right|^{i_{t,k}}e^{|I_t|\left(\varepsilon-\frac{\pi}{2}\right)|y_l|}(\sqrt{2\pi})^{|I_t|}}{\sum_{p=1}^{N_0}b_{0,p}(z_l)H_{0,p}(z_l)}$$

$$\begin{aligned}
&\leq e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}}\left(\sum_{p=1}^{N_1}|c_{1,p}|\right)\prod_{k=1}^{n}\left|(\log z_l)^k(1+o(1))\right|^{i_{1,k}}e^{|I_1|\left(\varepsilon-\frac{\pi}{2}\right)|y_l|}(\sqrt{2\pi})^{|I_1|}\\
&\quad\times e^{B_3}|z_l|^{B_2}e^{B_1|y_l|^{\delta_0}}\\
&\quad+e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}}\left(\sum_{p=1}^{N_2}|c_{2,p}|\right)\prod_{k=1}^{n}\left|(\log z_l)^k(1+o(1))\right|^{i_{2,k}}e^{|I_2|\left(\varepsilon-\frac{\pi}{2}\right)|y_l|}(\sqrt{2\pi})^{|I_2|}\\
&\quad\times e^{B_3}|z_l|^{B_2}e^{B_1|y_l|^{\delta_0}}+\cdots\\
&\quad+e^{(1+\varepsilon_1)^{\delta_0}|y_l|^{\delta_0}}\left(\sum_{p=1}^{N_t}|c_{t,p}|\right)\prod_{k=1}^{n}\left|(\log z_l)^k(1+o(1))\right|^{i_{t,k}}e^{|I_t|\left(\varepsilon-\frac{\pi}{2}\right)|y_l|}(\sqrt{2\pi})^{|I_t|}\\
&\quad\times e^{B_3}|z_l|^{B_2}e^{B_1|y_l|^{\delta_0}}\to 0\quad\text{as}\quad l\to+\infty.
\end{aligned}$$

This is a contradiction.

Subcase 2.2. Suppose that the infinite sequence of $\{z_l\}$ is a zero-sequence of the meromorphic function $Ha_{I_0}(z,\zeta,\zeta',\ldots,\zeta^{(m)})$ at most finitely many exceptions of the points of $\{z_l\}$ such that they satisfy (41). Without loss of generality, we suppose that the infinite sequence of $\{z_l\}$ is a zero-sequence of the meromorphic function $Ha_{I_0}(z,\zeta,\zeta',\ldots,\zeta^{(m)})$. Then,

$$\begin{aligned}
&H(z_l)a_{I_0}(z_l,\zeta(z_l),\zeta'(z_l),\ldots,\zeta^{(m)}(z_l))\\
&\quad=\sum_{p=1}^{N_0}H_{0,p}(z_l)\zeta^{i_{0,p,0}}(z_l)(\zeta'(z_l))^{i_{0,p,1}}\cdots(\zeta^{(m)}(z_l))^{i_{0,p,m}}=0.
\end{aligned}\tag{54}$$

By (54), the inequality of (30) and the uniqueness theorem of analytic functions (cf. [1, p. 127]) we can see that there exists an infinite sequence of $\{\hat{z}_l\}\subset\mathbb{C}$, and there exist two infinite decreasing sequences of positive numbers of $\{\delta_l\}$ and $\{\varepsilon_l\}$ such that

$$\delta_l\to 0\quad\text{and}\quad\varepsilon_l\to 0,\tag{55}$$

and

$$\begin{aligned}
&H(\hat{z}_l)a_{I_0}(\hat{z}_l,\zeta(\hat{z}_l),\zeta'(\hat{z}_l),\ldots,\zeta^{(m)}(\hat{z}_l))\\
&\quad=\sum_{p=1}^{N_0}H_{0,p}(\hat{z}_l)\zeta^{i_{0,p,0}}(\hat{z}_l)(\zeta'(\hat{z}_l))^{i_{0,p,1}}\cdots(\zeta^{(m)}(\hat{z}_l))^{i_{0,p,m}}\to 0
\end{aligned}\tag{56}$$

as $l\to+\infty$, and such that

$$0<|\hat{z}_l-z_l|<\delta_l,\tag{57}$$

$$
\begin{aligned}
(58)\qquad & H(\hat{z}_l)a_{I_0}(\hat{z}_l,\zeta(\hat{z}_l),\zeta'(\hat{z}_l),\ldots,\zeta^{(m)}(\hat{z}_l)) \\
&= \sum_{p=1}^{N_0} H_{0,p}(\hat{z}_l)\zeta^{i_{0,p,0}}(\hat{z}_l)(\zeta'(\hat{z}_l))^{i_{0,p,1}}\cdots(\zeta^{(m)}(\hat{z}_l))^{i_{0,p,m}} \neq 0
\end{aligned}
$$

and

$$
(59)\qquad \left|\zeta^{i_{j,p,0}}(\hat{z}_l)\prod_{k=1}^{m}(\zeta^{(k)}(\hat{z}_l))^{i_{j,p,k}} - \zeta^{i_{j,p,0}}(z_l)\prod_{k=1}^{m}(\zeta^{(k)}(z_l))^{i_{j,p,k}}\right| < \varepsilon_l
$$

with $0 \le j \le t$, $1 \le p \le N_j$ and $j, p \in \mathbb{Z}^+ \cup \{0\}$ for the large positive integer l. Moreover, by (34) and (59) we deduce for $0 \le j \le t$, $1 \le p \le N_j$ and $j, p \in \mathbb{Z}^+ \cup \{0\}$ that

$$
(60)\qquad \zeta^{i_{j,p,0}}(\hat{z}_l)(\zeta'(\hat{z}_l))^{i_{j,p,1}}\cdots(\zeta^{(m)}(\hat{z}_l))^{i_{j,p,m}} \to c_{j,p} \quad \text{as} \quad l \to +\infty.
$$

Next we replace the infinite sequence of $\{\hat{z}_l\}$ instead of the infinite sequence of $\{z_l\}$, and use (55)-(60) and the lines of the reasoning of Subcase 2.1, we can get a contradiction. Therefore, we prove

$$
(61)\qquad a_{I_0}(z,\zeta,\zeta',\ldots,\zeta^{(m)}) = 0
$$

identically in $\mathbb{C}$. By substituting (61) into (21), we can see that (21) can be rewritten into

$$
\begin{aligned}
(62)\qquad 0 &= a_{I_1}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{1,k}}\Gamma^{|I_1|} \\
&+ a_{I_2}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{2,k}}\Gamma^{|I_2|} + \cdots \\
&+ a_{I_t}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{t,k}}\Gamma^{|I_t|}
\end{aligned}
$$

identically in $z \in \mathbb{C}$.

By (62) and the supposition $0 = |I_0| < |I_1| < \cdots < |I_t|$ we deduce

$$
\begin{aligned}
(63)\qquad 0 &= a_{I_1}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{1,k}} \\
&+ a_{I_2}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{2,k}}\Gamma^{|I_2|-|I_1|} + \cdots \\
&+ a_{I_t}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)\prod_{k=1}^{n}\left(\frac{\Gamma^{(k)}}{\Gamma}\right)^{i_{t,k}}\Gamma^{|I_t|-|I_1|}
\end{aligned}
$$

identically in $z \in \mathbb{C}$. By (37) and (63) we deduce

$$
(64)\qquad 0 = a_{I_1}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)(\log z)^{\sum_{k=1}^{n} k i_{1,k}}(1+o(1))
$$

$$
\begin{aligned}
&+ a_{I_2}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)(\log z)^{\sum_{k=1}^{n} k i_{2,k}}\Gamma^{|I_2|-|I_1|}(1+o(1)) + \cdots \\
&+ a_{I_t}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)(\log z)^{\sum_{k=1}^{n} k i_{t,k}}\Gamma^{|I_t|-|I_1|}(1+o(1)),
\end{aligned}
$$

uniformly for any small $\varepsilon > 0$ and for all $z \in \mathbb{C} \setminus \{z : |\arg z - \pi| \le \varepsilon\}$ such that $z \to \infty$.

Next we use (64) and the lines of the reasoning in Case 2 to deduce

$$
a_{I_1}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right) = 0 \tag{65}
$$

identically in $\mathbb{C}$. By substituting (65) into (64) we have

$$
\begin{aligned}
0 = {} & a_{I_2}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)(\log z)^{\sum_{k=1}^{n} k i_{2,k}}\Gamma^{|I_2|-|I_1|}(1+o(1)) + \cdots \\
&+ a_{I_t}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)(\log z)^{\sum_{k=1}^{n} k i_{t,k}}\Gamma^{|I_t|-|I_1|}(1+o(1)),
\end{aligned} \tag{66}
$$

uniformly for any small $\varepsilon > 0$ and for all $z \in \mathbb{C} \setminus \{z : |\arg z - \pi| \le \varepsilon\}$ such that $z \to \infty$.

Next we use the supposition $0 = |I_0| < |I_1| < |I_2| < \cdots < |I_t|$ and the above same argument to deduce that all the coefficients $a_{I_t}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)$ with $2 \le j \le t$ in (66) are identically zero in $\mathbb{C}$. This together with (61) and (65) implies that all the coefficients $a_{I_t}\left(z,\zeta,\zeta',\ldots,\zeta^{(m)}\right)$ with $0 \le j \le t$ and $j \in \mathbb{Z}$ in (20) are identically zero in $\mathbb{C}$. Combining this with Lemma 2.9 and the result that $a_{I_j}(z,u_0,u_1,\ldots,u_m)$ with $0 \le j \le t$ is a polynomial in the variables $u_0,u_1,\ldots,u_m$ with coefficients being meromorphic functions of order less than one, we deduce that all the coefficients a_{I_j} with $0 \le j \le t$ and $j \in \mathbb{Z}$ in (18) are identically zero. This implies that the polynomial P is identically zero. Theorem 1.9 is thus completely proved.

References

[1] L. Ahlfors, *Complex Analysis*, McGraw-Hill, New York, 1979.

[2] S. B. Bank and R. P. Kaufman, *A note on Hölder's theorem concerning the gamma function*, Math. Ann. **232** (1978), no. 2, 115–120. https://doi.org/10.1007/BF01421399

[3] C. A. Berenstein and R. Gay, *Complex Variables*, Graduate Texts in Mathematics, 125, Springer, New York, 1991. https://doi.org/10.1007/978-1-4612-3024-3

[4] H. P. Cartan, *Sur les systèmes de fonctions holomorphes à variétés linéaires lacunaires et leurs applications*, Ann. Sci. École Norm. Sup. (3) **45** (1928), 255–346.

[5] W. Chen and Q. Wang, *On the differential and difference independence of* Γ *and* ζ, Acta Math. Sci. Ser. B (Engl. Ed.) **41** (2021), no. 2, 505–516. https://doi.org/10.1007/s10473-021-0214-1

[6] Y. M. Chiang and S. Feng, *Difference independence of the Riemann zeta function*, Acta Arith. **125** (2006), no. 4, 317–329. https://doi.org/10.4064/aa125-4-2

[7] G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 18, Stechert-Hafner, Inc., New York, 1964.

[8] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[9] D. Hilbert, *Mathematische Probleme*, Archiv f. Math. u. Phys. 3. Reihe. Bd.I, 1901, S. 44–63; S. 213–237.
[10] O. Hölder, *Üeber die Eigenschaft der Gammafunction keiner algevraischen Differentialgleichung zu genügen*, Math. Ann. **28** (1886), no. 1, 1–13.
[11] I. Laine, *Nevanlinna theory and complex differential equations*, De Gruyter Studies in Mathematics, 15, de Gruyter, Berlin, 1993. https://doi.org/10.1515/9783110863147
[12] B. Q. Li and Z. Ye, *On differential independence of the Riemann zeta function and the Euler gamma function*, Acta Arith. **135** (2008), no. 4, 333–337. https://doi.org/10.4064/aa135-4-2
[13] B. Q. Li and Z. Ye, *Algebraic differential equations concerning the Riemann zeta function and the Euler gamma function*, Indiana Univ. Math. J. **59** (2010), no. 4, 1405–1415. https://doi.org/10.1512/iumj.2010.59.3986
[14] B. Q. Li and Z. Ye, *On algebraic differential properties of the Riemann ζ-function and the Euler Γ-function*, Complex Var. Elliptic Equ. **56** (2011), no. 1-4, 137–145. https://doi.org/10.1080/17476930903394788
[15] B. Q. Li and Z. Ye, *Algebraic differential equations with functional coefficients concerning ζ and Γ*, J. Differential Equations **260** (2016), no. 2, 1456–1464. https://doi.org/10.1016/j.jde.2015.09.035
[16] L. Markus, *Differential independence of Γ and ζ*, J. Dynam. Differential Equations **19** (2007), no. 1, 133–154. https://doi.org/10.1007/s10884-006-9034-1
[17] J. Miles, *Quotient representations of meromorphic functions*, J. Analyse Math. **25** (1972), 371–388. https://doi.org/10.1007/BF02790046
[18] D. D. Mordykhai-Boltovskoi, *On hypertranscendence of the function $\xi(x, s)$*, Izv. Politekh. Inst. Warsaw **2** (1914), 1–16.
[19] A. M. Ostrowski, *Über Dirichletsche Reihen und algebraische Differentialgleichungen*, Math. Z. **8** (1920), no. 3-4, 241–298. https://doi.org/10.1007/BF01206530
[20] J. Steuding, *Value-distribution of L-functions*, Lecture Notes in Mathematics, 1877, Springer, Berlin, 2007.
[21] E. C. Titchmarsh, *The Theory of Functions*, second edition, Oxford Univ. Press, Oxford, 1939.
[22] S. Voronin, *The distribution of the nonzero values of the Riemann ζ-function*, Trudy Mat. Inst. Steklov. **128** (1972), 131–150, 260.
[23] Q. Y. Wang, Z. Li, M. Liu, and N. Li, *On algebraic differential equations concerning the Riemann-zeta function and the Euler-gamma function*, Complex Var. Elliptic Equ. **67** (2022), no. 10, 2507–2518. https://doi.org/10.1080/17476933.2021.1931849
[24] L. Yang, *Value Distribution Theory*, Springer-Verlag, Berlin Heidelberg, 1993.
[25] C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557, Kluwer Acad. Publ., Dordrecht, 2003.

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