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ISOTROPIC MEAN BERWALD FINSLER WARPED PRODUCT METRICS

Mehran Gabrani, Bahman Rezaei, and Esra Sengelen Sevim

Abstract. It is our goal in this study to present the structure of isotropic mean Berwald Finsler warped product metrics. We bring out the rich class of warped product Finsler metrics behaviour under this condition. We show that every Finsler warped product metric of dimension $n \geq 2$ is of isotropic mean Berwald curvature if and only if it is a weakly Berwald metric. Also, we prove that every locally dually flat Finsler warped product metric is weakly Berwaldian. Finally, we prove that every Finsler warped product metric is of isotropic Berwald curvature if and only if it is a Berwald metric.

1. Introduction

Warped product metrics have a remarkable and significant application in many areas in Mathematics, Riemannian geometry, Finsler geometry as well as Physics. Moreover, the class of warped product metrics can often be interpreted as key space models for general relativity theory and theory of space-time structure.

Recently, the significant and interesting warped product Finsler metric introduced by B. Chen, Z. Shen, and L. Zhao has attracted the attention of many authors [1]:

$$F(u,\ v) = \breve{\alpha}(\ \breve{u},\ \breve{v}\ )\phi\Big(u^1, \frac{v^1}{\breve{\alpha}(\ \breve{u},\ \breve{v}\ )}\Big), \tag{1}$$

where $u = (u^1, \breve{u})$, $v = v^1 \frac{\partial}{\partial u^1} + \breve{v}$. ϕ is a function on $\mathbb{R}^2$. They have used the notion of warped product which is equipped with a Riemannian metric. $M^n := I \times \breve{M}^{n-1}$, where I is an interval of $\mathbb{R}$ and $\breve{M}^{n-1}$ is an $(n-1)$-dimensional manifold. According to [1, Lemma 3.1], they proved that this class of Finsler metrics includes spherically symmetric Finsler metrics and also they obtained a formula of the flag curvature and Ricci curvature of Finsler warped product metrics to characterize these metrics to be Einstein [1]. H.

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Liu and X. Mo have given a local characterization of this metric with vanishing Douglas curvature [9]. In [6], M. Gabrani, B. Rezaei, and E. S. Sevim have obtained a differential equation that characterizes a Finsler warped product metric with isotropic E-curvature. Moreover, they have characterized the Landsberg Finsler warped product metrics [7].

Immediately after this study, P. Marcial and Z. Shen have introduced another significant type of warped product Finsler metrics, namely, it is similar to (1). They have been motivated to construct this type of warped product metric, inspiring the Schwarzschild metric, which is most likely getting the exact solution of the Einstein field equation; It determines the gravitational field around a static, spherically symmetric single body with no charge [3].

The new type class of warped product Finsler metrics is consistent with the form of metrics modeling static spacetimes and is simplified by spherical symmetry over spatial coordinates, which emerged from the Schwarzschild metric in isotropic coordinates, and it is given as follows [10]:

Consider the product manifold $M = \mathbb{R} \times \mathbb{R}^n$ with their Euclidean metrics dt^2 and α^2, respectively. Let $x = (x^0, \bar{x})$, where $\bar{x} = (x^1, \ldots, x^n)$ be a local coordinate system on M. From now on, our index conventions are as follows:

$$1 \leq A, B, \ldots \leq n, \quad 2 \leq i, j, \ldots \leq n.$$

Therefore, a vector y on M can be written as $y = y^A \partial/\partial x^A$. Now, we consider a Finsler warped product metric on M

$$F = \alpha\sqrt{\phi(z, \rho)}, \tag{2}$$

where $\alpha = |\bar{y}|$, $z = \frac{y^0}{|\bar{y}|}$, $\rho = |\bar{x}|$, and ϕ is a suitable function on $\mathbb{R}^2$.

This warped product Finsler metric given by (2) is very interesting and the study to be done on it constitutes an important part of both Mathematics and Physics. Understanding better and revealing the geometric behavior of this metric needs more concentration. For this reason, we present the following main theorems:

For the class of Finsler metrics, we first prove the following theorem:

Theorem 1.1. *Every Finsler warped product metric of dimension $n \geq 2$ is of isotropic mean Berwald curvature if and only if it is a weakly Berwald metric.*

Example 1.2. The following Finsler warped product metrics

$$F = \alpha \left(h(\rho)\right)^{-1} G(h(\rho)z)$$

are of Douglas type [12], where h and G are arbitrary positive functions such that

$$G - t(G)' > 0, \quad (G)'' > 0.$$

By (4), one easily find that $U = z\frac{(\ln h)'}{2\rho}$ and $V = W = -\frac{(\ln h)'}{2\rho}$. Thus, Eq. (21) holds. Hence, they are weakly Berwald metrics.

The Berwald curvature is a non-Riemannian quantity in Finsler geometry [11]. We know that Berwald metrics have isotropic Berwald curvature. However, the converse might not be true in general. This fact becomes a necessary and sufficient condition for the warped product Finsler metric given by (2). We present it as follows:

Theorem 1.3. *Let $F = \alpha\sqrt{\phi(z,\rho)}$, $z = \frac{y^0}{|\bar{y}|}$, $\rho = |\bar{x}|$ be a Finsler warped product metric. Then F is of isotropic Berwald curvature if and only if it is a Berwald metric.*

2. Preliminaries

Let G^A be the geodesic coefficients of a Finsler metric F on an n-dimensional manifold M, which are defined by

$$G^A := \frac{1}{4}g^{AB}\{[F^2]_{x^C y^B}y^C - [F^2]_{x^B}\},$$

where $g_{AB}(x,y) = \left[\frac{1}{2}F^2\right]_{y^A y^B}$ and $(g^{AB}) = (g_{AB})^{-1}$. Recently, P. Marcal and Z. Shen [10] have obtained the geodesic coefficients G^A of a Finsler warped product metric $F = \alpha\sqrt{\phi(z,\rho)}$, $z = \frac{y^0}{|\bar{y}|}$, $\rho = |\bar{x}|$ as follows:

$$G^0 = (U + zV)(x^m y^m)\alpha,\ G^i = (V + W)(x^m y^m)y^i - Wx^i\alpha^2, \tag{3}$$

where

$$U := \frac{1}{2\rho\Lambda}(2\phi\phi_{z\rho} - \phi_z\phi_\rho),\ V := \frac{1}{2\rho\Lambda}(\phi_\rho\phi_{zz} - \phi_z\phi_{z\rho}),\ W := \frac{1}{2\rho\Omega}\phi_\rho, \tag{4}$$

where

$$\Lambda := 2\phi\phi_{zz} - \phi_z^2, \quad \Omega := 2\phi - z\phi_z.$$

They also proved the following important lemma [10]:

Lemma 2.1. *A Finsler warped product metric is strongly convex if and only if*

$$2\phi - z\phi_z > 0, \quad 2\phi\phi_{zz} - \phi_z^2 > 0. \tag{5}$$

Considering the equation (4), we get the following important result:

Lemma 2.2. *If $W_z = 0$, then $V_z = 0$ and $U_{zz} = 0$.*

Proof. From (4), we have

$$U := \frac{2\phi\phi_{z\rho} - \phi_z\phi_\rho}{2\rho(2\phi\phi_{zz} - \phi_z^2)},\ V := \frac{\phi_\rho\phi_{zz} - \phi_z\phi_{z\rho}}{2\rho(2\phi\phi_{zz} - \phi_z^2)},\ W := \frac{\phi_\rho}{2\rho(2\phi - z\phi_z)}. \tag{6}$$

Let $W_z = 0$. Then $W = f(\rho)$. Therefore

$$\phi_\rho = f(\rho)[2\rho(2\phi - z\phi_z)], \quad \phi_{\rho z} = f(\rho)[2\rho(\phi_z - z\phi_{zz})]. \tag{7}$$

Hence, $U = -zf(\rho)$ and $V = f(\rho)$. Thus, $U_{zz} = 0$ and $V_z = 0$. □

Now, we need briefly to introduce some needed geometric quantities, and definitions to proof the main theorems in this paper.

Let (M, F) be a Finsler manifold. The E-curvature $\mathbf{E} = E_{AB}dx^A \otimes dx^B$ of F is defined by [8]

$$E_{AB} := \frac{1}{2}\frac{\partial^2}{\partial y^A \partial y^B}\Big(\frac{\partial G^C}{\partial y^C}\Big). \tag{8}$$

A Finsler metric with vanishing E-curvature is also a weakly Berwald metric. The Finsler metric F is of isotropic E-curvature if there is a scalar function $c = c(x)$ on M such that

$$\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}h, \tag{9}$$

where h is a family of bilinear forms $h_y = h_{AB}dx^A \otimes dx^B$, which are defined by $h_{AB} := FF_{y^A y^B}$ [4]. The Berwald curvature $\mathbf{B} = B_C{}^A{}_{DE}dx^C \otimes dx^D \otimes dx^E \otimes \frac{\partial}{\partial x^A}$ of a Finsler metric F is defined by

$$B_C{}^A{}_{DE} := \frac{\partial^3 G^A}{\partial y^C \partial y^D \partial y^E}.$$

F is called a Berwald metric if $\mathbf{B} = 0$. Furthermore, F is said to be of isotropic Berwald curvature if its Berwald curvature $B_C{}^A{}_{DE}$ satisfies

$$B_C{}^A{}_{DE} = \tau(x)(F_{y^C y^D}\delta^A_E + F_{y^C y^E}\delta^A_D + F_{y^D y^E}\delta^A_C + F_{y^C y^D y^E}y^A),$$

where $\tau(x)$ is a scalar function. Also,

$$\mathbf{D} = D^A_{BCD}dx^B \otimes dx^C \otimes dx^D$$

is a tensor on $TM \setminus \{0\}$ which is called the Douglas tensor, where

$$D^A_{BCD} := \frac{\partial^3}{\partial y^B \partial y^C \partial y^D}\Big(G^A - \frac{1}{n+1}\frac{\partial G^C}{\partial y^C}y^A\Big). \tag{10}$$

A Finsler metric F is called a Douglas metric if $\mathbf{D} = 0$.

We need the following important lemma that is obtained by X. Cheng and Z. Shen:

Lemma 2.3 ([2])**.** *For a Finsler metric F on an n-dimensional manifold M. The followings are equivalent.*

(a) *F is of isotropic Berwald curvature*;
(b) *$D_B{}^A{}_{CD} = 0$ and $E_{AB} = \frac{n+1}{2}\kappa F_{y^A y^B}$ for a scalar function $\kappa = \kappa(x)$ on M.*

Moreover, depending on the goal of this paper, we need to present briefly the notion of dually flat Finsler metrics:

A Finsler metric $F(x, y)$ on a manifold M is locally dually flat if and only if the function $F = F(x, y)$ satisfies the following equation in an adapted local coordinate system (x^A):

$$[F^2]_{x^B y^A}y^B - 2[F^2]_{x^A} = 0. \tag{11}$$

When we apply the dually flatness idea to the warped product Finsler metrics, we obtain a necessary and sufficient condition for a Finsler warped product metric $F=\alpha\sqrt{\phi(z,\rho)}$, $z:=\frac{y^0}{\alpha}$, $\alpha=:|\bar{y}|$, $\rho:=|\bar{x}|$ to be locally dually flat and we state the following proposition:

Proposition 2.4. *Let $F=\alpha\sqrt{\phi(z,\rho)}$, $z:=\frac{y^0}{\alpha}$, $\alpha=:|\bar{y}|$, $\rho:=|\bar{x}|$ be a Finsler warped product metric. Then F is locally dually flat if and only*

$$\phi_\rho=0. \tag{12}$$

Proof. By a direct computation of a Finsler warped product metric $F=\alpha\sqrt{\phi(z,\rho)}$, $z:=\frac{y^0}{\alpha}$, $\alpha=:|\bar{y}|$, $\rho:=|\bar{x}|$, we obtain

$$[F^2]_{x^0}=0,\qquad [F^2]_{x^j}=\frac{1}{\rho}\alpha^2\phi_\rho x^j, \tag{13}$$

$$[F^2]_{y^0x^B}y^B=\frac{1}{\rho}\alpha^2 s\phi_{z\rho},\qquad [F^2]_{y^ix^B}y^B=\frac{1}{\rho}\alpha s\Omega_\rho y^i. \tag{14}$$

Hence, one can see that (11) is equivalent to

$$0=[F^2]_{x^By^0}y^B-2[F^2]_{x^0}=\frac{1}{\rho}\alpha^2 s\phi_{z\rho}, \tag{15}$$

and

$$0=[F^2]_{x^By^i}y^B-2[F^2]_{x^i}=\frac{2}{\rho}\alpha\phi_\rho(sy^i-\alpha x^i)-\frac{z}{\rho}\alpha s\phi_{z\rho}y^i. \tag{16}$$

Hence, $\phi_\rho=0$. □

Now by Proposition 2.4 and [12, Theorem 5], we easily obtain the following:

Corollary 2.5. *Let $F=\alpha\sqrt{\phi(z,\rho)}$, $z:=\frac{y^0}{\alpha}$, $\alpha=:|\bar{y}|$, $\rho:=|\bar{x}|$ be a Finsler warped product metric. Then F is locally dually flat if and only it is locally projectively flat.*

By (12), we have that $\phi=\phi(z)$. Hence, by (6), $U=V=W=0$. So, $G^A=0$. By the above corollary and the proof of Theorem 1.1 in [13], one can see that if $\phi_\rho=0$, then F is a locally Minkowskian metric.

3. Isotropic mean Berwald Finsler warped product metrics

In this section, we will get the necessary and sufficient condition for a Finsler warped product metric to be of isotropic mean Berwald curvature. Then, we will use the locally dually flat condition and vanishing χ-curvature condition to simplify the if and only if equation of isotropic mean Berwald Finsler warped product metrics.

For a Finsler warped product metric, $F=\alpha\sqrt{\phi(z,\rho)}$, $z:=\frac{y^0}{\alpha}$, $\alpha=:|\bar{y}|$, $\rho:=|\bar{x}|$, we need the following important identities:

$$z_{y^0}=\frac{1}{\alpha},\ z_{y^i}=-\frac{zy^i}{\alpha^2},\ \alpha_{y^i}=\frac{y^i}{\alpha},\ z_{y^i}y^i=-z,$$

$$z_{y^i}x^i = -\frac{zy^ix^i}{\alpha^2},\ s_{y^i} = \frac{x^i\alpha - sy^i}{\alpha^2},\ s_{y^0} = 0.$$

By (3), we have

$$\frac{\partial G^0}{\partial y^0} = s\alpha(U_z + V + zV_z),$$

and

$$\frac{\partial G^i}{\partial y^i} = s\alpha[(n+1)V + (n-1)W - zV_z],$$

where $s := \frac{\langle x,y\rangle}{\alpha}$. Then, we have

$$\frac{\partial G^C}{\partial y^C} = \frac{\partial G^0}{\partial y^0} + \frac{\partial G^i}{\partial y^i} = s\alpha\big[U_z + (n+2)V + (n-1)W\big]. \tag{17}$$

Therefore, by (8) and (17), one can obtain that

$$E_{00} = \frac{1}{2}\big[U_{zzz} + (n+2)V_{zz} + (n-1)W_{zz}\big]\frac{s}{\alpha}, \tag{18}$$

$$\begin{aligned} E_{0j} = {} & \frac{1}{2}\big[U_{zz} + (n+2)V_z + (n-1)W_z\big]\frac{x^j}{\alpha} - \frac{1}{2}\big\{U_{zz} + (n+2)V_z \\ & + (n-1)W_z + z\big[U_{zzz} + (n+2)V_{zz} + (n-1)W_{zz}\big]\big\}\frac{s}{\alpha^2}y^j, \end{aligned} \tag{19}$$

$$\begin{aligned} E_{ij} = {} & -\frac{z}{2}\big[U_{zz} + (n+2)V_z + (n-1)W_z\big]\big(\frac{s}{\alpha}\delta^j_i + \frac{x^iy^j}{\alpha^2} + \frac{x^jy^i}{\alpha^2}\big) \\ & + \frac{z}{2}\big\{3\big[U_{zz} + (n+2)V_z + (n-1)W_z\big] + z\big[U_{zzz} + (n+2)V_{zz} \\ & + (n-1)W_{zz}\big]\big\}\frac{s}{\alpha}\frac{y^iy^j}{\alpha^2}. \end{aligned} \tag{20}$$

Thus, it is easy to see that the necessary and sufficient condition for a Finsler warped product metric $F = \alpha\sqrt{\phi(z,\rho)}$ to be weakly Berwaldian, $\mathbf{E} = 0$, namely

$$U_{zz} + (n+2)V_z + (n-1)W_z = 0. \tag{21}$$

By Proposition 2.4 and Lemma 2.2, we obtain the following proposition:

Proposition 3.1. *Every locally dually flat Finsler warped product metric on $M = \mathbb{R}\times\mathbb{R}^n$ is weakly Berwaldian,* $\mathbf{E} = 0$.

Proof. Let $F = \alpha\sqrt{\phi(z,\rho)}$ be a locally dually flat Finsler warped product metric. Then, by Proposition 2.4, we have

$$\phi_\rho = 0. \tag{22}$$

Thus, by (6), $U = V = W = 0$. So, the equation (21) holds. $\square$

In [10], P. Marcal and Z. Shen proved that:

Lemma 3.2 ([10])**.** *Let $F = \alpha\sqrt{\phi(z,\rho)}$, $z = \frac{y^0}{|\bar{y}|}$, $\rho = |\bar{x}|$ be a Finsler warped product metric on $M = \mathbb{R} \times \mathbb{R}^n$. Then the χ-curvature of F is given by*

$$\begin{aligned}(23)\quad \chi &= \chi_A dx^A \\ &= \chi_0 dx^0 + \chi_i dx^i \\ &= \big[s^2(\frac{1}{2\rho}\Pi_{z\rho} - U\Pi_{zz} - W\Pi_z) + \frac{1}{2}\left(2\,W\rho^2 + 1\right)\Pi_z\big](\alpha dx^0 - zy^i dx^i) \\ &\quad + s\,(zW + U)\,\Pi_z(sy^i - \alpha\, x^i)dx^i,\end{aligned}$$

where $\Pi := U_z + (n+2)V + (n-1)W$.

It is remarkable that the χ-curvature is obtained from S-curvature [5]. It is easy to see that $F = \alpha\sqrt{\phi(z,\rho)}$ has vanishing χ-curvature if the equation (21) holds. So, we have the following lemma:

Lemma 3.3. *Every weakly Berwald warped product Finsler metric* ($\Pi_z = 0$) *has vanishing χ-curvature $\chi = 0$.*

Proof of Theorem 1.1. Let $F = \alpha\sqrt{\phi(z,\rho)}$, $z := \frac{y^0}{\alpha}$, $\alpha =: |\bar{y}|$, $\rho := |\bar{x}|$ be a Finsler warped product metric given by (2). We have

$$(24)\qquad F_{y^0y^0} = 1/4\,\frac{2\,\phi\phi_{zz} - \phi_z^2}{\alpha\,\phi^{3/2}},$$

$$(25)\qquad F_{y^0y^j} = -1/4\,\frac{z\left(2\,\phi\phi_{zz} - \phi_z^2\right)y^j}{\alpha^2\phi^{3/2}},$$

$$(26)\qquad F_{y^iy^j} = 1/4\,\frac{-2(z\phi\phi_z - 2\phi^2)(\alpha^2\delta_i^j - y^iy^j) + z^2(2\phi\phi_{zz} - \phi_z^2)y^iy^j}{\alpha^3\phi^{3/2}}.$$

By (9), we get

$$(27)\qquad \frac{\partial}{\partial y^B}\frac{\partial}{\partial y^A}(\frac{\partial G^C}{\partial y^C}) = (n+1)cF_{y^Ay^B}.$$

Assume that F is of isotropic E-curvature. Then, by (18)$\sim$(20), (24)$\sim$(26) and (27), we obtain

$$\begin{aligned}(28)\qquad &\big[U_{zzz} + (n+2)V_{zz} + (n-1)W_{zz}\big]s \\ &= \frac{(n+1)c(2\,\phi\phi_{zz} - \phi_z^2)}{4\phi^{3/2}},\end{aligned}$$

$$\begin{aligned}(29)\qquad &\big[U_{zz} + (n+2)V_z + (n-1)W_z\big](\alpha x^j - sy^j) \\ &- z\big[U_{zzz} + (n+2)V_{zz} + (n-1)W_{zz}\big]sy^j \\ &= -\frac{(n+1)cz\left(2\,\phi\phi_{zz} - \phi_z^2\right)y^j}{4\phi^{3/2}},\end{aligned}$$

$$(30)\qquad -\alpha z\big[U_{zz} + (n+2)V_z + (n-1)W_z\big]\left(s\alpha\delta_i^j + x^iy^j + x^jy^i\right)$$

$$
\begin{aligned}
&+ z\big\{3\big[U_{zz} + (n+2)V_z + (n-1)W_z\big] \\
&+ z\big[U_{zzz} + (n+2)V_{zz} + (n-1)W_{zz}\big]\big\}sy^iy^j \\
= &\ \frac{(n+1)c[-2(z\phi\phi_z - 2\phi^2)(\alpha^2\delta_i^j - y^iy^j) + z^2(2\phi\phi_{zz} - \phi_z^2)y^iy^j]}{4\phi^{3/2}}.
\end{aligned}
$$

Substituting (28) into (29), we get

$$\big[U_{zz} + (n+2)V_z + (n-1)W_z\big](\alpha x^j - sy^j) = 0. \tag{31}$$

Then, by (28), $c = 0$ and therefore F is weakly Berwaldian, that is, $\mathbf{E} = 0$. The converse is clear. □

Now, we are going to prove Theorem 1.3. First, we need to state the following lemma that have been proved by N. Mayer and S. Chavez:

Lemma 3.4 ([12])**.** *Let $F = \alpha\sqrt{\phi(z,\rho)}$, $z = \frac{y^0}{|\bar{y}|}$, $\rho = |\bar{x}|$ be a Finsler warped product metric. Then F has vanishing Douglas curvature if and only if ϕ satisfies*

$$
\begin{aligned}
R - zR_z &= 0, \\
T_z &= 0, \\
W_z - zW_{zz} &= 0,
\end{aligned}
$$

where

$$R = U - \frac{z}{n+2}\left[U_z + (n-1)W\right], \ T = \frac{1}{n+2}(3W - U_z) \tag{32}$$

and W is given in (4)*.*

By Lemma 2.3, Theorem 1.1, and Lemma 3.4, we get the following:

Proposition 3.5. *Let $F = \alpha\sqrt{\phi(z,\rho)}$, $z = \frac{y^0}{|\bar{y}|}$, $\rho = |\bar{x}|$ be a Finsler warped product metric. Then F is isotropic Berwald if and only if ϕ satisfies*

$$R - zR_z = 0, \tag{33}$$

$$T_z = 0, \tag{34}$$

$$W_z - zW_{zz} = 0, \tag{35}$$

$$U_{zz} + (n+2)V_z + (n-1)W_z = 0. \tag{36}$$

Proof of Theorem 1.3. Suppose that $F = \alpha\sqrt{\phi(z,\rho)}$ is an isotropic Berwald metric. Then (33)-(36) hold. From (32), (33) and (34) we have

$$U - zU_z + z^2W_z = 0. \tag{37}$$

Differentiating (37) with respect to the variable z, we get

$$U_{zz} - 2W_z - zW_{zz} = 0. \tag{38}$$

By (35), (36), and (38) we have

$$W_z + V_z = 0. \tag{39}$$

By the proof of [12, Corollary 2], we can see that if (35), (37), and (39) hold, then F is a Berwald metric. The converse is trivial. □

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Mehran Gabrani
Department of Mathematics
Faculty of Science
Urmia University
Urmia, Iran
Email address: `m.gabrani@urmia.ac.ir`

Bahman Rezaei
Department of Mathematics
Faculty of Science
Urmia University
Urmia, Iran
Email address: `b.rezaei@urmia.ac.ir`

Esra Sengelen Sevim
Department of Mathematics
Istanbul Bilgi University
34060, Eski Silahtaraga Elektrik Santrali
Kazim Karabekir Cad. No: 2/13 Eyupsultan
Istanbul, Turkey
Email address: esra.sengelen@bilgi.edu.tr