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HOMOGENEOUS GEODESICS IN HOMOGENEOUS SUB-FINSLER MANIFOLDS

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Abstract. In this paper, we mainly study the problem of the existence of homogeneous geodesics in sub-Finsler manifolds. Firstly, we obtain a characterization of a homogeneous curve to be a geodesic. Then we show that every compact connected homogeneous sub-Finsler manifold and Carnot group admits at least one homogeneous geodesic through each point. Finally, we study a special class of ℓ^p-type bi-invariant metrics on compact semi-simple Lie groups. We show that every homogeneous curve in such a metric space is a geodesic. Moreover, we prove that the Alexandrov curvature of the metric space is neither non-positive nor non-negative.

1. Introduction

A geodesic in a homogeneous metric space is called a homogeneous geodesic if it is an orbit of a one-parameter subgroup of isometries. The problem of the existence of homogeneous geodesics in general homogeneous pseudo-Riemannian (Finsler) manifolds seems to be an interesting one, and some results have been established in several papers. Firstly, Kowalski and Szenthe [15] proved that any homogeneous Riemannian manifold admits at least one homogeneous geodesic through each point. Then in [13], Dušek proved that any homogeneous affine manifold admits at least one homogeneous geodesic through each point. Finally, Yan and Huang [20] proved that any homogeneous Finsler space admits at least one homogeneous geodesic through each point. We refer the readers to [4, 12] and the references therein for more information.

Recently, Podobryaev [18] studied homogeneous geodesics in sub-Riemannian manifolds. He obtained a criterion for a geodesic to be homogeneous in terms of its initial momentum and a broad condition for the existence of at least

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one homogeneous geodesic. In this paper, we study homogeneous geodesics in homogeneous sub-Finsler manifolds and will prove the following main result.

Theorem 1.1. *Every compact connected homogeneous sub-Finsler manifold and Carnot group admits at least one homogeneous geodesic through each point.*

Remark 1.2. In this paper, the sub-Finsler metrics are reversible. In fact, Theorem 1.1 is still valid for irreversible homogeneous sub-Finsler metrics.

The arrangement of this paper is as follows. In Sections 2, 3, and 4, we recall some terminologies and results on metric spaces and homogeneous sub-Finsler manifolds. In Section 5, we study the problem of the existence of homogeneous geodesics in sub-Finsler manifolds and prove Theorem 1.1. In Section 6, we study a special class of ℓ^p-type bi-invariant metrics on compact semi-simple Lie groups. We will show that every homogeneous curve in such a metric space is a geodesic.

2. Metric spaces

In this section, we recall some results about metric spaces, which can be found in [6].

Definition 2.1. Let X be a set and $d : X \times X \to [0, \infty)$ be a function on X. The pair (X, d) is called a metric space if for any $x, y, z \in X$ satisfies the following properties:
(1) non-negativity: $d(x, y) \geq 0$ with equality if and only if $x = y$;
(2) symmetry: $d(x, y) = d(y, x)$;
(3) triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Given $x \in X$ and $r > 0$, the open ball and closed ball of radius r about x shall be denoted by

$$B(x, r) = \{y \in X \mid d(x, y) < r\}, \quad \bar{B}(x, r) = \{y \in X \mid d(x, y) \leq r\},$$

respectively. Let $\mathcal{T}$ denote the topology on (X, d) induced by the open balls.

Definition 2.2. Let (X, d) be a metric space.

- A sequence $\{x_i\}$ in X is called a Cauchy sequence if, for each $\varepsilon > 0$, there exists $N > 0$ satisfying when $m \geq n > N$, then $d(x_n, x_m) < \varepsilon$.
- Given $\varepsilon > 0$, a subset $A \subset X$ is called an ε-net of X if, for each $x \in X$, there exists $a_x \in A$ such that $d(a_x, x) < \varepsilon$.
- (X, d) is called complete if every Cauchy sequence in X converges in X with respect to $\mathcal{T}$.
- (X, d) is called totally bounded if it has a finite ε-net for each $\varepsilon > 0$.
- (X, d) is called boundedly compact if every bounded closed ball is compact.

Theorem 2.3. *Let (X, d) be a metric space. Then the following statements are equivalent*:

- (X, d) *is compact.*
- (X, d) *is sequentially compact.*
- (X, d) *is complete and totally bounded.*

Definition 2.4. Let (X, d) be a metric space and $\gamma : [a, b] \to X$ be a continuous curve. The length of γ with respect to the metric d, denoted by $L_d(\gamma)$, is defined by the supremum of the sums

$$\Sigma(P) = \sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i))$$

over all the partitions $P = \{t_0, \dots, t_N\}$ of $[a, b]$, where $a = t_0 \leq t_1 \leq \cdots \leq t_N = b$. A continuous curve is said to be rectifiable if its length is finite.

Given two points $x, y \in X$, define the associated metric d_L of d as follows:

$$\begin{aligned} &d_L(x, y) \\ &= \inf\{L_d(\gamma) \mid \gamma : [0, T] \to X,\ \gamma \text{ is continuous}, \gamma(0) = x, \gamma(T) = y\}. \end{aligned} \tag{2.1}$$

We have the following statements.

Theorem 2.5 ([6], Proposition 2.3.12)**.** *Let (X, d) be a metric space with the associated metric d_L defined by* (2.1)*.*
(1) $d_L(x, y) \geq d(x, y)$, $\forall x, y \in X$.
(2) $L_{d_L}(\gamma) = L_d(\gamma)$ *for any rectifiable curve γ in (X, d).*
(3) $(d_L)_L = d_L$.

In particular, d is called an intrinsic metric if $d = d_L$.

Definition 2.6. Let (X, d) be a metric space.

- A continuous curve $\gamma : [0, T] \to X$ is called a shortest path if its length is minimal among the continuous curves with the same endpoints; in other words, $L_d(\gamma') \geq L_d(\gamma)$ for any continuous curve γ' from $\gamma(0)$ to $\gamma(T)$.
- A continuous curve $\gamma : [0, T] \to X$ is called a geodesic if for every $t \in [0, T]$, there exists an open interval (a, b) containing t in $[0, T]$ such that $L_d(\gamma|_{[a,b]}) = d(\gamma(a), \gamma(b))$.
- A continuous curve $\gamma : [0, T] \to X$ is called a minimizing geodesic if $L_d(\gamma|_{[a,b]}) = d(\gamma(a), \gamma(b))$ for every closed interval $[a, b] \subset [0, T]$.
- A continuous curve $\gamma : [0, T] \to X$ is said to have constant speed if there exists a constant $C > 0$ such that
$$L_d(\gamma|_{[a,b]}) = C(b - a), \quad \forall [a, b] \subset [0, T].$$
In particular, if $C = 1$, then γ is said to be parameterized by arc length.

Note that a minimizing geodesic is always a shortest path, but not vice versa unless (X, d) is an intrinsic space. It is known that every rectifiable curve can be parameterized by arc length, see Proposition 2.5.9 in [6].

Theorem 2.7 (Hopf-Rinow-Cohn-Vossen Theorem, [6], Theorem 2.5.28)**.** *Let (X, d) be a locally compact intrinsic metric space. Then the following assertions are equivalent*:

(1) *(X, d) is complete.*

(2) *(X, d) is boundedly compact, i.e., every bounded closed ball is compact.*

(3) *Each constant speed geodesic $\gamma : [0, T) \to X$ can be extended to a continuous curve $\overline{\gamma} : [0, T] \to X$.*

(4) *There is a point $x \in X$ such that every constant speed minimizing geodesic $\gamma : [0, T) \to X$ with $\gamma(0) = x$ can be extended to a continuous curve $\overline{\gamma} : [0, T] \to X$.*

Furthermore, if any of the above holds, every two points in X can be connected by a minimizing geodesic. In this case, the metric space (X, d) is called a geodesic space.

Now we recall the notion of Alexandrov curvature in a geodesic space. A geodesic triangle $\triangle$ in a metric space (X, d) consists of three points $p, q, r \in X$, its vertices, and a choice of three geodesic segments $[p, q]$, $[q, r]$, $[r, p]$, joining them, its sides. Such a geodesic triangle will be denoted by $\triangle(p, q, r)$.

For each geodesic triangle $\triangle(p, q, r)$ in (X, d), we construct a comparison triangle $\bar{\triangle}(\bar{p}, \bar{q}, \bar{r})$ in the Euclidean plane with the same length of sides, i.e.,

$$d(p, q) = |\bar{p}\bar{q}|, \quad d(q, r) = |\bar{q}\bar{r}|, \quad d(r, p) = |\bar{r}\bar{p}|,$$

where $|\cdot|$ denotes the usual Euclidean distance on the Euclidean plane. It is clear that a comparison triangle is uniquely defined up to a rigid motion of Euclidean plane.

Definition 2.8. A metric space (X, d) is said to be non-positively (resp. non-negatively) curved if in some neighborhood of each point the following holds:

- For every geodesic triangle $\triangle(p, q, r)$ and every point $x \in [p, r]$, one has $d(x, q) \leq |\bar{x}\bar{q}|$ (resp. $d(x, q) \geq |\bar{x}\bar{q}|$), where $\bar{x}$ is the point on the side $[\bar{p}, \bar{r}]$ of a comparison triangle $\bar{\triangle}(\bar{p}, \bar{q}, \bar{r})$ such that $d(p, x) = |\bar{p}\bar{x}|$.

The definition given above was introduced by A. D. Alexandrov [1]. It provides a good notion of an upper bound on curvature in an arbitrary metric space. Classical comparison theorems in differential geometry show that a Riemannian manifold is non-positively (resp. non-negatively) curved in the above sense if and only if all of its sectional curvatures are non-positive (resp. non-negative).

Definition 2.9. A metric space (X, d) is called a CAT(0) space if for every geodesic triangle $\triangle$ and comparison triangle $\bar{\triangle}$, for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,

$$d(x, y) \leq |\bar{x}\bar{y}|.$$

The terminology "CAT(0)" was coined by M. Gromov ([14], p. 119). The initials are in honour of E. Cartan, A. D. Alexandrov and V. A. Toponogov, each of whom considered similar conditions in varying degrees of generality.

Theorem 2.10 (Cartan-Hadamard Theorem). *A complete simply connected intrinsic non-positively curved metric space is a* CAT(0) *space.*

3. Sub-Finsler manifolds

In this section we introduce some basic facts about sub-Finsler manifolds.

Definition 3.1. Let V be an n-dimensional real vector space. A norm $F : V \to \mathbb{R}$ is a function with the following properties:
(1) $F(y) \geq 0, \forall y \in V$. $F(y) = 0$ if and only if $y = 0$.
(2) $F(\lambda y) = |\lambda| F(y)$, $\forall \lambda \in \mathbb{R}$, $y \in V$.
(3) $F(y + z) \leq F(y) + F(z)$, $\forall y, z \in V$.
In particular, a norm F is called an Euclidean norm if $F = \sqrt{\langle \cdot, \cdot \rangle}$ for some inner product $\langle \cdot, \cdot \rangle$ on V.

There are many non-Euclidean norms on a vector space. Here we present some examples which will be used in this paper.

Example 3.2 (ℓ^p norm). Given $p \geq 1$ and a vector $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, the ℓ^p norm of x is defined by:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

Clearly, the ℓ^p norm $\| \cdot \|_p$ is Euclidean if and only if $p = 2$.

Example 3.3 (ℓ^p-type norm). Let F_i be norms on vector spaces V_i, $i = 1, \ldots, m$, $m \in \mathbb{N}^+$. Then the function $F^p : V = V_1 \oplus \cdots \oplus V_m \to [0, \infty)$, $p \geq 1$, defined by

$$F^p(y) = \left(\sum_{i=1}^{m} (F_i(y_i))^p \right)^{\frac{1}{p}}, \quad \forall y = y_1 + \cdots + y_m \in V,$$

is a norm. Moreover, when F_i are all Euclidean norms and $p = 2$, the resulting norm F^2 is also an Euclidean norm.

Notice that a norm F on V is continuous, we have:

Proposition 3.4. *Let F and $\| \cdot \|$ be norms on V. Then there exist constants $C_2 \geq C_1 > 0$ such that*

$$C_1 \|y\| \leq F(y) \leq C_2 \|y\|, \quad \forall y \in V.$$

Definition 3.5. Let M be a connected smooth n-dimensional manifold and $\mathbf{S}$ be a smooth bracket-generating distribution on M with rank k, $k \leq n$, i.e., $\mathbf{S}$ is a subbundle of TM and the vector fields which are sections of $\mathbf{S}$ together with all brackets span every tangent space $T_x M$, $x \in M$. A sub-Finsler metric on M is a real continuous function $F : \mathbf{S} \to [0, \infty)$ such that the restriction of F to any $\mathbf{S}_x$, $x \in M$, is a norm. In this case, we say that $(M, \mathbf{S}, F)$ is a sub-Finsler manifold.

In particular, when F is induced by a Riemannian metric on M, $(M, \mathbf{S}, F)$ is called a sub-Riemannian manifold. Moreover, when $S = TM$, the tangent bundle of M, the corresponding sub-Finsler metrics are called Finsler metrics.

Remark 3.6. We should mention that, in classical Finsler geometry [2, 8, 19], a Finsler metric on a connected smooth manifold M is a real continuous function $F : TM \to [0, \infty)$ such that

1. F is smooth on the slit tangent bundle $TM \setminus \{0\}$.
2. The restriction of F to any T_xM, $x \in M$, is a Minkowski norm.

Let $(M, \mathbf{S}, F)$ be a sub-Finsler manifold, a curve $\gamma : [0, T] \to M$ is called admissible if it is absolutely continuous and $\dot{\gamma}(t) \in \mathbf{S}_{\gamma(t)}$ for almost every $t \in [0, T]$. The length of an admissible curve γ is defined by

$$L_F(\gamma) = \int_0^T F(\dot{\gamma}(t))dt.$$

Define the distance function $d_F : M \times M \to [0, \infty)$ by

$$d_F(p, q) = \inf\{L_F(\gamma) \mid \gamma \text{ is admissible, } \gamma(0) = p, \ \gamma(T) = q\}. \tag{3.1}$$

We have the following statements.

Theorem 3.7 ([6]). *Let $(M, \mathbf{S}, F)$ be a sub-Finsler manifold. Then we have*

(1) *(M, d_F) is an intrinsic metric space.*

(2) *The topology $\mathcal{T}$ induced by (M, d_F) is exactly the original topology of M.*

4. Homogeneous sub-Finsler manifolds

Definition 4.1. Let $(M, \mathbf{S}, F)$ be a sub-Finsler manifold.

- A diffeomorphism ϕ of M is called an isometry if for any $x \in M$, $X \in \mathbf{S}_x$, $F(d\phi(X)) = F(X)$.
- A mapping ϕ of M onto itself is called a distance isometry if for any $p, q \in M$, $d_F(\phi(p), \phi(q)) = d_F(p, q)$.

It is easily seen that, the set of isometries of $(M, \mathbf{S}, F)$ forms a group, denoted by $\mathcal{I}(M)$, which is clearly contained in the group $I(M, d_F)$ of distance isometries. Let $I_p(M, d_F)$ be the isotropy subgroup of $I(M, d_F)$ at $p \in M$. Van Dantzig and van der Waerden [9] proved that $I(M, d_F)$ is a locally compact topological transformation group on M with respect to the compact-open topology and $I_p(M, d_F)$ is compact. Bochner and Montgomery [5] proved that a locally compact group of differentiable transformations of a manifold is a Lie transformation group. Combing the above results, in fact, we have proved the following

Theorem 4.2. *Let $(M, \mathbf{S}, F)$ be a sub-Finsler manifold. Then the group of isometries $\mathcal{I}(M)$ of M is a Lie transformation group of M. Let $p \in M$ and $\mathcal{I}_p(M)$ be the isotropy subgroup of $\mathcal{I}(M)$ at p. Then $\mathcal{I}_p(M)$ is compact.*

In the Riemannian (classical Finsler) geometry, it was proved in [11,17] that, the group $I(M, d_F)$ coincides with the group of isometries $\mathcal{I}(M)$. In general case, even in the sub-Riemannian setting, it is still unknown whether $I(M, d_F)$ is a Lie transformation group of M [7].

Definition 4.3. A connected sub-Finsler manifold $(M, \mathbf{S}, F)$ is called a homogeneous sub-Finsler manifold if its isometries group $\mathcal{I}(M)$ acts transitively on M, namely, for every pair $p, q \in M$, there exists an isometry $\phi \in \mathcal{I}(M)$ such that $\phi(p) = q$.

Now we recall some important results due to Berestovskiĭ [3]. Let G be a connected Lie group which acts effectively and transitively on a connected smooth manifold M and let H be the isotropy group of G at a fixed point $p \in M$. Then M can be viewed as the coset space G/H by sending $g \cdot p$ to gH, $\forall g \in G$. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Denote by K the Killing form of $\mathfrak{g}$. Since H is compact and the restriction of K to $\mathfrak{h}$ is non-degenerate [15], we have a K-orthogonal reductive decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $K(\mathfrak{h}, \mathfrak{m}) = 0$ and $\mathrm{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m}$, $\forall h \in H$, here Ad denotes the adjoint representation of G. Moreover, one can identify $\mathfrak{m}$ with the tangent space T_pM of M at the point p via the mapping $\pi : X \to \frac{d}{dt}\big|_{t=0} \exp(tX) \cdot p$, where exp denotes the exponential map of $\mathfrak{g}$. Let $\mathrm{Pr}_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$ denote the natural projection. Then for any $g \in G$ and $X \in \mathfrak{g}$, we have

$$\begin{aligned} dg^{-1}\left(\frac{d}{dt}\Big|_{t=0} \exp(tX)g \cdot p\right) &= \frac{d}{dt}\Big|_{t=0} \exp(t\mathrm{Ad}g^{-1}(X)) \cdot p \\ &= \pi \circ \mathrm{Pr}_{\mathfrak{m}}\left(\mathrm{Ad}g^{-1}(X)\right). \end{aligned}$$

Hence, the tangent space $T_{g\cdot p}(M)$ at the point $g \cdot p$ can be identified as the subspace $\mathrm{Ad}g(\mathfrak{m})$.

Definition 4.4. A subspace $\mathfrak{m}_0 \subset \mathfrak{m}$ is called bracket-generating if the smallest Lie algebra generated by $\mathfrak{h} + \mathfrak{m}_0$ is $\mathfrak{g}$ itself.

Now let $\mathfrak{m}_0 \subset \mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant bracket-generating subspace, $\mathcal{D}_{gH} = g_*(\mathfrak{m}_0) \subset T_{gH}(G/H)$ be the subspace of $T_{gH}(G/H)$ associated to $\mathfrak{m}_0$ via the left action of g on G/H, $\forall g \in G$. Then $\mathbf{S}_0 := \cup_{g\in G}\mathcal{D}_{gH}$ is a smooth bracket-generating G-invariant distribution on G/H [3]. Assume F_o is an $\mathrm{Ad}(H)$-invariant norm on $\mathfrak{m}_0$ and let $F_0 : \mathbf{S}_0 \to \mathbb{R}$ be the sub-Finsler metric on G/H defined by

$$F_0(g_*(X)) = F_o(X), \quad \forall g \in G, X \in \mathfrak{m}_0.$$

Theorem 4.5 ([3], Theorem 2)**.** *Let $(M = G/H, \mathbf{S}_0, F_0)$ be a sub-Finsler manifold and d_0 be the metric defined by* (3.1)*. Then $(M = G/H, d_0)$ is a G-invariant intrinsic metric space and the metric topology $\mathcal{T}$ coincides with the manifold topology.*

Conversely, we have:

Theorem 4.6 ([3], Theorem 3). *Any locally compact, locally contractible homogeneous space with intrinsic metric d_0 is isometric to a homogeneous sub-Finsler manifold $(G/H, \mathbf{S}_0, F_0)$ with respect to a bracket-generating subspace $\mathfrak{m}_0$ and an $\mathrm{Ad}(H)$-invariant norm F_o on $\mathfrak{m}_0$, which is given by*

$$F_o(X) = \lim_{t\to 0} \frac{d_0(p, \exp tX \cdot p)}{|t|}, \quad \forall X \in \mathfrak{m}_0. \tag{4.1}$$

Moreover, given a quotient space G/H, a correspondence $(\mathfrak{m}_0, F_o) \leftrightarrow d_0$ is a bijection of the set of all such pairs onto the set of all G-invariant intrinsic metrics on G/H.

Combing Theorem 2.7 and Theorem 4.5, we immediately have:

Theorem 4.7. *Every connected homogeneous sub-Finsler manifold is a complete geodesic space. Namely, every two points can be connected by a minimizing geodesic.*

For convenience, sometimes the homogeneous sub-Finsler manifold $(G/H, \mathbf{S}_0, F_0)$ will be denoted by $(G/H, \mathfrak{m}_0, F_o)$. In particular, when $\mathfrak{m}_0 = \mathfrak{m}$ and F_o is a Minkowski norm, $(G/H, \mathfrak{m}, F_o)$ is a homogeneous classical Finsler manifold, which was widely studied in the last two decades [10].

5. Homogeneous geodesics

In this section, we mainly study the problem of existence of homogeneous geodesics in homogeneous sub-Finsler manifolds and prove Theorem 1.1.

Definition 5.1. Let $(M = G/H, \mathbf{S}_0, F_0)$ be a homogeneous sub-Finsler manifold, $p = eH$. A geodesic $\gamma(t)$ through p is called a homogeneous geodesic if there exists a vector $X \in \mathfrak{g}$ such that $\gamma(t) = \exp tX \cdot p$. Moreover, such a vector X is called a geodesic vector.

It is easily seen that a homogeneous geodesic $\gamma(t) = \exp tX \cdot p$ is defined on the whole $\mathbb{R}$ and $\mathrm{Pr}_{\mathfrak{m}}(X) \in \mathfrak{m}_0$, according to Lemma 5 in [3].

There is a well known criterion for homogeneous geodesics in the Riemannian case due to Kowalski and Vanhecke [16].

Lemma 5.2 (Geodesic lemma). *Let $(M = G/H, TM, g)$ be a homogeneous Riemannian manifold. Then a curve $\gamma(t) = \exp tX \cdot p$ is a homogeneous geodesic if and only if the following equation holds*:

$$\langle \mathrm{Pr}_{\mathfrak{m}}(X), \mathrm{Pr}_{\mathfrak{m}}([X, Z])\rangle = 0, \quad \forall Z \in \mathfrak{m},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathfrak{m}$ induced by g.

It was proved in [15] that every homogeneous Riemannian manifold admits at least one homogeneous geodesic. Recently, Podobryaev studied homogeneous geodesics in sub-Riemannian manifolds and proved the following theorem.

Theorem 5.3 ([18], Theorem 5)**.** *Let $(G/H, \mathfrak{m}_0, F_o)$ be a homogeneous sub-Riemannian manifold. If* $\ker K = \mathfrak{m}$ *or* $K|_{\mathfrak{m}_0} \neq 0$*, where K denotes the Killing form of the Lie algebra of G, then there exists a homogeneous sub-Riemannian geodesic.*

To study homogeneous geodesics in sub-Finsler manifolds, we first prove the following result.

Proposition 5.4. *Let $(G/H, \mathfrak{m}_0, F_o)$ be a homogeneous sub-Finsler manifold and $F_\mathfrak{m}$ be an* $\mathrm{Ad}(H)$*-invariant norm on $\mathfrak{m}$ such that $F_o(X) \geq F_\mathfrak{m}(X)$, $\forall X \in \mathfrak{m}_0$. Assume $X \in \mathfrak{g}$ is a geodesic vector of $(G/H, \mathfrak{m}, F_\mathfrak{m})$ satisfying* $\mathrm{Pr}_\mathfrak{m}(X) \in \mathfrak{m}_0$ *and* $F_o(\mathrm{Pr}_\mathfrak{m}(X)) = F_\mathfrak{m}(\mathrm{Pr}_\mathfrak{m}(X))$*. Then $X \in \mathfrak{g}$ is also a geodesic vector of $(G/H, \mathfrak{m}_0, F_o)$. Namely, $\gamma(t) = \exp tX \cdot H$ is a homogeneous geodesic in $(G/H, \mathfrak{m}_0, F_o)$.*

Proof. Notation as above. Let d_F and F be the distance function and G-invariant Finsler metric on G/H corresponding to $(G/H, \mathfrak{m}, F_\mathfrak{m})$, respectively. Since $\gamma(t) = \exp tX \cdot H$ is a homogeneous geodesic in $(G/H, \mathfrak{m}, F_\mathfrak{m})$, there exists a constant $\varepsilon > 0$ such that

$$\varepsilon F_\mathfrak{m}(\mathrm{Pr}_\mathfrak{m}(X)) = \int_0^\varepsilon F(\dot\gamma(t))dt = L_F(\gamma|_{[0,\varepsilon]}) = d_F(H, \exp \varepsilon X \cdot H).$$

On the other hand, by the assumption that $d_0 \geq d_F$, we obtain

$$\begin{aligned} d_F(H, \exp \varepsilon X \cdot H) &\leq d_0(H, \exp \varepsilon X \cdot H) \leq L_{d_0}(\gamma|_{[0,\varepsilon]}) \\ &= \int_0^\varepsilon F_0(\dot\gamma(t))dt = \varepsilon F_o(\mathrm{Pr}_\mathfrak{m}(X)) \\ &= \varepsilon F_\mathfrak{m}(\mathrm{Pr}_\mathfrak{m}(X)). \end{aligned}$$

This asserts that $d_0(H, \exp \varepsilon X \cdot H) = L_{d_0}(\gamma|_{[0,\varepsilon]})$. Hence $\gamma(t) = \exp tX \cdot H$ is a homogeneous geodesic in $(G/H, \mathfrak{m}_0, F_o)$. ☐

Now we state the main result in this section.

Theorem 5.5. *Every compact connected homogeneous sub-Finsler manifold $(M = G/H, \mathbf{S}_0, F_0)$ admits a homogeneous geodesic.*

Proof. Since M is compact, G is compact and the Lie algebra $\mathfrak{g}$ of G has a decomposition $\mathfrak{g} = \mathfrak{g}_s + \mathfrak{g}_a$, where $\mathfrak{g}_s$ is compact semi-simple and $\mathfrak{g}_a$ is Abelian. Let B be the negative of Killing form of $\mathfrak{g}_s$ and $\langle \cdot, \cdot \rangle_0$ be any inner product on $\mathfrak{g}_a$. Set an Euclidean inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by

$$\langle X + Y, X + Y \rangle = B(X, X) + \langle Y, Y \rangle_0, \quad \forall X \in \mathfrak{g}_s, Y \in \mathfrak{g}_a.$$

It is clear that $\langle \cdot, \cdot \rangle$ is $\mathrm{Ad}(G)$-invariant. Let g be the G-invariant Riemannian metric on M generated by $\langle \cdot, \cdot \rangle|_\mathfrak{m}$ on $\mathfrak{m}$ and denote by d_g the associated Riemannian distance function defined by (3.1). It follows from Lemma 5.2 that, for every vector $X \in \mathfrak{m}$, the curve $\exp tX \cdot p$ is a homogeneous geodesic in (M, d_g).

Now assume $\mathfrak{m}_0$ is an $\mathrm{Ad}(H)$-invariant bracket-generating subspace in $\mathfrak{m}$ and F_o is an $\mathrm{Ad}(H)$-invariant norm on $\mathfrak{m}_0$. We will show that the homogeneous sub-Finsler manifold $(M = G/H, d_0)$ admits a homogeneous geodesic. Let $f(z) = \frac{F_o^2(z)}{\langle z,z\rangle}$, $\forall z \in \mathfrak{m}_0 \setminus \{0\}$. By homogeneity, $f(z)$ can be viewed as defined on the sphere $I = \{z \in \mathfrak{m}_0 \mid \langle z, z\rangle = 1\}$, hence it must attain its minimum and maximum at some points. Since $f > 0$, we may suppose that $f(z)$ attains the minimal value at $y \in I$ and $\lambda = f(y) > 0$. Hence $F_o^2(z) \geq \lambda\langle z, z\rangle$, $\forall z \in \mathfrak{m}_0$. Let g_λ and d_{g_λ} be the Riemannian metric and distance function on M induced by $\lambda\langle\cdot,\cdot\rangle|_{\mathfrak{m}}$ on $\mathfrak{m}$, respectively. Clearly, $d_{g_\lambda} = \sqrt{\lambda} d_g$ and $\gamma(t) = \exp ty \cdot p$ is a homogeneous geodesic in (M, d_{g_λ}). By Proposition 5.4, we obtain that $\gamma(t) = \exp ty \cdot p$ is a homogeneous geodesic in (M, d_0), which completes the proof of the theorem. □

Definition 5.6. A homogeneous sub-Finsler manifold $(G/H, \mathfrak{m}_0, F_o)$ is called a Carnot group if $H = \{e\}$, G is a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits a direct sum decomposition in nontrivial vector subspaces

$$\mathfrak{g} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s, \quad s \geq 1,$$

such that

$$[\mathfrak{m}_0, \mathfrak{m}_j] = \mathfrak{m}_{j+1}, \quad j = 0, \ldots, s-1, \quad [\mathfrak{m}_0, \mathfrak{m}_s] = 0.$$

Proposition 5.7. *Let $(G, \mathfrak{m}_0, \langle\cdot,\cdot\rangle_0)$ be a sub-Riemannian Carnot group, where $\langle\cdot,\cdot\rangle_0$ is an inner product on $\mathfrak{m}_0$. Then for every vector $X \in \mathfrak{m}_0$, $\gamma(t) = \exp tX$, $t \in \mathbb{R}$ is a homogeneous geodesic.*

Proof. We choose an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ such that

$$\langle\cdot,\cdot\rangle|_{\mathfrak{m}_0} = \langle\cdot,\cdot\rangle_0, \quad \langle\mathfrak{m}_i, \mathfrak{m}_j\rangle = 0, \quad 0 \leq i < j \leq s.$$

Notice that $\langle\mathfrak{m}_0, [\mathfrak{g}, \mathfrak{g}]\rangle = 0$. Then by Lemma 5.2, it is easily seen that every vector $X \in \mathfrak{m}_0$ is a geodesic vector of the homogeneous Riemannian manifold $(G, \mathfrak{g}, \langle\cdot,\cdot\rangle)$. Thus the proposition follows from Proposition 5.4. □

Combing Propositions 5.4, 5.7 and the argument in the proof of Theorem 5.5, we immediately have the following result, which completes the proof of Theorem 1.1.

Theorem 5.8. *Every Carnot group admits a homogeneous geodesic.*

Proof. It follows from Proposition 3.4 that, for any norm F_o on $\mathfrak{m}_0$, there exists an inner product $\langle\cdot,\cdot\rangle_0$ on $\mathfrak{m}_0$ such that $F_o^2(X) \geq \langle X, X\rangle_0$ for all $X \in \mathfrak{m}_0$ and $F_o^2(Y) = \langle Y, Y\rangle_0$ for some nonzero vector $Y \in \mathfrak{m}_0$. It is clear that Y is a geodesic vector of Carnot group $(G, \mathfrak{m}_0, F_o)$, according to Propositions 5.7. □

6. Bi-invariant metrics on compact Lie groups

In this section, we study a special class of bi-invariant metrics on compact semi-simple Lie groups.

Let $m \in \mathbb{N}$, $m \geq 2$. For any $i \in \{1, \ldots, m\}$, let G_i be connected and simple connected compact simple Lie group with unit e_i and corresponding Lie algebra $\mathfrak{g}_i$. Denote by B_i the negative of Killing form of $\mathfrak{g}_i$, g_i the bi-invariant Riemannian metric on G_i induced by B_i, and d_i the distance function of (G_i, g_i). The following results are true.

• d_i is bi-invariant. That is, for any $x, y, g \in G_i$,

$$d_i(gx, gy) = d_i(xg, yg) = d_i(x, y).$$

• Every geodesic in (G_i, d_i) is homogeneous. That is, if γ is a geodesic and $\gamma(0) = e_i$, then there exists a vector $X \in \mathfrak{g}_i$ such that $\gamma(t) = \exp tX$, $t \in \mathbb{R}$.

• The sectional curvature of (G_i, g_i) is non-negative and hence (G_i, d_i) is non-negatively curved in the sense of Definition 2.8.

Now set $G = G_1 \times G_2 \times \cdots \times G_m$, $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_m$, $B = B_1 \oplus B_2 \oplus \cdots \oplus B_m$. Define a class of functions d^p, $p \geq 1$ on G by

$$d^p\left((x_1, \ldots, x_m), (y_1, \ldots, y_m)\right) = \left(\sum_{i=1}^{m} d_i^p(x_i, y_i)\right)^{\frac{1}{p}},$$

$\forall x_i, y_i \in G_i$, $i = 1, \ldots, m$.

Theorem 6.1. *Notation as above.* $\forall p \geq 1$, *we have*

(1) d^p *is a bi-invariant intrinsic metric on* G.

(2) *The norm* F^p *on* $\mathfrak{g}$ *determined by* (G, d^p) *is given by*

$$F^p(X) = \left(\sum_{i=1}^{m} (B_i(X_i, X_i))^{\frac{p}{2}}\right)^{\frac{1}{p}}, \quad \forall X = (X_1, \ldots, X_m) \in \mathfrak{g}.$$

(3) $\forall X \in \mathfrak{g}$, $\gamma(t) = \exp tX$, $t \in \mathbb{R}$ *is a geodesic in* (G, d^p).

(4) *If* $p \neq 2$, (G, d^p) *is neither non-positively curved nor non-negatively curved.*

Proof. (1) It is easy to see that d^p is a bi-invariant metric. We now prove d^p is an intrinsic metric. Notice that (G, d^p) is a compact metric space. Then by Theorem 2.4.16 in [6], it suffices to show that, for every two points $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in G$, there exists a midpoint $z \in G$ satisfying $d^p(x, z) = d^p(y, z) = \frac{1}{2} d^p(x, y)$. For any $i \in \{1, \ldots, m\}$, as the metric space (G_i, d_i) is a compact homogeneous Riemannian manifold, then for the points $x_i, y_i \in G_i$, there exists a midpoint $z_i \in G_i$ such that $d_i(x_i, z_i) = d_i(y_i, z_i) = \frac{1}{2} d_i(x_i, y_i)$. Now set $z = (z_1, \ldots, z_m) \in G$, a quick calculation yields

$$d^p(x, z) = \left(\sum_{i=1}^{m} d_i^p(x_i, z_i)\right)^{\frac{1}{p}} = \frac{1}{2}\left(\sum_{i=1}^{m} d_i^p(x_i, y_i)\right)^{\frac{1}{p}} = \frac{1}{2} d^p(x, y)$$

and

$$d^p(y,z) = \left(\sum_{i=1}^m d_i^p(y_i,z_i)\right)^{\frac{1}{p}} = \frac{1}{2}\left(\sum_{i=1}^m d_i^p(x_i,y_i)\right)^{\frac{1}{p}} = \frac{1}{2}d^p(x,y),$$

which implies that z is a midpoint between points x,y in (G,d^p).

(2) Notice that

$$\lim_{t\to 0}\frac{d_i(e_i,\exp tX_i)}{|t|} = (B_i(X_i,X_i))^{\frac{1}{2}}, \quad \forall X_i \in \mathfrak{g}_i.$$

It follows from (4.1) that, $\forall X=(X_1,\ldots,X_m)\in\mathfrak{g}$,

$$\begin{aligned}F^p(X) &= \lim_{t\to 0}\frac{d^p\left((e_1,\ldots,e_m),(\exp tX_1,\ldots,\exp tX_m)\right)}{|t|}\\ &= \lim_{t\to 0}\frac{1}{|t|}\left(\sum_{i=1}^m d_i^p(e_i,\exp tX_i)\right)^{\frac{1}{p}}\\ &= \left(\sum_{i=1}^m\left(\lim_{t\to 0}\frac{d_i(e_i,\exp tX_i)}{|t|}\right)^p\right)^{\frac{1}{p}}\\ &= \left(\sum_{i=1}^m (B_i(X_i,X_i))^{\frac{p}{2}}\right)^{\frac{1}{p}}.\end{aligned}$$

(3) Given $X=(X_1,\ldots,X_m)\in\mathfrak{g}$, let $\gamma(t)=\exp tX=(\gamma_1(t),\ldots,\gamma_m(t))$, where $\gamma_i(t)=\exp tX_i$. Since for all $i=1,\ldots,m$, $\gamma_i(t)=\exp tX_i$ is a geodesic in (G_i,d_i), there exists a constant $\varepsilon>0$ such that

$$d_i(e_i,\exp\varepsilon X_i) = L_{d_i}(\gamma_i|_{[0,\varepsilon]}) = \varepsilon(B_i(X_i,X_i))^{\frac{1}{2}}.$$

Hence

$$\begin{aligned}L_{d^p}(\gamma|_{[0,\varepsilon]}) &= \int_0^\varepsilon F^p(\dot\gamma(t))dt\\ &= \varepsilon\left(\sum_{i=1}^m (B_i(X_i,X_i))^{\frac{p}{2}}\right)^{\frac{1}{p}}\\ &= \left(\sum_{i=1}^m d_i^p(e_i,\exp\varepsilon X_i)\right)^{\frac{1}{p}}\\ &= d^p(\gamma(0),\gamma(\varepsilon)).\end{aligned}$$

This implies that $\gamma|_{[0,\varepsilon]}$ is a shortest path in (G,d^p) and thus $\gamma(t)=\exp tX$ is a geodesic in (G,d^p).

(4) We consider two geodesic triangles $\triangle_1(a,b_1,c_1)$ and $\triangle_2(a,b_2,c_2)$ in (G,d^p), where $a=(e_1,\ldots,e_m)$, $b_1=(\exp\varepsilon X_1,e_2,\ldots,e_m)$, $c_1=(e_1,\exp\varepsilon X_2,e_3,\ldots,e_m)$, $b_2=(\exp\varepsilon X_1,\exp\varepsilon X_2,e_3,\ldots,e_m)$, $c_2=(\exp\varepsilon X_1,\exp(-\varepsilon X_2),$

$e_3, \dots, e_m)$, $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$, $B_1(X_1, X_1) = B_2(X_2, X_2) = 1$, $\varepsilon > 0$ is sufficiently small. We have

$$d^p(a, b_1) = d^p(a, c_1) = \varepsilon, \quad d^p(b_1, c_1) = 2^{\frac{1}{p}}\varepsilon,$$
$$d^p(a, b_2) = d^p(a, c_2) = 2^{\frac{1}{p}}\varepsilon, \quad d^p(b_2, c_2) = 2\varepsilon.$$

Now choose the midpoints $x_1 = (\exp \frac{\varepsilon}{2} X_1, \exp \frac{\varepsilon}{2} X_2, e_3, \dots, e_m) \in [b_1, c_1]$ and $x_2 = (\exp \varepsilon X_1, e_2, \dots, e_m) \in [b_2, c_2]$, we get

$$d^p(a, x_1) = \frac{1}{2} \cdot 2^{\frac{1}{p}}\varepsilon, \quad d^p(a, x_2) = \varepsilon. \tag{6.1}$$

In the comparison triangles $\bar{\triangle}_1(\bar{a}, \bar{b}_1, \bar{c}_1)$ and $\bar{\triangle}_2(\bar{a}, \bar{b}_2, \bar{c}_2)$, a direct calculation shows

$$|\bar{a}\bar{x}_1| = \varepsilon\sqrt{1 - (\frac{1}{2} \cdot 2^{\frac{1}{p}})^2}, \quad |\bar{a}\bar{x}_2| = \varepsilon\sqrt{(2^{\frac{1}{p}})^2 - 1}. \tag{6.2}$$

Combing (6.1) and (6.2), we obtain

$$\begin{cases} d^p(a, x_1) > |\bar{a}\bar{x}_1|, & 1 \le p < 2, \\ d^p(a, x_1) = |\bar{a}\bar{x}_1|, & p = 2, \\ d^p(a, x_1) < |\bar{a}\bar{x}_1|, & p > 2, \end{cases}$$

and

$$\begin{cases} d^p(a, x_2) < |\bar{a}\bar{x}_2|, & 1 \le p < 2, \\ d^p(a, x_2) = |\bar{a}\bar{x}_2|, & p = 2, \\ d^p(a, x_2) > |\bar{a}\bar{x}_2|, & p > 2. \end{cases}$$

Clearly, if $p \neq 2$, (G, d^p) is neither non-positively curved nor non-negatively curved. □

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