HOMOGENEOUS GEODESICS IN HOMOGENEOUS 
SUB-FINSLER MANIFOLDS

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Abstract. In this paper, we mainly study the problem of the existence 
of homogeneous geodesics in sub-Finsler manifolds. Firstly, we obtain 
a characterization of a homogeneous curve to be a geodesic. Then we 
show that every compact connected homogeneous sub-Finsler manifold 
and Carnot group admits at least one homogeneous geodesic through 
each point. Finally, we study a special class of $\ell^p$-type bi-invariant metrics 
on compact semi-simple Lie groups. We show that every homogeneous 
curve in such a metric space is a geodesic. Moreover, we prove that 
the Alexandrov curvature of the metric space is neither non-positive nor 
non-negative.

1. Introduction

A geodesic in a homogeneous metric space is called a homogeneous geodesic if 
it is an orbit of a one-parameter subgroup of isometries. The problem of the ex- 
istence of homogeneous geodesics in general homogeneous pseudo-Riemannian 
(Finsler) manifolds seems to be an interesting one, and some results have been 
established in several papers. Firstly, Kowalski and Szenthe [15] proved that 
any homogeneous Riemannian manifold admits at least one homogeneous ge- 
odesic through each point. Then in [13], Dušek proved that any homogeneous 
affine manifold admits at least one homogeneous geodesic through each point. 
Finally, Yan and Huang [20] proved that any homogeneous Finsler space admits 
at least one homogeneous geodesic through each point. We refer the readers to 
[4, 12] and the references therein for more information.

Recently, Podobryaev [18] studied homogeneous geodesics in sub-Riemann- 
ian manifolds. He obtained a criterion for a geodesic to be homogeneous in 
terms of its initial momentum and a broad condition for the existence of at least

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one homogeneous geodesic. In this paper, we study homogeneous geodesics in homogeneous sub-Finsler manifolds and will prove the following main result.

**Theorem 1.1.** Every compact connected homogeneous sub-Finsler manifold and Carnot group admits at least one homogeneous geodesic through each point.

**Remark 1.2.** In this paper, the sub-Finsler metrics are reversible. In fact, Theorem 1.1 is still valid for irreversible homogeneous sub-Finsler metrics.

The arrangement of this paper is as follows. In Sections 2, 3, and 4, we recall some terminologies and results on metric spaces and homogeneous sub-Finsler manifolds. In Section 5, we study the problem of the existence of homogeneous geodesics in sub-Finsler manifolds and prove Theorem 1.1. In Section 6, we study a special class of \( \ell^p \)-type bi-invariant metrics on compact semi-simple Lie groups. We will show that every homogeneous curve in such a metric space is a geodesic.

2. Metric spaces

In this section, we recall some results about metric spaces, which can be found in [6].

**Definition 2.1.** Let \( X \) be a set and \( d : X \times X \to [0, \infty) \) be a function on \( X \). The pair \((X, d)\) is called a metric space if for any \( x, y, z \in X \) satisfies the following properties:

1. non-negativity: \( d(x, y) \geq 0 \) with equality if and only if \( x = y \);
2. symmetry: \( d(x, y) = d(y, x) \);
3. triangle inequality: \( d(x, z) \leq d(x, y) + d(y, z) \).

Given \( x \in X \) and \( r > 0 \), the open ball and closed ball of radius \( r \) about \( x \) shall be denoted by

\[
B(x, r) = \{ y \in X | d(x, y) < r \}, \quad \bar{B}(x, r) = \{ y \in X | d(x, y) \leq r \},
\]

respectively. Let \( T \) denote the topology on \((X, d)\) induced by the open balls.

**Definition 2.2.** Let \((X, d)\) be a metric space.

- A sequence \( \{x_i\} \) in \( X \) is called a Cauchy sequence if, for each \( \varepsilon > 0 \), there exists \( N > 0 \) satisfying when \( m \geq n > N \), then \( d(x_n, x_m) < \varepsilon \).
- Given \( \varepsilon > 0 \), a subset \( A \subseteq X \) is called an \( \varepsilon \)-net of \( X \) if, for each \( x \in X \), there exists \( a_x \in A \) such that \( d(a_x, x) < \varepsilon \).
- \((X, d)\) is called complete if every Cauchy sequence in \( X \) converges in \( X \) with respect to \( T \).
- \((X, d)\) is called totally bounded if it has a finite \( \varepsilon \)-net for each \( \varepsilon > 0 \).
- \((X, d)\) is called boundedly compact if every bounded closed ball is compact.

**Theorem 2.3.** Let \((X, d)\) be a metric space. Then the following statements are equivalent:
• \((X, d)\) is compact.
• \((X, d)\) is sequentially compact.
• \((X, d)\) is complete and totally bounded.

**Definition 2.4.** Let \((X, d)\) be a metric space and \(\gamma : [a, b] \to X\) be a continuous curve. The length of \(\gamma\) with respect to the metric \(d\), denoted by \(L_d(\gamma)\), is defined by the supremum of the sums

\[
\Sigma(P) = \sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i))
\]

over all the partitions \(P = \{t_0, \ldots, t_N\}\) of \([a, b]\), where \(a = t_0 \leq t_1 \leq \cdots \leq t_N = b\). A continuous curve is said to be rectifiable if its length is finite.

Given two points \(x, y \in X\), define the associated metric \(d_L\) of \(d\) as follows:

\[
d_L(x, y) = \inf \{ L_d(\gamma) | \gamma : [0, T] \to X, \gamma \text{ is continuous, } \gamma(0) = x, \gamma(T) = y \}.
\]

We have the following statements.

**Theorem 2.5** ([6], Proposition 2.3.12). Let \((X, d)\) be a metric space with the associated metric \(d_L\) defined by (2.1).

1. \(d_L(x, y) \geq d(x, y)\), \(\forall x, y \in X\).
2. \(L_{d_L}(\gamma) = L_d(\gamma)\) for any rectifiable curve \(\gamma\) in \((X, d)\).
3. \((d_L)_L = d_L\).

In particular, \(d\) is called an intrinsic metric if \(d = d_L\).

**Definition 2.6.** Let \((X, d)\) be a metric space.

- A continuous curve \(\gamma : [0, T] \to X\) is called a shortest path if its length is minimal among the continuous curves with the same endpoints; in other words, \(L_d(\gamma') \geq L_d(\gamma)\) for any continuous curve \(\gamma'\) from \(\gamma(0)\) to \(\gamma(T)\).
- A continuous curve \(\gamma : [0, T] \to X\) is called a geodesic if for every \(t \in [0, T]\), there exists an open interval \((a, b)\) containing \(t\) in \([0, T]\) such that \(L_d(\gamma|_{[a,b]}) = d(\gamma(a), \gamma(b))\).
- A continuous curve \(\gamma : [0, T] \to X\) is called a minimizing geodesic if \(L_d(\gamma|_{[a,b]}) = d(\gamma(a), \gamma(b))\) for every closed interval \([a, b] \subset [0, T]\).
- A continuous curve \(\gamma : [0, T] \to X\) is said to have constant speed if there exists a constant \(C > 0\) such that

\[
L_d(\gamma|_{[a,b]}) = C(b - a), \quad \forall [a, b] \subset [0, T].
\]

In particular, if \(C = 1\), then \(\gamma\) is said to be parameterized by arc length.

Note that a minimizing geodesic is always a shortest path, but not vice versa unless \((X, d)\) is an intrinsic space. It is known that every rectifiable curve can be parameterized by arc length, see Proposition 2.5.9 in [6].
Theorem 2.7 (Hopf-Rinow-Cohn-Vossen Theorem, [6], Theorem 2.5.28). Let \((X, d)\) be a locally compact intrinsic metric space. Then the following assertions are equivalent:

1. \((X, d)\) is complete.
2. \((X, d)\) is boundedly compact, i.e., every bounded closed ball is compact.
3. Each constant speed geodesic \(\gamma : [0, T) \to X\) can be extended to a continuous curve \(\overline{\gamma} : [0, T] \to X\).
4. There is a point \(x \in X\) such that every constant speed minimizing geodesic \(\gamma : [0, T) \to X\) with \(\gamma(0) = x\) can be extended to a continuous curve \(\overline{\gamma} : [0, T] \to X\).

Furthermore, if any of the above holds, every two points in \(X\) can be connected by a minimizing geodesic. In this case, the metric space \((X, d)\) is called a geodesic space.

Now we recall the notion of Alexandrov curvature in a geodesic space. A geodesic triangle \(\Delta\) in a metric space \((X, d)\) consists of three points \(p, q, r \in X\), its vertices, and a choice of three geodesic segments \([p, q], [q, r], [r, p]\), joining them, its sides. Such a geodesic triangle will be denoted by \(\Delta(p, q, r)\).

For each geodesic triangle \(\Delta(p, q, r)\) in \((X, d)\), we construct a comparison triangle \(\overline{\Delta}(\overline{p}, \overline{q}, \overline{r})\) in the Euclidean plane with the same length of sides, i.e.,

\[
d(p, q) = |\overline{p} \overline{q}|, \quad d(q, r) = |\overline{q} \overline{r}|, \quad d(r, p) = |\overline{r} \overline{p}|,
\]

where \(| \cdot |\) denotes the usual Euclidean distance on the Euclidean plane. It is clear that a comparison triangle is uniquely defined up to a rigid motion of Euclidean plane.

Definition 2.8. A metric space \((X, d)\) is said to be non-positively (resp. non-negatively) curved if in some neighborhood of each point the following holds:

- For every geodesic triangle \(\Delta(p, q, r)\) and every point \(x \in [p, r]\), one has \(d(x, q) \leq |\overline{x} \overline{q}|\) (resp. \(d(x, q) \geq |\overline{x} \overline{q}|\)), where \(\overline{x}\) is the point on the side \([\overline{p}, \overline{r}]\) of a comparison triangle \(\overline{\Delta}(\overline{p}, \overline{q}, \overline{r})\) such that \(d(p, x) = |\overline{p} \overline{x}|\).

The definition given above was introduced by A. D. Alexandrov [1]. It provides a good notion of an upper bound on curvature in an arbitrary metric space. Classical comparison theorems in differential geometry show that a Riemannian manifold is non-positively (resp. non-negatively) curved in the above sense if and only if all of its sectional curvatures are non-positive (resp. non-negative).

Definition 2.9. A metric space \((X, d)\) is called a \(\text{CAT}(0)\) space if for every geodesic triangle \(\Delta\) and comparison triangle \(\overline{\Delta}\), for all \(x, y \in \Delta\) and all comparison points \(\overline{x}, \overline{y} \in \overline{\Delta}\),

\[
d(x, y) \leq |\overline{x} \overline{y}|.
\]

The terminology “\(\text{CAT}(0)\)” was coined by M. Gromov ([14], p. 119). The initials are in honour of E. Cartan, A. D. Alexandrov and V. A. Toponogov, each of whom considered similar conditions in varying degrees of generality.
Theorem 2.10 (Cartan-Hadamard Theorem). A complete simply connected intrinsic non-positively curved metric space is a CAT(0) space.

3. Sub-Finsler manifolds

In this section we introduce some basic facts about sub-Finsler manifolds.

Definition 3.1. Let $V$ be an $n$-dimensional real vector space. A norm $F : V \to \mathbb{R}$ is a function with the following properties:

1. $F(y) \geq 0, \forall y \in V$. $F(y) = 0$ if and only if $y = 0$.
2. $F(\lambda y) = |\lambda|F(y)$, $\forall \lambda \in \mathbb{R}, y \in V$.
3. $F(y + z) \leq F(y) + F(z)$, $\forall y, z \in V$.

In particular, a norm $F$ is called an Euclidean norm if $F = \sqrt{\langle \cdot, \cdot \rangle}$ for some inner product $\langle \cdot, \cdot \rangle$ on $V$.

There are many non-Euclidean norms on a vector space. Here we present some examples which will be used in this paper.

Example 3.2 ($\ell_p$ norm). Given $p \geq 1$ and a vector $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, the $\ell_p$ norm of $x$ is defined by:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}.$$

Clearly, the $\ell_p$ norm $\| \cdot \|_p$ is Euclidean if and only if $p = 2$.

Example 3.3 ($\ell_p$-type norm). Let $F_i$ be norms on vector spaces $V_i$, $i = 1, \ldots, m$, $m \in \mathbb{N}^+$. Then the function $F^p : V = V_1 \oplus \cdots \oplus V_m \to [0, \infty)$, $p \geq 1$, defined by

$$F^p(y) = \left(\sum_{i=1}^{m} (F_i(y_i))^p\right)^{\frac{1}{p}}, \quad \forall y = y_1 + \cdots + y_m \in V,$$

is a norm. Moreover, when $F_i$ are all Euclidean norms and $p = 2$, the resulting norm $F^2$ is also an Euclidean norm.

Notice that a norm $F$ on $V$ is continuous, we have:

Proposition 3.4. Let $F$ and $\| \cdot \|$ be norms on $V$. Then there exist constants $C_2 \geq C_1 > 0$ such that

$$C_1\|y\| \leq F(y) \leq C_2\|y\|, \quad \forall y \in V.$$

Definition 3.5. Let $M$ be a connected smooth $n$-dimensional manifold and $\mathbf{S}$ be a smooth bracket-generating distribution on $M$ with rank $k$, $k \leq n$, i.e., $\mathbf{S}$ is a subbundle of $TM$ and the vector fields which are sections of $\mathbf{S}$ together with all brackets span every tangent space $T_xM$, $x \in M$. A sub-Finsler metric on $M$ is a real continuous function $F : \mathbf{S} \to [0, \infty)$ such that the restriction of $F$ to any $\mathbf{S}_x$, $x \in M$, is a norm. In this case, we say that $(M, \mathbf{S}, F)$ is a sub-Finsler manifold.
In particular, when $F$ is induced by a Riemannian metric on $M$, $(M, S, F)$ is called a sub-Riemannian manifold. Moreover, when $S = TM$, the tangent bundle of $M$, the corresponding sub-Finsler metrics are called Finsler metrics.

**Remark 3.6.** We should mention that, in classical Finsler geometry [2, 8, 19], a Finsler metric on a connected smooth manifold $M$ is a real continuous function $F : TM \to [0, \infty)$ such that

1. $F$ is smooth on the slit tangent bundle $TM \setminus \{0\}$.
2. The restriction of $F$ to any $T_x M$, $x \in M$, is a Minkowski norm.

Let $(M, S, F)$ be a sub-Finsler manifold, a curve $\gamma : [0, T] \to M$ is called admissible if it is absolutely continuous and $\dot{\gamma}(t) \in S_{\gamma(t)}$ for almost every $t \in [0, T]$. The length of an admissible curve $\gamma$ is defined by

$$L_F(\gamma) = \int_0^T F(\dot{\gamma}(t))dt.$$ 

Define the distance function $d_F : M \times M \to [0, \infty)$ by

$$d_F(p, q) = \inf\{L_F(\gamma) | \gamma \text{ is admissible, } \gamma(0) = p, \gamma(T) = q\}. \quad (3.1)$$

We have the following statements.

**Theorem 3.7** ([6]). Let $(M, S, F)$ be a sub-Finsler manifold. Then we have

1. $(M, d_F)$ is an intrinsic metric space.
2. The topology $T$ induced by $(M, d_F)$ is exactly the original topology of $M$.

### 4. Homogeneous sub-Finsler manifolds

**Definition 4.1.** Let $(M, S, F)$ be a sub-Finsler manifold.

- A diffeomorphism $\phi$ of $M$ is called an isometry if for any $x \in M$, $X \in S_x$, $F(d\phi(X)) = F(X)$.
- A mapping $\phi$ of $M$ onto itself is called a distance isometry if for any $p, q \in M$, $d_F(\phi(p), \phi(q)) = d_F(p, q)$.

It is easily seen that, the set of isometries of $(M, S, F)$ forms a group, denoted by $I(M)$, which is clearly contained in the group $I(M, d_F)$ of distance isometries. Let $I_p(M, d_F)$ be the isotropy subgroup of $I(M, d_F)$ at $p \in M$. Van Dantzig and van der Waerden [9] proved that $I(M, d_F)$ is a locally compact topological transformation group on $M$ with respect to the compact-open topology and $I_p(M, d_F)$ is compact. Bochner and Montgomery [5] proved that a locally compact group of differentiable transformations of a manifold is a Lie transformation group. Combining the above results, in fact, we have proved the following

**Theorem 4.2.** Let $(M, S, F)$ be a sub-Finsler manifold. Then the group of isometries $I(M)$ of $M$ is a Lie transformation group of $M$. Let $p \in M$ and $I_p(M)$ be the isotropy subgroup of $I(M)$ at $p$. Then $I_p(M)$ is compact.
In the Riemannian (classical Finsler) geometry, it was proved in [11,17] that, the group \( I(M, d_F) \) coincides with the group of isometries \( \mathcal{I}(M) \). In general case, even in the sub-Riemannian setting, it is still unknown whether \( I(M, d_F) \) is a Lie transformation group of \( M \) [7].

**Definition 4.3.** A connected sub-Finsler manifold \((M, S, F)\) is called a homogeneous sub-Finsler manifold if its isometries group \( \mathcal{I}(M) \) acts transitively on \( M \), namely, for every pair \( p, q \in M \), there exists an isometry \( \phi \in \mathcal{I}(M) \) such that \( \phi(p) = q \).

Now we recall some important results due to Berestovskii [3]. Let \( G \) be a connected Lie group which acts effectively and transitively on a connected smooth manifold \( M \) and let \( H \) be the isotropy group of \( G \) at a fixed point \( p \in M \). Then \( M \) can be viewed as the coset space \( G/H \) by sending \( g \cdot p \) to \( gH \), \( \forall g \in G \). Let \( \mathfrak{g} \) and \( \mathfrak{h} \) denote the Lie algebras of \( G \) and \( H \), respectively. Denote by \( K \) the Killing form of \( \mathfrak{g} \). Since \( H \) is compact and the restriction of \( K \) to \( \mathfrak{h} \) is non-degenerate [15], we have a \( K \)-orthogonal reductive decomposition

\[
\mathfrak{g} = \mathfrak{h} + \mathfrak{m},
\]

where \( \mathfrak{m} \subset \mathfrak{g} \) is a vector subspace such that \( K(\mathfrak{h}, \mathfrak{m}) = 0 \) and \( \text{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m} \), \( \forall h \in H \), here \( \text{Ad} \) denotes the adjoint representation of \( G \). Moreover, one can identify \( \mathfrak{m} \) with the tangent space \( T_p M \) of \( M \) at the point \( p \) via the mapping \( \pi : X \rightarrow \frac{d}{dt} \bigg|_{t=0} \exp(tX) \cdot p \), where \( \exp \) denotes the exponential map of \( \mathfrak{g} \). Let \( \text{Pr}_m : \mathfrak{g} \rightarrow \mathfrak{m} \) denote the natural projection. Then for any \( g \in G \) and \( X \in \mathfrak{g} \), we have

\[
dg^{-1} \left( \frac{d}{dt} \bigg|_{t=0} \exp(tX)g \cdot p \right) = \frac{d}{dt} \bigg|_{t=0} \exp(t\text{Ad}_g^{-1}(X)) \cdot p = \pi \circ \text{Pr}_m (\text{Ad}_g^{-1}(X)).
\]

Hence, the tangent space \( T_g p(M) \) at the point \( g \cdot p \) can be identified as the subspace \( \text{Ad}_g(\mathfrak{m}) \).

**Definition 4.4.** A subspace \( \mathfrak{m}_0 \subset \mathfrak{m} \) is called bracket-generating if the smallest Lie algebra generated by \( \mathfrak{h} + \mathfrak{m}_0 \) is \( \mathfrak{g} \) itself.

Now let \( \mathfrak{m}_0 \subset \mathfrak{m} \) be an \( \text{Ad}(H) \)-invariant bracket-generating subspace, \( \mathcal{D}_{gH} = g_*(\mathfrak{m}_0) \subset T_{gH} (G/H) \) be the subspace of \( T_{gH} (G/H) \) associated to \( \mathfrak{m}_0 \) via the left action of \( g \) on \( G/H \), \( \forall g \in G \). Then \( \mathfrak{S}_0 := \cup_{g \in G} \mathcal{D}_{gH} \) is a smooth bracket-generating \( G \)-invariant distribution on \( G/H \) [3]. Assume \( F_0 \) is an \( \text{Ad}(H) \)-invariant norm on \( \mathfrak{m}_0 \) and let \( F_0 : \mathfrak{S}_0 \rightarrow \mathbb{R} \) be the sub-Finsler metric on \( G/H \) defined by

\[
F_0(g_*(X)) = F_0(X), \quad \forall g \in G, X \in \mathfrak{m}_0.
\]

**Theorem 4.5** ([3], Theorem 2). Let \( (M = G/H, \mathfrak{S}_0, F_0) \) be a sub-Finsler manifold and \( d_0 \) be the metric defined by (3.1). Then \( (M = G/H, d_0) \) is a \( G \)-invariant intrinsic metric space and the metric topology \( \mathcal{T} \) coincides with the manifold topology.
Conversely, we have:

**Theorem 4.6** ([3], Theorem 3). Any locally compact, locally contractible homogeneous space with intrinsic metric $d_0$ is isometric to a homogeneous sub-Finsler manifold $(G/H, S_0, F_0)$ with respect to a bracket-generating subspace $m_0$ and an $\text{Ad}(H)$-invariant norm $F_o$ on $m_0$, which is given by

$$F_o(X) = \lim_{t \to 0} \frac{d_0(p, \exp tX \cdot p)}{|t|}, \quad \forall X \in m_0.$$  

Moreover, given a quotient space $G/H$, a correspondence $(m_0, F_o) \leftrightarrow d_0$ is a bijection of the set of all such pairs onto the set of all $G$-invariant intrinsic metrics on $G/H$.

Combining Theorem 2.7 and Theorem 4.5, we immediately have:

**Theorem 4.7.** Every connected homogeneous sub-Finsler manifold is a complete geodesic space. Namely, every two points can be connected by a minimizing geodesic.

For convenience, sometimes the homogeneous sub-Finsler manifold $(G/H, S_0, F_0)$ will be denoted by $(G/H, m_0, F_o)$. In particular, when $m_0 = m$ and $F_o$ is a Minkowski norm, $(G/H, m, F_o)$ is a homogeneous classical Finsler manifold, which was widely studied in the last two decades [10].

## 5. Homogeneous geodesics

In this section, we mainly study the problem of existence of homogeneous geodesics in homogeneous sub-Finsler manifolds and prove Theorem 1.1.

**Definition 5.1.** Let $(M = G/H, S_0, F_0)$ be a homogeneous sub-Finsler manifold, $p = eH$. A geodesic $\gamma(t)$ through $p$ is called a homogeneous geodesic if there exists a vector $X \in \mathfrak{g}$ such that $\gamma(t) = \exp tX \cdot p$. Moreover, such a vector $X$ is called a geodesic vector.

It is easily seen that a homogeneous geodesic $\gamma(t) = \exp tX \cdot p$ is defined on the whole $\mathbb{R}$ and $\text{Pr}_m(X) \in m_0$, according to Lemma 5 in [3].

There is a well known criterion for homogeneous geodesics in the Riemannian case due to Kowalski and Vanhecke [16].

**Lemma 5.2 (Geodesic lemma).** Let $(M = G/H, TM, g)$ be a homogeneous Riemannian manifold. Then a curve $\gamma(t) = \exp tX \cdot p$ is a homogeneous geodesic if and only if the following equation holds:

$$\langle \text{Pr}_m(X), \text{Pr}_m([X, Z]) \rangle = 0, \quad \forall Z \in m,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $m$ induced by $g$.

It was proved in [15] that every homogeneous Riemannian manifold admits at least one homogeneous geodesic. Recently, Podobryaev studied homogeneous geodesics in sub-Riemannian manifolds and proved the following theorem.
**Theorem 5.3** ([18], Theorem 5). Let \((G/H, m_0, F_0)\) be a homogeneous sub-Riemannian manifold. If \(\ker K = m\) or \(K|_{m_0} \neq 0\), where \(K\) denotes the Killing form of the Lie algebra of \(G\), then there exists a homogeneous sub-Riemannian geodesic.

To study homogeneous geodesics in sub-Finsler manifolds, we first prove the following result.

**Proposition 5.4.** Let \((G/H, m_0, F_0)\) be a homogeneous sub-Finsler manifold and \(F_m\) be an \(\text{Ad}(H)\)-invariant norm on \(m\) such that \(F_o(X) \geq F_m(X)\), \(\forall X \in m_0\). Assume \(X \in g\) is a geodesic vector of \((G/H, m, F_m)\) satisfying \(\operatorname{Pr}_m(X) \in m_0\) and \(F_o(\operatorname{Pr}_m(X)) = F_m(\operatorname{Pr}_m(X))\). Then \(X \in g\) is also a geodesic vector of \((G/H, m_0, F_0)\). Namely, \(\gamma(t) = \exp tX \cdot H\) is a homogeneous geodesic in \((G/H, m_0, F_0)\).

**Proof.** Notation as above. Let \(d_F\) and \(F\) be the distance function and \(G\)-invariant Finsler metric on \(G/H\) corresponding to \((G/H, m, F_m)\), respectively. Since \(\gamma(t) = \exp tX \cdot H\) is a homogeneous geodesic in \((G/H, m, F_m)\), there exists a constant \(\varepsilon > 0\) such that

\[
\varepsilon F_m(\operatorname{Pr}_m(X)) = \int_0^\varepsilon F(\dot{\gamma}(t))dt = L_F(\gamma|_{[0,\varepsilon]}) = d_F(H, \exp \varepsilon X \cdot H).
\]

On the other hand, by the assumption that \(d_0 \geq d_F\), we obtain

\[
d_F(H, \exp \varepsilon X \cdot H) \leq d_0(H, \exp \varepsilon X \cdot H) \leq L_{d_0}(\gamma|_{[0,\varepsilon]})
\]

\[
= \int_0^\varepsilon F_0(\dot{\gamma}(t))dt = \varepsilon F_o(\operatorname{Pr}_m(X)) = \varepsilon F_m(\operatorname{Pr}_m(X)).
\]

This asserts that \(d_0(H, \exp \varepsilon X \cdot H) = L_{d_0}(\gamma|_{[0,\varepsilon]})\). Hence \(\gamma(t) = \exp tX \cdot H\) is a homogeneous geodesic in \((G/H, m_0, F_0)\). \(\square\)

Now we state the main result in this section.

**Theorem 5.5.** Every compact connected homogeneous sub-Finsler manifold \((M = G/H, S_0, F_0)\) admits a homogeneous geodesic.

**Proof.** Since \(M\) is compact, \(G\) is compact and the Lie algebra \(g\) of \(G\) has a decomposition \(g = g_s + g_a\), where \(g_s\) is compact semi-simple and \(g_a\) is Abelian. Let \(B\) be the negative of Killing form of \(g_s\) and \(\langle \cdot, \cdot \rangle_0\) be any inner product on \(g_a\). Set an Euclidean inner product \(\langle \cdot, \cdot \rangle\) on \(g\) by

\[
\langle X + Y, X + Y \rangle = B(X, X) + \langle Y, Y \rangle_0, \quad \forall X \in g_s, Y \in g_a.
\]

It is clear that \(\langle \cdot, \cdot \rangle\) is \(\text{Ad}(G)\)-invariant. Let \(g\) be the \(G\)-invariant Riemannian metric on \(M\) generated by \(\langle \cdot, \cdot \rangle|_m\) on \(m\) and denote by \(d_g\) the associated Riemannian distance function defined by (3.1). It follows from Lemma 5.2 that, for every vector \(X \in m\), the curve \(\exp tX \cdot p\) is a homogeneous geodesic in \((M, d_g)\).
Now assume $m_0$ is an $\text{Ad}(H)$-invariant bracket-generating subspace in $m$ and $F_0$ is an $\text{Ad}(H)$-invariant norm on $m_0$. We will show that the homogeneous sub-Finsler manifold $(M = G/H, d_0)$ admits a homogeneous geodesic. Let $f(z) = \frac{F_0^2(z)}{\langle z, z \rangle}$, $\forall z \in m_0 \setminus \{0\}$. By homogeneity, $f(z)$ can be viewed as defined on the sphere $I = \{z \in m_0 | \langle z, z \rangle = 1\}$, hence it must attain its minimum and maximum at some points. Since $f > 0$, we may suppose that $f$ attains the minimal value at $y \in I$ and $\lambda = f(y) > 0$. Hence $F_0^2(z) \geq \lambda \langle z, z \rangle$, $\forall z \in m_0$. Let $g_\lambda$ and $d_{g_\lambda}$ be the Riemannian metric and distance function on $M$ induced by $\lambda \langle \cdot, \cdot \rangle_{m_0}$ on $m$. Clearly, $d_{g_\lambda} = \sqrt{d_0}$ and $\gamma(t) = \exp ty \cdot p$ is a homogeneous geodesic in $(M, d_{g_\lambda})$. By Proposition 5.4, we obtain that $\gamma(t) = \exp ty \cdot p$ is a homogeneous geodesic in $(M, d_0)$, which completes the proof of the theorem.

**Definition 5.6.** A homogeneous sub-Finsler manifold $(G/H, m_0, F_0)$ is called a Carnot group if $H = \{e\}$, $G$ is a connected and simply connected nilpotent Lie group whose Lie algebra $g$ admits a direct sum decomposition in nontrivial vector subspaces

$$g = m_0 \oplus m_1 \oplus \cdots \oplus m_s, \quad s \geq 1,$$

such that

$$[m_0, m_j] = m_{j+1}, \quad j = 0, \ldots, s - 1, \quad [m_0, m_s] = 0.$$

**Proposition 5.7.** Let $(G, m_0, \langle \cdot, \cdot \rangle_0)$ be a sub-Riemannian Carnot group, where $\langle \cdot, \cdot \rangle_0$ is an inner product on $m_0$. Then for every vector $X \in m_0$, $\gamma(t) = \exp tX$, $t \in \mathbb{R}$ is a homogeneous geodesic.

**Proof.** We choose an inner product $\langle \cdot, \cdot \rangle$ on $g$ such that

$$\langle \cdot, \cdot \rangle_{m_0} = \langle \cdot, \cdot \rangle_0, \quad \langle m_i, m_j \rangle = 0, \quad 0 \leq i < j \leq s.$$ 

Notice that $(m_0, [g, g]) = 0$. Then by Lemma 5.2, it is easily seen that every vector $X \in m_0$ is a geodesic vector of the homogeneous Riemannian manifold $(G, g, \langle \cdot, \cdot \rangle)$. Thus the proposition follows from Proposition 5.4.

Combining Propositions 5.4, 5.7 and the argument in the proof of Theorem 5.5, we immediately have the following result, which completes the proof of Theorem 1.1.

**Theorem 5.8.** Every Carnot group admits a homogeneous geodesic.

**Proof.** It follows from Proposition 3.4 that, for any norm $F_0$ on $m_0$, there exists an inner product $\langle \cdot, \cdot \rangle_0$ on $m_0$ such that $F_0^2(X) \geq \langle X, X \rangle_0$ for all $X \in m_0$ and $F_0^2(Y) = \langle Y, Y \rangle_0$ for some nonzero vector $Y \in m_0$. It is clear that $Y$ is a geodesic vector of Carnot group $(G, m_0, F_0)$, according to Propositions 5.7.
6. Bi-invariant metrics on compact Lie groups

In this section, we study a special class of bi-invariant metrics on compact semi-simple Lie groups.

Let $m \in \mathbb{N}$, $m \geq 2$. For any $i \in \{1, \ldots, m\}$, let $G_i$ be connected and simple connected compact simple Lie group with unit $e_i$ and corresponding Lie algebra $\mathfrak{g}_i$. Denote by $B_i$ the negative of Killing form of $\mathfrak{g}_i$, $g_i$ the bi-invariant Riemannian metric on $G_i$ induced by $B_i$, and $d_i$ the distance function of $(G_i, g_i)$. The following results are true.

- $d_i$ is bi-invariant. That is, for any $x, y, g \in G_i$,
  \[ d_i(gx, gy) = d_i(xg, yg) = d_i(x, y). \]

- Every geodesic in $(G_i, d_i)$ is homogeneous. That is, if $\gamma$ is a geodesic and $\gamma(0) = e_i$, then there exists a vector $X \in \mathfrak{g}_i$, such that $\gamma(t) = \exp tX$, $t \in \mathbb{R}$.

- The sectional curvature of $(G_i, g_i)$ is non-negative and hence $(G_i, d_i)$ is non-negatively curved in the sense of Definition 2.8.

Now set $G = G_1 \times G_2 \times \ldots \times G_m$, $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \times \ldots \times \mathfrak{g}_m$, $B = B_1 \oplus B_2 \oplus \cdots \oplus B_m$.

Define a class of functions $d^p$, $p \geq 1$ on $G$ by

\[ d^p((x_1, \ldots, x_m), (y_1, \ldots, y_m)) = \left( \sum_{i=1}^{m} d_i^p(x_i, y_i) \right)^{\frac{1}{p}}, \]

$\forall x_i, y_i \in G_i$, $i = 1, \ldots, m$.

**Theorem 6.1.** Notation as above. $\forall p \geq 1$, we have

1. $d^p$ is a bi-invariant intrinsic metric on $G$.
2. The norm $F^p$ on $\mathfrak{g}$ determined by $(G, d^p)$ is given by
   \[ F^p(X) = \left( \sum_{i=1}^{m} B_i(X_i, X_i) \right)^{\frac{1}{p}}, \quad \forall X = (X_1, \ldots, X_m) \in \mathfrak{g}. \]

3. $\forall X \in \mathfrak{g}$, $\gamma(t) = \exp tX$, $t \in \mathbb{R}$ is a geodesic in $(G, d^p)$.
4. If $p \neq 2$, $(G, d^p)$ is neither non-positively curved nor non-negatively curved.

**Proof.** (1) It is easy to see that $d^p$ is a bi-invariant metric. We now prove $d^p$ is an intrinsic metric. Notice that $(G, d^p)$ is a compact metric space. Then by Theorem 2.4.16 in [6], it suffices to show that, for every two points $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in G$, there exists a midpoint $z \in G$ satisfying $d^p(x, z) = d^p(y, z) = \frac{1}{2}d^p(x, y)$. For any $i \in \{1, \ldots, m\}$, as the metric space $(G_i, d_i)$ is a compact homogeneous Riemannian manifold, then for the points $x_i, y_i \in G_i$, there exists a midpoint $z_i \in G_i$ such that $d_i(x_i, z_i) = d_i(y_i, z_i) = \frac{1}{2}d_i(x_i, y_i)$. Now set $z = (z_1, \ldots, z_m) \in G$, a quick calculation yields

\[ d^p(x, z) = \left( \sum_{i=1}^{m} d_i^p(x_i, z_i) \right)^{\frac{1}{p}} = \frac{1}{2} \left( \sum_{i=1}^{m} d_i^p(x_i, y_i) \right)^{\frac{1}{p}} = \frac{1}{2} d^p(x, y). \]
and
\[
d^p(y, z) = \left( \sum_{i=1}^{m} d_i^p(y_i, z_i) \right)^{\frac{1}{p}} = \frac{1}{2} \left( \sum_{i=1}^{m} d_i^p(x_i, y_i) \right)^{\frac{1}{2}} = \frac{1}{2} d^p(x, y),
\]
which implies that \( z \) is a midpoint between points \( x, y \) in \((G, d^p)\).

(2) Notice that
\[
\lim_{t \to 0} \frac{d_i(e_i, \exp tX_i)}{|t|} = (B_i(X_i, X_i))^{\frac{1}{2}}, \quad \forall X_i \in g_i.
\]
It follows from (4.1) that, \( \forall X = (X_1, \ldots, X_m) \in g \),
\[
F^p(X) = \lim_{t \to 0} \frac{d^p((e_1, \ldots, e_m), (\exp tX_1, \ldots, \exp tX_m))}{|t|}
= \lim_{t \to 0} \frac{1}{|t|} \left( \sum_{i=1}^{m} d_i^p(e_i, \exp tX_i) \right)^{\frac{1}{p}}
= \left( \sum_{i=1}^{m} \left( \lim_{t \to 0} \frac{d_i(e_i, \exp tX_i)}{|t|} \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}
= \left( \sum_{i=1}^{m} (B_i(X_i, X_i))^{\frac{1}{2}} \right)^{\frac{1}{p}}.
\]

(3) Given \( X = (X_1, \ldots, X_m) \in g \), let \( \gamma(t) = \exp tX = (\gamma_1(t), \ldots, \gamma_m(t)) \), where \( \gamma_i(t) = \exp tX_i \). Since for all \( i = 1, \ldots, m \), \( \gamma_i(t) = \exp tX_i \) is a geodesic in \((G_i, d_i)\), there exists a constant \( \varepsilon > 0 \) such that
\[
d_i(e_i, \exp \varepsilon X_i) = L_{d_i}(\gamma_i|_{[0,\varepsilon]}) = \varepsilon (B_i(X_i, X_i))^{\frac{1}{2}}.
\]
Hence
\[
L_{d^p}(\gamma|_{[0,\varepsilon]}) = \int_0^\varepsilon F^p(\dot{\gamma}(t))dt
= \varepsilon \left( \sum_{i=1}^{m} (B_i(X_i, X_i))^{\frac{1}{2}} \right)^{\frac{1}{p}}
= \left( \sum_{i=1}^{m} d_i^p(e_i, \exp \varepsilon X_i) \right)^{\frac{1}{p}}
= d^p(\gamma(0), \gamma(\varepsilon)).
\]
This implies that \( \gamma|_{[0,\varepsilon]} \) is a shortest path in \((G, d^p)\) and thus \( \gamma(t) = \exp tX \) is a geodesic in \((G, d^p)\).

(4) We consider two geodesic triangles \( \triangle_1(a, b_1, c_1) \) and \( \triangle_2(a, b_2, c_2) \) in \((G, d^p)\), where \( a = (e_1, \ldots, e_m) \), \( b_1 = (\exp \varepsilon X_1, e_2, \ldots, e_m) \), \( c_1 = (e_1, \exp \varepsilon X_2, e_3, \ldots, e_m) \), \( b_2 = (\exp \varepsilon X_1, \exp \varepsilon X_2, e_3, \ldots, e_m) \), \( c_2 = (\exp \varepsilon X_1, \exp (-\varepsilon X_2), \exp (-\varepsilon X_1)) \).
e_3, \ldots, e_m). X_1 \in g_1, X_2 \in g_2, B_1(X_1, X_1) = B_2(X_2, X_2) = 1, \varepsilon > 0 is sufficiently small. We have
\begin{align*}
  d^p(a, b_1) = d^p(a, c_1) = \varepsilon, & \quad d^p(b_1, c_1) = 2^\frac{2}{p} \varepsilon, \\
  d^p(a, b_2) = d^p(a, c_2) = 2^\frac{2}{p} \varepsilon, & \quad d^p(b_2, c_2) = 2\varepsilon.
\end{align*}

Now choose the midpoints \( x_1 = (\exp \frac{\varepsilon}{2} X_1, \exp \frac{\varepsilon}{2} X_2, e_3, \ldots, e_m) \in [b_1, c_1] \) and \( x_2 = (\exp \varepsilon X_1, e_2, \ldots, e_m) \in [b_2, c_2] \), we get
\begin{equation}
  (6.1) \quad d^p(a, x_1) = \frac{1}{2} \cdot 2^\frac{2}{p} \varepsilon, \quad d^p(a, x_2) = \varepsilon.
\end{equation}

In the comparison triangles \( \triangle_1(\bar{a}, \bar{b}_1, \bar{c}_1) \) and \( \triangle_2(\bar{a}, \bar{b}_2, \bar{c}_2) \), a direct calculation shows
\begin{equation}
  (6.2) \quad |\bar{a} \bar{x}_1| = \varepsilon \sqrt{1 - \left( \frac{1}{2} \cdot 2^\frac{2}{p} \right)^2}, \quad |\bar{a} \bar{x}_2| = \varepsilon \sqrt{(2^\frac{1}{p})^2 - 1}.
\end{equation}

Combing (6.1) and (6.2), we obtain
\begin{equation*}
\begin{cases}
  d^p(a, x_1) > |\bar{a} \bar{x}_1|, & 1 \leq p < 2, \\
  d^p(a, x_1) = |\bar{a} \bar{x}_1|, & p = 2, \\
  d^p(a, x_1) < |\bar{a} \bar{x}_1|, & p > 2,
\end{cases}
\end{equation*}

and
\begin{equation*}
\begin{cases}
  d^p(a, x_2) < |\bar{a} \bar{x}_2|, & 1 \leq p < 2, \\
  d^p(a, x_2) = |\bar{a} \bar{x}_2|, & p = 2, \\
  d^p(a, x_2) > |\bar{a} \bar{x}_2|, & p > 2.
\end{cases}
\end{equation*}

Clearly, if \( p \neq 2 \), \((G, d^p)\) is neither non-positively curved nor non-negatively curved. □

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