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BREDON HOMOLOGY OF WALLPAPER GROUPS

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Abstract. In this paper we compute the Bredon homology of wallpaper groups with respect to the family of finite groups and with coefficients in the complex representation ring. We provide explicit bases of the homology groups in terms of irreducible characters of the stabilizers.

1. Introduction

Bredon homology is one of the main instances of equivariant homology theory. Roughly speaking (see definitions in Section 2), given a G-space X and a family $\mathcal{F}$ of stabilizers of the action, the values of the homology are determined by a coefficient module N which takes values in abelian groups and takes account of the structure of the family $\mathcal{F}$. After their development by G. Bredon in the sixties for the case of G finite, different choices of $\mathcal{F}$ have showed different roles of this homology theory in contexts as equivariant obstruction theory [17], partition complexes [2], stable homotopy [8] or group dimension theory [10]. The computation of the Bredon homology groups has been particularly important in relation with the isomorphism conjectures (see a survey in [13]), as they may permit the computation of K-theory groups via an equivariant version of the Atiyah-Hirzebruch spectral sequence. In this framework, when X is the classifying space $\underline{E}G$ for proper actions of G (see Section 2.2 below), the Bredon homology of $\underline{E}G$ is an invariant of the group G, and in this case we will directly say the "Bredon homology of G".

We deal in the sequel with the Bredon homology of the crystallographic groups of the plane (also called *wallpaper groups*), with respect to the family of finite subgroups and with coefficients in the complex representation ring. This coefficient module codifies the complex representation theory of the finite subgroups of the group, and is relevant in relation with the Baum-Connes conjecture, the best known of the isomorphism conjectures. Recall that this

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statement identifies the equivariant K-homology of the space $\underline{E}G$ with the K-theory of the reduced C^*-algebra of G. The conjecture is true for wallpaper groups, as they are solvable (see Section 2.1), and the corresponding values of $K_*(C_r^*G)$ were computed in his thesis by Yang ([23], see also Lück-Stamm [14], where in particular a little mistake in Yang's results is corrected). Other computations of Bredon homology in the context of Baum-Connes conjecture may be found for example in [1, 11, 20].

The main goal of this paper is to offer *explicit* computations of the Bredon homology group of wallpaper groups. By "explicit" we mean giving bases of the homology groups (as abelian groups) in terms of irreducible characters of representations of finite stabilizers of the action of the groups, as well as a detailed description of the Bredon complex and the corresponding differentials. Aside the information which is obtained in this way about the representation theory of the group, the motivation of the study has come from the following problem. Consider a group G which is a colimit of wallpaper groups, and try to compute the left-hand side of Baum-Connes for G (this is, for example, the case of different extensions of $SL(2,\mathbb{Z})$ by $\mathbb{Z}^2$). A possible strategy is to obtain the Bredon homology of G out of the Bredon homologies of the wallpaper groups involved in the colimit, but this computation involve a precise knowledge of the induced homomorphisms in homology, and in particular of concrete generators of each group in the diagram, which is the kind of information that our study provides. Moreover, we also expect that our results may permit a sharp description of the Baum-Connes' assembly map for wallpaper groups, as in for example [3, 7, 19]. Finally, we are aware that Yang and Lück-Stamm results, and also the computational approach by Bui-Ellis [4], provide the isomorphism type (as abstract groups) of some of these Bredon homology groups, but as said above, we believe that the main contribution of the present paper is the explicit description of the groups.

The structure of the paper is as follows.

In Section 2 we give the necessary information about wallpaper groups, classifying spaces, Bredon homology, representation theory and Smith normal forms in order to make the paper as self-contained as possible.

In particular, Section 2.4 contains all the relevant information about the representation theory of the stabilizers.

Section 3 contains all the computations of Bredon homology groups, with a little introduction in which we explain the steps we follow in each calculation.

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2. Preliminaries

2.1. Wallpaper groups

In this subsection we recall the main features of the wallpaper groups, which are the main object of study of this paper. Nice surveys of the theory can be found in [15, 21], while presentations by generators and relations for all the wallpaper groups are given in [6].

We start with the definition:

Definition. A discrete group G of isometries of the plane $\mathbb{R}^2$ is called a *wallpaper group* if the action of G on the plane is properly discontinuous and the quotient $\mathbb{R}^2/G$ is compact.

There are exactly seventeen non-isomorphic wallpaper groups, as was independently proved by Fedorov and Schoenflies. Every such group G is in particular defined by an extension:

$$\mathbb{Z}^2 \to G \to F,$$

where F is a finite group, called the *point group* of the wallpaper group. The generators of the free abelian group correspond to two independent translations, and the images of a certain compact pattern of the plane by these translations tessellate it (this is the reason of the name *wallpaper*). In Figure 2.1 such patterns are pictured for all these groups, and the fact that they tessellate, and then contain a fundamental domain for the action, will be often used implicitly in Section 4, when a representative of class of equivariant 2-cells for a G-CW complex structure is defined for every wallpaper group.

These groups can also possess rotations, reflections and glide-reflections, corresponding in particular rotations and reflections with torsion elements of the groups. In Figure 2.1 the rotation centers contained in the pattern can be observed, as well as the reflection and glide-reflection axes. We will use this picture as a major source of information in the computations of Section 3.

In Table 1 we have compiled some relevant information about the wallpaper groups that will useful for us. In the second column there and throughout the paper we will denote by D_n the dihedral group of $2n$ elements. In turn, the cyclic group of n elements will be denoted indistinctly by C_n or $\mathbb{Z}/n$ in all the sequel. The third column of the table explain if the extension that defines the group splits or not. In the fourth the torsion primes of every group can be found: observe that **p1** and **pg** are the only torsion-free instances. Finally, in the fifth column, we distinguish 2- and 4- rotations when there are 2-rotations which in the groups that have no roots; and same for 2-, 3- and 6-rotations in **p6** and **p6m**.

In next section we will define the classifying space for proper actions, which is fundamental in our computations, and in particular we will recall the appropriate model of this space for wallpaper groups.

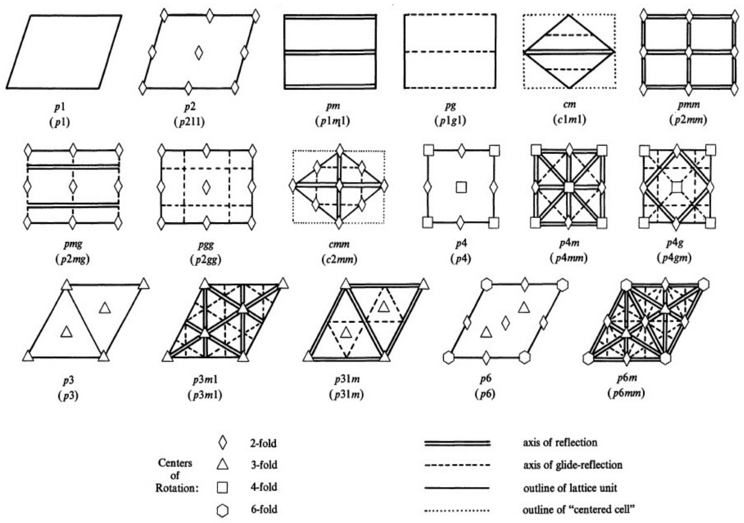

FIGURE 2.1. Patterns for the wallpaper groups

TABLE 1. Wallpaper groups

Group	Point group	Split	Torsion	Rotations	Reflections	Glide-reflections
p1	$\{1\}$	-	No	No	No	No
p2	C_2	Yes	2	2-	No	No
pm	C_2	Yes	2	No	Yes	No
pg	C_2	No	No	No	No	Yes
cm	C_2	Yes	2	No	Yes	Yes
pmm	D_2	Yes	2	2-	Yes	No
pmg	D_2	No	2	2-	Yes	Yes
pgg	D_2	No	2	2-	No	Yes
cmm	D_2	Yes	2	2-	Yes	Yes
p4	C_4	Yes	2	2-,4-	No	No
p4m	D_4	No	2	2-,4-	Yes	Yes
p4g	D_4	Yes	2	2-,4-	Yes	Yes
p3	C_3	Yes	3	3-	No	No
p3m1	D_3	Yes	2,3	3-	Yes	Yes
p31m	D_3	Yes	2,3	3-	Yes	Yes
p6	C_6	Yes	2,3	2-,3-,6-	No	No
p6m	D_6	Yes	2,3	2-,3-,6-	Yes	Yes

2.2. Classifying space for proper actions

The main geometric object in this paper is the classifying space for proper actions, so we will recall the necessary definitions here. We refer the reader to [12] for a thorough exposition about the subject, and to [5] for generalities about group actions and G-CW-complexes.

Definition. Let G be a discrete group, $\mathcal{F}$ a family of subgroups closed under conjugation and subgroups. Then a G-CW-complex $E_{\mathcal{F}}G$ is called a *classifying space for the family* $\mathcal{F}$ if given a subgroup $H < G$, the fixed point set $E_{\mathcal{F}}G^H$ is contractible if $H \in \mathcal{F}$ and empty otherwise.

Observe that the definition implies that the stabilizers of the action of G belong to $\mathcal{F}$. It can be proved that this space is unique up to homotopy equivalence.

When $\mathcal{F}$ is the trivial family, $E_{\mathcal{F}}G = EG$, the universal space for G-principal bundles. On the other hand, when $\mathcal{F}$ is the family of finite subgroups of G, $E_{\mathcal{F}}G$ is usually denoted by $\underline{E}G$, and called the *classifying space for proper actions of* G. As stated in the introduction, this is the main object of interest of the left-hand side of Baum-Connes, and the computation of its Bredon homology for wallpaper groups G is the main goal of this paper. In fact, these computations are feasible because there is a very simple model available for $E_{\mathcal{F}}G$ in this case:

Theorem 2.1. *Let G be a wallpaper group, and consider the usual action of G on the plane via isometries. Then $\mathbb{R}^2$ is a model for $\underline{E}G$.*

Proof. See Section 4 of [14]. ☐

The good knowledge of the actions of these groups of the plane will make possible the computation of the Bredon homology group. In next section we will recall the necessary definitions of this equivariant homology theory.

2.3. Bredon homology

We recall in this section the main facts concerning Bredon homology, which is the main invariant we will deal with in this paper. We only review the topological version, following the approach of [20]; a good exposition that includes the algebraic version can be found in [18].

Let G be a discrete group, $\mathcal{F}$ a family of groups which is closed under conjugation and taking subgroups. Consider the *orbit category* $O_{\mathcal{F}}(G)$, whose objects are the homogeneous spaces G/K, $K \subset G$ with $K \in \mathcal{F}$, and whose morphisms are the G-equivariant maps. Then a *left Bredon module* N over $O_{\mathcal{F}}(G)$ is a covariant functor

$$N : O_{\mathcal{F}}(G) \to \mathbf{Ab},$$

where $\mathbf{Ab}$ is the category of abelian groups.

Now consider a left Bredon module N and a G-CW-complex X, and assume that all the stabilizers of the G-action belong to the family $\mathcal{F}$. Then the *Bredon chain complex* (C_n, Φ_n) can be defined in the following way. For every $d \geq 0$, consider a set $\{e_i^d\}_{i\in I}$ of representatives of orbits of d-cells in X, and denote by $stab(e_i^d)$ the stabilizer of e_i^d. Then we define the *n-th group of Bredon chains* as $C_d = \bigoplus_{i\in I} N(G/stab(e_i^d))$.

Now, consider a $(d-1)$-face of e_i^d, which can be given as ge for a certain $(d-1)$-cell e. Then we have an inclusion of stabilizers $g^{-1}stab(e_i^d)g \subseteq stab(e)$. As $g^{-1}stab(e_i^d)g$ and $stab(e_i^d)$ are isomorphic, the previous inclusion induces an equivariant G-map $f : G/stab(e_i^d) \to G/stab(e)$. In turn, as N is a functor, we have an induced homomorphism $N(f) : N(G/stab(e_i^d)) \to N(G/stab(e))$. Taking into account that the boundary of e_i^d can be written as $\partial e_i^d = \sum_{j=1}^{n} e_j^{d-1}g_j$ for certain $g_j \in G$ and using linear extension to all representatives of equivariant d-cells, we obtain a differential $\Phi_d : C_d \to C_{d-1}$ for every $d > 0$. So have the following definition:

Definition. The homology groups of the chain complex (C_i, Φ_i) will be denoted by $H_i^{\mathcal{F}}(X, N)$ and called *Bredon homology groups* of G with coefficients in N with respect to the family $\mathcal{F}$.

These groups are an invariant of the G-homotopy type of X.

In this paper we are interested in Bredon homology with coefficients in the complex representation ring. In the next section we recall the definition of this coefficient module.

2.4. Representation theory

We refer the reader to the classic book of Serre [22] for all the basic concepts concerning complex representation theory of finite groups and their characters.

As said above, we compute Bredon homology with respect to the family of the finite subgroups with coefficients in the representation ring Bredon module $\mathcal{R}_{\mathbb{C}}$. This module is defined in the following way. Given a group G, the functor $\mathcal{R}_{\mathbb{C}} : Or(G) \to \mathbf{Ab}$ is defined over objects as $\mathcal{R}_{\mathbb{C}}(G/K) = R_{\mathbb{C}}(K)$ the complex representation ring of K. To define the functor over morphisms, observe that for any equivariant map $f : G/K \to G/H$ there exists $g \in G$ such that $gKg^{-1} \subseteq H$. As $R_{\mathbb{C}}(gKg^{-1}) = R_{\mathbb{C}}(K)$, we can define $\mathcal{R}(f) : R_{\mathbb{C}}(K) \to R_{\mathbb{C}}(K)$ by induction from the subgroup inclusion $gKg^{-1} \subseteq H$. A detailed exposition about the properties of this functor can be found in [18, Section 3].

To compute the differentials in the Bredon chain complex, we will need to know the homomorphisms between representation rings that are induced by inclusion of stabilizers in the wallpaper groups. In order to do so, we recall the structure of these rings as abelian groups and bases of irreducible characters in Table 2. There, the first element of each basis will always represent the trivial representation. In the case of D_3, which is isomorphic to the symmetric group S_3, χ_2 stands for the sign representation and χ_3 for the standard representation. For the dihedral groups, ϕ_i stand for the characters that correspond to 2-dimensional irreducible representations.

Finally, in Table 3 we describe explicitly the homomorphisms between the representation rings of the stabilizers, which are easily obtained using the character tables of the groups and Frobenius reciprocity (see [22, Ch. 2]). For the groups in the left-hand side of the homomorphisms (always the trivial group or

Table 2. Representation theory of stabilizers

Group	Representation ring	Basis
C_n	$\mathbb{Z}^n$	$\langle \chi_1, \dots, \chi_n \rangle$
D_2	$\mathbb{Z}^4$	$\langle \chi_1, \chi_2, \chi_3, \chi_4 \rangle$
D_3	$\mathbb{Z}^3$	$\langle \chi_1, \chi_2, \chi_3 \rangle$
D_4	$\mathbb{Z}^5$	$\langle \chi_1, \chi_2, \chi_3, \chi_4, \phi \rangle$
D_6	$\mathbb{Z}^6$	$\langle \chi_1, \chi_2, \chi_3, \chi_4, \phi_1, \phi_2 \rangle$

C_2) the generators will be denoted by the letter ρ, being ρ_1 the trivial representation in the case of C_2. The notation $\rho \uparrow$ means, in each line, that we are giving the character induced by ρ via the group inclusion of the left. In lines 10-17, C_2^1 is the conjugacy class of an order 2 element with non-trivial roots in D_n, while C_2^2 corresponds to an element with no non-trivial roots. For the characters in the right-hand side we keep the notation of Table 2.

Table 3. Induced characters on stabilizers

	Inclusion	Induced character	Image
1	$\{1\} \hookrightarrow C_n$	$\rho \uparrow$	$\chi_1 + \dots + \chi_n$
2	$\{1\} \hookrightarrow D_2$	$\rho \uparrow$	$\chi_1 + \chi_2 + \chi_3 + \chi_4$
3	$\{1\} \hookrightarrow D_3$	$\rho \uparrow$	$\chi_1 + \chi_2 + 2\chi_3$
4	$\{C_2\} \hookrightarrow C_2$	$\rho_1 \uparrow$	χ_1
5	$\{C_2\} \hookrightarrow C_2$	$\rho_2 \uparrow$	χ_2
6	$\{C_2\} \hookrightarrow D_2$	$\rho_1 \uparrow$	$\chi_1 + \chi_2$
7	$\{C_2\} \hookrightarrow D_2$	$\rho_2 \uparrow$	$\chi_3 + \chi_4$
8	$\{C_2\} \hookrightarrow D_3$	$\rho_1 \uparrow$	$\chi_1 + \chi_3$
9	$\{C_2\} \hookrightarrow D_3$	$\rho_2 \uparrow$	$\chi_2 + \chi_3$
10	$\{C_2^1\} \hookrightarrow D_4$	$\rho_1 \uparrow$	$\chi_1 + \chi_2 + \chi_3 + \chi_4$
11	$\{C_2^1\} \hookrightarrow D_4$	$\rho_2 \uparrow$	2ϕ
12	$\{C_2^2\} \hookrightarrow D_4$	$\rho_1 \uparrow$	$\chi_1 + \chi_3 + \phi$
13	$\{C_2^2\} \hookrightarrow D_4$	$\rho_2 \uparrow$	$\chi_2 + \chi_4 + \phi$
14	$\{C_2^1\} \hookrightarrow D_6$	$\rho_1 \uparrow$	$\chi_1 + \chi_2 + 2\phi_2$
15	$\{C_2^1\} \hookrightarrow D_6$	$\rho_2 \uparrow$	$\chi_3 + \chi_4 + 2\phi_1$
16	$\{C_2^2\} \hookrightarrow D_6$	$\rho_1 \uparrow$	$\chi_1 + \chi_3 + \phi_1 + \phi_2$
17	$\{C_2^2\} \hookrightarrow D_6$	$\rho_2 \uparrow$	$\chi_2 + \chi_4 + \phi_1 + \phi_2$

2.5. Smith normal form

When computing the Bredon homology of wallpaper groups, the computation of the Smith normal form of a matrix is necessary, in order to describe the groups and also for obtaining explicit bases, which is one of the main goals of this paper. A thorough treatment on the subject can be found in [9], and we recall briefly here the main results that are used in the paper.

Let A be an $m \times n$ matrix with integer entries. Then there always exist invertible matrices P and Q of size m and n, respectively, such that the matrix $D = PAQ$ has the following shape:

- For a certain $k \leq \min(m, n)$ and for every $i \leq k$, the entries d_{ii} of the matrix D are nonzero integers.
- For every $i \leq k$, d_{ii} divides $d_{i+1,i+1}$.
- The remaining entries of D are zero.

The matrix D is called the *Smith normal form* of A (usually abbreviated SNF) and is unique up to signs of the d_{ii}. The non-trivial entries are called the *invariant factors* or the *elementary divisors* of A.

We will use the Smith normal form to describe kernels and cokernels of homomorphisms between free abelian groups. Consider then a homomorphism $f : \mathbb{Z}^n \to \mathbb{Z}^m$, and the associated matrix A of size $m \times n$. Let $SNF(A) = PAQ$ be a decomposition of the Smith normal form, and $(d_1, \dots, d_k)$ the invariant factors. Then we have the following:

- Assume $|d_j| > 1$, $|d_{j-1}| = 1$, or $d_j = d_1$ if $|d_1| > 1$. Then the cokernel of f is isomorphic to $\mathbb{Z}/(d_j) \times \cdots \times \mathbb{Z}/(d_k) \times \mathbb{Z}^{m-k}$.
- The images of the last $m-k$ column vectors of P^{-1} under the projection $\mathbb{Z}^m \to Coker f$ produce a basis of the torsion-free part of $Coker f$.
- The last $n - k$ column vectors of Q provide a basis for the kernel of f.

These results will be essential when computing the Bredon homology groups.

Remark 2.2. There are different algorithms to compute the Smith normal form of a matrix. The computations for this paper have been performed used the algorithm implemented in [16]. The outcome of our computations, including Smith normal forms and auxiliary matrices, is available on request.

3. Bredon homology of wallpaper groups

In this section we undertake the main goal of this paper, which is the explicit computation of the Bredon homology of the wallpaper groups, with respect to the family of finite subgroups. Some notation will be required at this point. In general, for any of the wallpaper groups, a representative of a class of equivariant i-cells will be denoted by e_i^j. When there is only one equivariant i-cell the superscript will be suppressed. The irreducible characters in the representations rings of the stabilizers of 0-cells will be denoted by α, of stabilizers of 1-cells by the letter β, and of stabilizers of 2-cells by the letter γ. The homology classes in every chain group will be denoted by brackets. If $H < K$ is an inclusion of stabilizers and χ is a character on H, then the induced character on K is denoted by $(\chi \uparrow K)$. In the exposition we will sometimes refer to Figure 2.1 without express mention.

In the computation of the Bredon homology of every wallpaper group we undertake the following strategy. Starting from the pattern of the group of Figure 2.1 and taking into account that the pattern contains a fundamental domain for the group, we describe a G-CW-complex structure in the plane with a unique class of equivariant 2-cells and we compute the boundaries of the 2-cells and 1-cells. Then, we describe the stabilizers of the cells and form the corresponding Bredon chain complex. After that, using the previously computed boundaries and the induced representations that are listed in Table 3 and taking account of the orientations, we describe the differentials of the Bredon complex. We conclude by using the Smith normal forms of the matrices of the differentials and their auxiliary matrices to describe the homology groups and their bases, in terms of irreducible characters of the stabilizers.

3.1. The group p1

As the group is generated by the two translations, a representative e_2 for the equivariant 2-cells will be given by the polygon in Figure 2.1. Call O the lower vertex of the polygon D in its left-hand side. Then the remaining vertices, going clockwise, are $P = g_1O$, $Q = g_2O$ and $R = g_3O$ for three elements $g_i \in \mathbf{p1}$. We then consider just one class of **p1**-equivariant 0-cells in the plane, and we choose the representative e_0 corresponding to the vertex O. In this structure, there are two equivariant 1-cells, which are represented by the edges $e_1^0 = OP$ and $e_1^2 = PQ$. Hence, the boundaries with respect to the orbit representatives are given by:

$$\begin{aligned}\partial e_2 &= e_1^0 + e_1^1 + g_3e_1^0 + g_1e_1^1,\\ \partial e_1^0 &= g_1e_0 - e_0,\\ \partial e_1^1 &= g_2e_0 - g_1e_0.\end{aligned}$$

As this group is torsion-free, all the stabilizers of the equivariant cells are trivial. Then the Bredon chain complex has the shape:

$$0 \to \mathbb{Z}\gamma \to \mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_2 \to \mathbb{Z}\alpha \to 0.$$

Now we have:

$$\begin{aligned}&\Phi_2(\gamma) = (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) - (\gamma \uparrow \mathrm{stab}(e_1^0)) - (\gamma \uparrow \mathrm{stab}(e_1^1)) = 0,\\ &\Phi_1(\beta_1) = (\beta_1 \uparrow \mathrm{stab}(e_0^0)) - (\beta_1 \uparrow \mathrm{stab}(e_0^0)) = 0,\\ &\Phi_1(\beta_2) = (\beta_2 \uparrow \mathrm{stab}(e_0^0)) - (\beta_2 \uparrow \mathrm{stab}(e_0^0)) = 0.\end{aligned}$$

Hence, the differentials in the Bredon complex are trivial, and we obtain $H_2^{\mathcal{F}}(\mathbf{p1}, R_{\mathbb{C}}) = \mathbb{Z}[\gamma]$, $H_1^{\mathcal{F}}(\mathbf{p1}, R_{\mathbb{C}}) = \mathbb{Z}[\beta_1] \oplus \mathbb{Z}[\beta_2]$ and $H_0^{\mathcal{F}}(\mathbf{p1}, R_{\mathbb{C}}) = \mathbb{Z}[\alpha]$.

3.2. The group p2

Consider the lower half of the polygon in Figure 2.1, which will be a representative for the equivariant 2-cell e_2. We consider the five vertices in this lower half of the picture, and also the center of the polygon. Starting from the lower left-hand side vertex and counting clockwise, we denote the vertices by O, P, Q, R, S and T. We consider four classes of 0-cells, with representatives e_0^0, e_0^1, e_0^2 and e_0^3 that correspond, respectively, to the vertices O, P, Q and T. Observe that R is the image of P under the rotation r_1 of center Q, and S is the image of O under the rotation r_2 of center T. In turn, there are three classes of 1-cells, with representatives e_1^0, e_1^1 and e_1^2 that correspond to the edges OP, PQ and ST. Hence, if t is the horizontal translation, we have the boundaries:

$$\begin{aligned}
\partial(e_2) &= e_1^0 + e_1^1 + r_1 e_1^1 + t e_1^0 + e_1^2 + r_2 e_1^2,\\
\partial(e_1^0) &= e_0^1 - e_0^0,\\
\partial(e_1^1) &= e_0^2 - e_0^1,\\
\partial(e_1^2) &= e_0^3 - e_0^0.
\end{aligned}$$

The only nontrivial stabilizers correspond to the 0-cells (which are 2-centers of rotation), and are all isomorphic to C_2. Then the chain complex is:

$$0 \to \mathbb{Z}\gamma \to \bigoplus_{i=0}^{2} \mathbb{Z}\beta_i \to \bigoplus_{i=0}^{3} (\mathbb{Z}\alpha_i^1 \oplus \mathbb{Z}\alpha_i^2) \to 0.$$

Taking account of line 1 in Table 3, the differentials are defined in the following way:

$$\begin{aligned}
\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) - (\gamma \uparrow \mathrm{stab}(e_1^1))\\
&\quad - (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^2)) - (\gamma \uparrow \mathrm{stab}(e_1^2)) = 0,\\
\Phi_1(\beta_0) &= (\beta_0 \uparrow \mathrm{stab}(e_0^1)) - (\beta_1 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^2 - \alpha_0^1 - \alpha_0^2,\\
\Phi_1(\beta_1) &= (\beta_1 \uparrow \mathrm{stab}(e_0^2)) - (\beta_2 \uparrow \mathrm{stab}(e_0^1)) = \alpha_2^1 + \alpha_2^2 - \alpha_1^1 - \alpha_1^2,\\
\Phi_1(\beta_2) &= (\beta_2 \uparrow \mathrm{stab}(e_0^3)) - (\beta_2 \uparrow \mathrm{stab}(e_0^0)) = \alpha_3^1 + \alpha_3^2 - \alpha_0^1 - \alpha_0^2.
\end{aligned}$$

Now computing the Smith normal form of the matrix of Φ_1 we obtain that the invariant factors of Φ_1 are $(1, 1, 1)$. As Φ_2 is trivial, this implies that $H_2^{\mathcal{F}}(\mathbf{p2}, R_{\mathbb{C}}) = \mathbb{Z}$, $H_1^{\mathcal{F}}(\mathbf{p2}, R_{\mathbb{C}}) = 0$ and $H_0^{\mathcal{F}}(\mathbf{p2}, R_{\mathbb{C}}) = \mathbb{Z}^5$.

It is clear that a basis for $H_2^{\mathcal{F}}(\mathbf{p2}, R_{\mathbb{C}})$ is $[\gamma]$. In turn, the matrix Q obtained in the computation of the SNF shows that a basis of $H_0^{\mathcal{F}}(\mathbf{p2}, R_{\mathbb{C}})$ is given by $([\alpha_0^1], [\alpha_0^2], [\alpha_1^2], [\alpha_2^2], [\alpha_3^2])$.

3.3. The group pm

Again in this case a representative e_2 for the unique 2-cell will be given by the lower half of the rectangle. Consider O, P, Q and R the vertices of this little rectangle, starting from the lower in the left-hand side edge. There will be two classes of 0-cells, with representatives e_0^0 and e_0^1, that correspond, respectively, to the vertices O and P, with Q in the class of P and R in the class of O (the identification given by the horizontal translation t). In turn, there are two classes of 1-cells, with representatives e_1^0, e_1^1 and e_1^2, given by the edges OP, PQ and RP, respectively. Observe that RQ is the image of OP under t. Now we can compute the boundaries:

$$\begin{aligned}
\partial(e_2) &= e_1^0 + e_1^1 + te_1^0 + e_1^2,\\
\partial(e_1^0) &= e_0^1 - e_0^0,\\
\partial(e_1^1) &= te_0^1 - e_0^1,\\
\partial(e_1^2) &= e_0^0 - te_0^0.
\end{aligned}$$

Now the vertices and the edges e_1^1 and e_1^2 are in rotation axes, so their stabilizers are isomorphic to C_2, while the stabilizer of the remaining edge is trivial. The the Bredon chain complex takes the following shape:

$$0 \to \mathbb{Z}\gamma \to \mathbb{Z}\beta_0 \oplus \mathbb{Z}\beta_1^1 \oplus \mathbb{Z}\beta_1^2 \oplus \mathbb{Z}\beta_2^1 \oplus \mathbb{Z}\beta_2^2 \ \to \bigoplus_{i=0}^{1}(\mathbb{Z}\alpha_i^1 \oplus \mathbb{Z}\alpha_i^2) \to 0.$$

Now we can compute the differentials, taking account of line 1 in Table 3:

$$\begin{aligned}
\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) - (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^2))\\
&= \beta_1^1 + \beta_1^2 + \beta_2^1 + \beta_2^2,\\
\Phi_1(\beta_0) &= (\beta_0 \uparrow \mathrm{stab}(e_0^1)) - (\beta_1 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^2 - \alpha_0^1 - \alpha_0^2,\\
\Phi_1(\beta_1^1) &= (\beta_1^1 \uparrow \mathrm{stab}(e_0^1)) - (\beta_1^1 \uparrow \mathrm{stab}(e_0^1)) = 0,\\
\Phi_1(\beta_1^2) &= (\beta_1^2 \uparrow \mathrm{stab}(e_0^1)) - (\beta_1^2 \uparrow \mathrm{stab}(e_0^1)) = 0,\\
\Phi_1(\beta_2^1) &= (\beta_2^1 \uparrow \mathrm{stab}(e_0^0)) - (\beta_1^1 \uparrow \mathrm{stab}(e_0^0)) = 0,\\
\Phi_1(\beta_2^2) &= (\beta_2^2 \uparrow \mathrm{stab}(e_0^0)) - (\beta_1^2 \uparrow \mathrm{stab}(e_0^0)) = 0.
\end{aligned}$$

We compute the SNF of the matrices of Φ_2 and Φ_1 and we, respectively, obtain that the invariant factors are (1) and (1). This implies that $H_2^{\mathcal{F}}(\mathbf{pm}, R_{\mathbb{C}}) = 0$, $H_1^{\mathcal{F}}(\mathbf{pm}, R_{\mathbb{C}}) = \mathbb{Z}^3$ and $H_0^{\mathcal{F}}(\mathbf{pm}, R_{\mathbb{C}}) = \mathbb{Z}^3$.

Now, the matrix Q obtained in the computation of the SNF for Φ_1 and the definition of Φ_2 show that a basis of $H_1^{\mathcal{F}}(\mathbf{pm}, R_{\mathbb{C}})$ is given by $([\beta_1^1], [\beta_1^2], [\beta_2^1])$, while a basis for $H_0^{\mathcal{F}}(\mathbf{pm}, R_{\mathbb{C}})$ is given by $([\alpha_0^2], [\alpha_1^1], [\alpha_1^2])$.

3.4. The group pg

Here we divide the (big) rectangle in Figure 2.1 in two equal rectangles by a vertical line; then the left one will be a representative e_2 of the unique class of 2-cells. Consider the vertices O, P, Q, R of this rectangle, counting clockwise from the left-hand lower vertex O. Observe that P is the image of O under vertical translation, Q under glide-reflection, and R under the composition of both. Hence, there will also be a unique class of 0-cells, and we denote by e_0 the representative given by O. In turn, there are two classes of 1-cells, identified by OP and PQ, which we, respectively, denote e_1^0 and e_1^1. Hence, if we call t the vertical translation (going upwards) and g the glide reflection, the boundaries of the representatives are defined in the following way:

$$\begin{aligned}\partial(e_2) &= e_1^0 + e_1^1 + ge_1^0 + t^{-1}e_1^1,\\ \partial(e_1^0) &= te_0^0 - e_0^0,\\ \partial(e_1^1) &= ge_0^0 - te_0^0.\end{aligned}$$

The group **pg** is torsion-free, and hence all its stabilizers are trivial. This time the chain complex is quite simple:

$$0 \to \mathbb{Z}\gamma \to \mathbb{Z}\beta_0 \oplus \mathbb{Z}\beta_1 \to \mathbb{Z}\alpha \to 0.$$

Now we compute the differentials of the complex:

$$\begin{aligned}\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) + (\gamma \uparrow \mathrm{stab}(e_1^0)) - (\gamma \uparrow \mathrm{stab}(e_1^1))\\ &= 2\beta_0,\\ \Phi_1(\beta_0) &= (\beta_0 \uparrow \mathrm{stab}(e_0^0)) - (\beta_0 \uparrow \mathrm{stab}(e_0^0)) = 0,\\ \Phi_1(\beta^1) &= (\beta_1 \uparrow \mathrm{stab}(e_0^0)) - (\beta^1 \uparrow \mathrm{stab}(e_0^0)) = 0.\end{aligned}$$

As Φ_1 is trivial, $H_0^{\mathcal{F}}(\mathbf{pg}, R_{\mathbb{C}}) = \mathbb{Z}$. On the other hand, the invariant factor of the SNF of the matrix of Φ_2 is (2), so $H_1^{\mathcal{F}}(\mathbf{pg}, R_{\mathbb{C}}) = \mathbb{Z} \oplus \mathbb{Z}/2$ and $H_2^{\mathcal{F}}(\mathbf{pg}, R_{\mathbb{C}}) = 0$.

By construction it is easy to see here that a basis for $H_0^{\mathcal{F}}(\mathbf{pg}, R_{\mathbb{C}})$ is given by $[\alpha]$, while $[\beta_0]$ and $[\beta_1]$ generate, respectively, the torsion part and the free part of $H_1^{\mathcal{F}}(\mathbf{pg}, R_{\mathbb{C}})$.

3.5. The group cm

Here our representative e_2 of the class of equivariant 2-cells will be given by the lower half of the rhombus. We denote its vertices by O, P and Q, starting from the one in the left and counting anti-clockwise. The horizontal translation t takes O to Q, while a glide-reflection sends O to P and P to Q. Then, again we consider a unique class of equivalence of 0-cells, whose representative e_0 is identified with O. There are also two classes of 1-cells, whose representatives e_1^0 and e_1^1 we identify with OP and QR (observe that PQ is the image of OP under the glide-reflection g). Now the boundaries are given by:

$$\begin{aligned}\partial(e_2) &= e_1^0 + ge_1^0 + e_1^1,\\ \partial(e_1^0) &= ge_0^0 - e_0^0,\\ \partial(e_1^1) &= e_0^0 - g^2e_0^0.\end{aligned}$$

Observe that both e_0^0 and e_1^1 lie in a reflection axis, so their stabilizers are isomorphic to C_2. On the other hand, the stabilizer of the other edge is trivial, and hence we have the following Bredon chain complex:

$$0 \to \mathbb{Z}\gamma \to \mathbb{Z}\beta_0 \oplus \mathbb{Z}\beta_1^1 \oplus \mathbb{Z}\beta_1^2 \to \mathbb{Z}\alpha^1 \oplus \mathbb{Z}\alpha^2 \to 0.$$

The differentials are quite simple in this case, taking again account of line 1 in Table 3:

$$\begin{aligned}\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) = 2\beta_0 + \beta_1^1 + \beta_1^2,\\ \Phi_1(\beta_0) &= (\beta_0 \uparrow \mathrm{stab}(e_0^0)) - (\beta_0 \uparrow \mathrm{stab}(e_0^0)) = 0,\\ \Phi_1(\beta^1) &= (\beta_1 \uparrow \mathrm{stab}(e_0^0)) - (\beta^1 \uparrow \mathrm{stab}(e_0^0)) = 0.\end{aligned}$$

Again the triviality of Φ_1 immediately implies $H_0^{\mathcal{F}}(\mathbf{cm}, R_{\mathbb{C}}) = \mathbb{Z}^2$. On the other hand, the unique invariant factor of the SNF of the matrix of Φ_2 is (1), so $H_1^{\mathcal{F}}(\mathbf{cm}, R_{\mathbb{C}}) = \mathbb{Z}^2$ and $H_2^{\mathcal{F}}(\mathbf{cm}, R_{\mathbb{C}}) = 0$.

It is clear that a basis of $H_0^{\mathcal{F}}(\mathbf{cm}, R_{\mathbb{C}})$ is given by $[\alpha^1]$ and $[\alpha^2]$, while the definition of Φ_2 implies that $[\beta_1^1]$ and $[\beta_1^2]$ form a basis for $H_1^{\mathcal{F}}(\mathbf{cm}, R_{\mathbb{C}})$.

3.6. The group pmm

Here the lower left small square can be taken a representative e_2 of the equivariant class of 2-cells under the action of **pmm**. We consider the four vertices O, P, Q and R of this small square, starting as always in the lower one of the left-hand side, and counting clockwise. Each of this edges will correspond, respectively, to a representatives of different classes of 0-cells, say e_0^0, e_0^1, e_0^2 and e_0^3. In turn, there will be also four representatives of classes of 1-cells, namely e_1^0, e_1^1, e_1^2 and e_1^3, which we, respectively, identify with the edges OP, PQ, QR and RP. The boundaries in this case are easy, because the group makes no identifications inside the small square:

$$\begin{aligned}\partial(e_2) &= e_1^0 + e_1^1 + e_1^2 + e_1^3,\\ \partial(e_1^0) &= e_0^1 - e_0^0,\\ \partial(e_1^1) &= e_0^2 - e_0^1,\\ \partial(e_1^2) &= e_0^3 - e_0^2,\\ \partial(e_1^3) &= e_0^0 - e_0^3.\end{aligned}$$

In this model all the edges lie on reflection axes, and there no other relevant isometries. Hence, all the stabilizers of the edges are isomorphic to C_2. In turn, every vertex lie in two different reflection axes, so the stabilizers of the vertices are isomorphic to D_2. The Bredon complex takes then the following shape:

$$0 \to \mathbb{Z}\gamma \to \bigoplus_{i=0}^{3}(\beta_i^1 \oplus \beta_i^2) \to \bigoplus_{i=0}^{3}(\alpha_i^1 \oplus \alpha_i^2 \oplus \alpha_i^3 \oplus \alpha_i^4) \to 0.$$

Taking account of lines 1, 6 and 7 in Table 3, we compute the differentials:

$$\begin{aligned}\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) + (\gamma \uparrow \mathrm{stab}(e_1^2)) + (\gamma \uparrow \mathrm{stab}(e_1^3))\\ &= \sum_{i=0}^{3}\sum_{j=1}^{2}\beta_i^j,\\ \Phi_1(\beta_0^1) &= (\beta_0^1 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^2 - \alpha_0^1 - \alpha_0^2,\end{aligned}$$

$$\begin{aligned}
\Phi_1(\beta_0^2) &= (\beta_0^2 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^3 + \alpha_1^4 - \alpha_0^3 - \alpha_0^4,\\
\Phi_1(\beta_1^1) &= (\beta_1^1 \uparrow \mathrm{stab}(e_0^2)) - (\beta_0 \uparrow \mathrm{stab}(e_0^1)) = \alpha_2^1 + \alpha_2^2 - \alpha_1^1 - \alpha_1^3,\\
\Phi_1(\beta_1^2) &= (\beta_1^2 \uparrow \mathrm{stab}(e_0^2)) - (\beta_0 \uparrow \mathrm{stab}(e_0^1)) = \alpha_2^3 + \alpha_2^4 - \alpha_1^2 - \alpha_1^4,\\
\Phi_1(\beta_2^1) &= (\beta_2^1 \uparrow \mathrm{stab}(e_0^3)) - (\beta_0 \uparrow \mathrm{stab}(e_0^2)) = \alpha_3^1 + \alpha_3^2 - \alpha_2^1 - \alpha_2^3,\\
\Phi_1(\beta_2^2) &= (\beta_2^2 \uparrow \mathrm{stab}(e_0^3)) - (\beta_0 \uparrow \mathrm{stab}(e_0^2)) = \alpha_3^3 + \alpha_3^4 - \alpha_2^2 - \alpha_2^4,\\
\Phi_1(\beta_3^1) &= (\beta_3^1 \uparrow \mathrm{stab}(e_0^0)) - (\beta_0 \uparrow \mathrm{stab}(e_0^3)) = \alpha_0^1 + \alpha_0^3 - \alpha_3^1 - \alpha_3^3,\\
\Phi_1(\beta_3^2) &= (\beta_3^2 \uparrow \mathrm{stab}(e_0^0)) - (\beta_0 \uparrow \mathrm{stab}(e_0^3)) = \alpha_0^2 + \alpha_0^4 - \alpha_3^2 - \alpha_3^4.
\end{aligned}$$

Observe that when we described the differentials, we have taken into account that two coincident edges define different subgroups (isomorphic to C_2) in the stabilizer of the common vertex.

We now compute the SNF of the matrices of Φ_2 and Φ_1 and we obtain that the invariant factors are (1) and $(1,1,1,1,1,1,1)$, respectively. This implies that $H_2^{\mathcal{F}}(\mathbf{pmm}, R_{\mathbb{C}}) = 0$, $H_1^{\mathcal{F}}(\mathbf{pmm}, R_{\mathbb{C}}) = 0$ and $H_0^{\mathcal{F}}(\mathbf{pmm}, R_{\mathbb{C}}) = \mathbb{Z}^9$.

Finally, the last columns of the matrix Q obtained in the computation of the SNF for Φ_1 show that a basis for $H_0^{\mathcal{F}}(\mathbf{pm}, R_{\mathbb{C}})$ is given by $([\alpha_0^3], [\alpha_1^3], [\alpha_1^4], [\alpha_2^3], [\alpha_2^4], [\alpha_3^1], [\alpha_3^2], [\alpha_3^3], [\alpha_3^4])$.

3.7. The group pmg

A representative e_2 for the class of equivariant 2-cells in **pmg** will given by any of the two rectangles of the picture in Figure 2.1 whose horizontal edges are reflection axes, so we choose for example the left one. We consider six vertices on it: the four vertices given by the corners of the rectangle, an the marked rotation centers in the middle points of the vertical sides. Starting from the lower vertex of the left-hand side of the rectangle and going clockwise, we call these vertices O, P, Q, R, S and T. Then, representatives of the four classes e_0^0, e_0^1, e_0^2 and e_0^3 of equivariant 0-cells will be, respectively, given by the vertices O, P, R and S. Remark that Q is the image of O under the rotation r_1 of center P, and T is the image of R under the rotation r_2 of center S. There will also four classes of equivariant 1-cells, whose representatives e_1^0, e_1^1, e_1^2 and e_1^3 are identified with OP, QR, RS and TP. Observe that $r_1(OP) = QP$ and $r_2(RS) = TS$. We are now ready to compute the boundaries:

$$\begin{aligned}
\partial(e_2) &= e_1^0 + r_1 e_1^0 + e_1^1 + e_1^2 + r_2 e_2 + e_1^3,\\
\partial(e_1^0) &= e_0^1 - e_0^0,\\
\partial(e_1^1) &= e_0^2 - r_1 e_0^0,\\
\partial(e_1^2) &= e_0^3 - e_0^2,\\
\partial(e_1^3) &= e_0^0 - r_2 e_0^2.
\end{aligned}$$

Now, the horizontal edges of the rectangle are in reflection axes, and same happens to the vertices e_0^0 and e_0^2, so the stabilizers of the corresponding cells are isomorphic to C_2. As e_0^1 and e_0^3 are centers of 2-rotation the stabilizers are also isomorphic to C_2. Finally, the group acts freely over the classes of e_1^0 and e_1^0, so we can form the Bredon chain complex:

$$\begin{aligned} 0 \to \mathbb{Z}\gamma \to & \mathbb{Z}\beta_0 \oplus \mathbb{Z}\beta_1^1 \oplus \mathbb{Z}\beta_1^2 \oplus \mathbb{Z}\beta_2^1 \oplus \mathbb{Z}\beta_2^2 \oplus \mathbb{Z}\beta_3 \\ & \to \bigoplus_{i=1}^{2} \mathbb{Z}\alpha_0^i \oplus \bigoplus_{i=1}^{2} \mathbb{Z}\alpha_1^i \bigoplus_{i=1}^{2} \mathbb{Z}\alpha_2^i \bigoplus_{i=1}^{2} \mathbb{Z}\alpha_3^i \to 0. \end{aligned}$$

Let us compute the differentials of the complex, taking account of line 1 in Table 3:

$$\begin{aligned} \Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) - (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) + (\gamma \uparrow \mathrm{stab}(e_1^2)) \\ &\quad - (\gamma \uparrow \mathrm{stab}(e_1^2)) + (\gamma \uparrow \mathrm{stab}(e_1^3)) = \beta_1^1 + \beta_1^2 + \beta_2^1 + \beta_2^2, \\ \Phi_1(\beta_0) &= (\beta_0 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^2 - \alpha_0^1 - \alpha_0^2, \\ \Phi_1(\beta_1^1) &= (\beta_0^2 \uparrow \mathrm{stab}(e_0^2)) - (\beta_1^1 \uparrow \mathrm{stab}(e_0^0)) = \alpha_2^1 + \alpha_0^1, \\ \Phi_1(\beta_1^2) &= (\beta_1^2 \uparrow \mathrm{stab}(e_0^2)) - (\beta_1^2 \uparrow \mathrm{stab}(e_0^0)) = \alpha_2^2 + \alpha_0^2, \\ \Phi_1(\beta_2^1) &= (\beta_2^1 \uparrow \mathrm{stab}(e_0^3)) - (\beta_2^1 \uparrow \mathrm{stab}(e_0^2)) = \alpha_3^1 + \alpha_3^2 - \alpha_2^1 - \alpha_2^2, \\ \Phi_1(\beta_2^2) &= (\beta_2^2 \uparrow \mathrm{stab}(e_0^3)) - (\beta_2^2 \uparrow \mathrm{stab}(e_0^2)) = \alpha_0^1 + \alpha_2^1, \\ \Phi_1(\beta_3) &= (\beta_3 \uparrow \mathrm{stab}(e_0^0)) - (\beta_3 \uparrow \mathrm{stab}(e_0^2)) = \alpha_0^2 + \alpha_2^2. \end{aligned}$$

From the SNF of the matrices of Φ_2 and Φ_1 it is obtained that the invariant factors are (1) and $(1,1,1,1)$, respectively. This implies that $H_2^{\mathcal{F}}(\mathbf{pmg}, R_{\mathbb{C}}) = 0$, $H_1^{\mathcal{F}}(\mathbf{pmg}, R_{\mathbb{C}}) = \mathbb{Z}$ and $H_0^{\mathcal{F}}(\mathbf{pmg}, R_{\mathbb{C}}) = \mathbb{Z}^4$.

In turn, the matrices P and Q obtained in the computation of the Smith normal forms show that a basis for $H_0^{\mathcal{F}}(\mathbf{pmg}, R_{\mathbb{C}})$ is given $([\alpha_1^2], [\alpha_2^1], [\alpha_2^2], [\alpha_3^2])$ and a basis for $H_1^{\mathcal{F}}(\mathbf{pmg}, R_{\mathbb{C}})$ is given by $[\beta_1^1 + \beta_2^1]$.

3.8. The group pgg

For this group, a representative e_2 for the class of equivariant 2-cells will be given for example by the triangle determined by the middle points of the vertical sides and the center of the lower horizontal side of the (big) rectangle of the picture in Figure 2.1. Consider these three vertices and the center of the rectangle, and call them O, P, Q and R, starting from the middle point of the left vertical side and counting clockwise. Then representatives e_0^0 and e_0^1 for the classes of 0-cells will be given by O and P, being Q the image of O under a rotation r of center P and R the image of O under a glide-reflection g. Representatives e_1^0 and e_1^1 for the classes of 1-cells are given by RO and OP, respectively, being $QP = r(OP)$ and $RQ = g(OR)$. Now the boundaries are defined in the following way:

$$\begin{aligned} \partial(e_2) &= e_1^1 + re_1^1 + ge_1^0 + e_1^0, \\ \partial(e_1^0) &= ge_0^0 - e_0^0, \\ \partial(e_1^1) &= e_0^1 - e_0^0. \end{aligned}$$

The two representatives of 0-cells are centers of 2-rotation, and then their stabilizers are isomorphic to C_2. On the other hand, the group acts freely over the classes of e_1^0 and e_1^1, so we obtain the following Bredon complex:

$$0 \to \mathbb{Z}\gamma \to \mathbb{Z}\beta_0 \oplus \mathbb{Z}\beta_1 \to \mathbb{Z}\alpha_0^1 \oplus \mathbb{Z}\alpha_0^2 \oplus \mathbb{Z}\alpha_1^1 \oplus \mathbb{Z}\alpha_1^2.$$

The differentials of the chain complex are now given in the following way, taking account of line 1 in Table 3:

$$\begin{aligned}\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^1)) - (\gamma \uparrow \mathrm{stab}(e_1^1)) + (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^0)) \\ &= 2\beta_1, \\ \Phi_1(\beta_0) &= (\beta_0 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^2 - \alpha_0^1 - \alpha_0^2, \\ \Phi_1(\beta_1) &= (\beta_1 \uparrow \mathrm{stab}(e_0^0)) - (\beta_1 \uparrow \mathrm{stab}(e_0^0)) = 0.\end{aligned}$$

Now the Smith normal form of the matrices of Φ_2 and Φ_1 gives, respectively, the invariant factors (2) and (1). Hence, we have $H_2^{\mathcal{F}}(\mathbf{pgg}, R_{\mathbb{C}}) = 0$, $H_1^{\mathcal{F}}(\mathbf{pgg}, R_{\mathbb{C}}) = \mathbb{Z}/2$ and $H_0^{\mathcal{F}}(\mathbf{pgg}, R_{\mathbb{C}}) = \mathbb{Z}^3$.

Also, the matrices P and Q obtained in the computation of the Smith normal forms show that a basis for $H_0^{\mathcal{F}}(\mathbf{pgg}, R_{\mathbb{C}})$ is given by $([\alpha_1^2], [\alpha_2^1], [\alpha_2^2])$ and a generator for $H_1^{\mathcal{F}}(\mathbf{pgg}, R_{\mathbb{C}})$ is given by $[\beta_0]$.

3.9. The group cmm

Consider the four triangles inside the rhombus in the figure. A representative e_2 for the equivalence class of 2-cells will be given by the lower left-hand side triangle. Starting from the left and going clockwise, denote by O, P and Q the vertices of this triangle, and by R the middle point of the diagonal side. Then there will be three representatives e_0^0, e_0^1 and e_0^2 of the classes of 0-cells, corresponding, respectively, to the vertices O and P and R; observe that Q is the image of P under the 2-rotation r whose center is R. In turn, there are three classes of 1-cells, with representatives e_1^0, e_1^1 and e_1^2 identified, respectively, with OP, PQ and QR. For the remaining edge we have $r(QR) = RP$. Let us describe now the boundaries:

$$\begin{aligned}\partial(e_2) &= e_1^0 + e_1^1 + e_1^2 + re_1^2, \\ \partial(e_1^0) &= e_0^1 - e_0^0, \\ \partial(e_1^1) &= re_0^0 - e_0^0, \\ \partial(e_1^2) &= e_0^2 - re_0^0.\end{aligned}$$

Now, in both e_0^0 and e_0^1 two reflection axes cross, an hence the stabilizers of these two cells are isomorphic to D_2. As e_0^2 is a center of 2-rotation, its stabilizer is C_2. Concerning the 1-cells, e_1^0 and e_1^1 are in reflection axes, so their stabilizer is C_2, while the group act freely over the class of e_1^2. Hence we have the chain complex:

$$0 \to \mathbb{Z}\beta_0^1 \oplus \mathbb{Z}\beta_0^2 \oplus \mathbb{Z}\beta_1^1 \oplus \mathbb{Z}\beta_1^2 \oplus \mathbb{Z}\beta_2 \to \bigoplus_{i=1}^{4} \mathbb{Z}\alpha_0^i \bigoplus_{i=1}^{4} \mathbb{Z}\alpha_1^i \bigoplus_{i=1}^{2} \mathbb{Z}\alpha_2^i \to 0.$$

Taking account of lines 1, 2, 6 and 7 in Table 3, the differentials in this case are given by:

$$\begin{aligned}
\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) + (\gamma \uparrow \mathrm{stab}(e_1^2)) - (\gamma \uparrow \mathrm{stab}(e_1^2)) \\
&= \beta_0^1 + \beta_0^2 + \beta_1^1 + \beta_1^2, \\
\Phi_1(\beta_0^1) &= (\beta_0^1 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0^1 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^2 - \alpha_0^1 - \alpha_0^2, \\
\Phi_1(\beta_0^2) &= (\beta_0^2 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0^2 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^3 + \alpha_1^4 - \alpha_0^3 - \alpha_0^4, \\
\Phi_1(\beta_1^1) &= (\beta_1^1 \uparrow \mathrm{stab}(e_0^0)) - (\beta_1^1 \uparrow \mathrm{stab}(e_0^1)) = \alpha_0^1 + \alpha_0^3 - \alpha_1^1 - \alpha_1^3, \\
\Phi_1(\beta_1^2) &= (\beta_1^2 \uparrow \mathrm{stab}(e_0^0)) - (\beta_1^2 \uparrow \mathrm{stab}(e_0^1)) = \alpha_0^2 + \alpha_0^4 - \alpha_1^2 - \alpha_1^4, \\
\Phi_1(\beta_2) &= (\beta_2 \uparrow \mathrm{stab}(e_0^2)) - (\beta_2 \uparrow \mathrm{stab}(e_0^0)) = \alpha_2^1 + \alpha_2^2 - \alpha_0^1 - \alpha_0^2 - \alpha_0^3 - \alpha_0^4.
\end{aligned}$$

The SMF of the matrices of Φ_2 and Φ_1 give the invariant factors (2) and $(1, 1, 1, 1)$, respectively. This implies $H_2^{\mathcal{F}}(\mathbf{cmm}, R_{\mathbb{C}}) = 0$, $H_1^{\mathcal{F}}(\mathbf{cmm}, R_{\mathbb{C}}) = 0$ and $H_0^{\mathcal{F}}(\mathbf{cmm}, R_{\mathbb{C}}) = \mathbb{Z}^6$.

Using again the matrix Q associated to the SMF, we obtain that a basis of $H_0^{\mathcal{F}}(\mathbf{cmm}, R_{\mathbb{C}})$ is given by $([\alpha_0^1 + \alpha_0^2], [\alpha_0^3], [\alpha_0^4], [\alpha_1^1], [\alpha_1^3], [\alpha_2^2])$.

3.10. The group p4

We divide the square in the picture in four equal squares, using the vertical segment defined by the middle points of the horizontal sides and the horizontal segment defined by the middle points of the vertical sides. The representative e_2 for the class of 2-equivariant cells will be the lower little square in the left-hand side. Denote by O, P, Q and R the vertices of this little square, starting on the upper-left and going clockwise. There are three classes of 0-cells, whose representatives e_0^0, e_0^1 and e_0^2 correspond, respectively, to O, P and R; observe that if t is the 4-rotation (counterclockwise) whose rotation center is P, $t(O) = Q$. Moreover, there are two classes of 1-cells, with representatives e_1^0 and e_1^1 corresponding to the sides RO and OP; the two remaining sides can be obtained as $QP = t(OP)$ and $RQ = t(RO)$. We have the boundaries:

$$\begin{aligned}
\partial(e_2) &= e_1^0 + e_1^1 + te_1^1 + te_1^0, \\
\partial(e_1^0) &= e_0^0 - e_0^2, \\
\partial(e_1^1) &= e_0^1 - e_0^0.
\end{aligned}$$

Concerning the stabilizers, e_0^1 and e_0^2 are centers of 4-rotation, and hence their stabilizers are isomorphic to C_4; while e_0^2 is a center of rotation of 180 degrees, so its stabilizer is C_2. The group **p4** acts freely on each class of 1-cells, so the Bredon chain complex has the following shape:

$$0 \to \mathbb{Z}\gamma \to \mathbb{Z}\beta_0 \oplus \mathbb{Z}\beta_1 \to \bigoplus_{i=1}^{2} \mathbb{Z}\alpha_0^i \bigoplus_{i=1}^{4} \mathbb{Z}\alpha_1^i \bigoplus_{i=1}^{4} \mathbb{Z}\alpha_2^i \to 0.$$

Now we can describe the differentials of the complex, taking account of line 1 in Table 3:

$$\begin{aligned}
\Phi_2(\gamma) &= (\gamma \uparrow \operatorname{stab}(e_1^0)) + (\gamma \uparrow \operatorname{stab}(e_1^1)) - (\gamma \uparrow \operatorname{stab}(e_1^1)) - (\gamma \uparrow \operatorname{stab}(e_1^0)) = 0,\\
\Phi_1(\beta_0) &= (\beta_0 \uparrow \operatorname{stab}(e_0^0)) - (\beta_0 \uparrow \operatorname{stab}(e_0^2)) = \alpha_0^1 + \alpha_0^2 - \alpha_2^1 - \alpha_2^2 - \alpha_2^3 - \alpha_2^4,\\
\Phi_1(\beta_1) &= (\beta_1 \uparrow \operatorname{stab}(e_0^1)) - (\beta_1 \uparrow \operatorname{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^2 + \alpha_1^3 + \alpha_1^4 - \alpha_0^1 - \alpha_0^2.
\end{aligned}$$

As $\Phi_2 = 0$, it is deduced immediately that $H_2^{\mathcal{F}}(\mathbf{p4}, R_{\mathbb{C}}) = \mathbb{Z}$. In turn, the SMF of the matrix of Φ_1 gives $(1,1)$ as invariant factors, so $H_1^{\mathcal{F}}(\mathbf{p4}, R_{\mathbb{C}}) = 0$ and $H_0^{\mathcal{F}}(\mathbf{cmm}, R_{\mathbb{C}}) = \mathbb{Z}^8$.

It is clear that $[\gamma]$ is a basis of $H_2^{\mathcal{F}}(\mathbf{p4}, R_{\mathbb{C}})$, while the shape of the matrix Q associated to the SNF of the matrix of Φ_1 implies that a basis for $H_0^{\mathcal{F}}(\mathbf{p4}, R_{\mathbb{C}})$ is given by $([\alpha_0^2], [\alpha_1^2], [\alpha_1^3], [\alpha_1^4], [\alpha_2^1], [\alpha_2^2], [\alpha_2^3], [\alpha_2^4])$.

3.11. The group p4m

All the eight triangles in the picture in Figure 2.1 whose edges are reflection will be elements of the class of equivariant 2-cells. We choose as a representative e_2 the only one whose lower side is the left half of the lower horizontal side of the square. Starting from the left vertex of this half and counting clockwise, we denote by O, P and Q the vertices of the triangle. Then the representatives e_0^0, e_0^1 and e_0^2 of the classes of 0-cells will be, respectively, identified with this three points. There are also three classes for 1-cells, so we make the segments OP, PQ and QO correspond to the representatives e_1^0, e_1^1 and e_1^2. We compute the boundaries for these cells:

$$\begin{aligned}
\partial(e_2) &= e_1^0 + e_1^1 + e_1^2,\\
\partial(e_1^0) &= e_0^1 - e_0^0,\\
\partial(e_1^1) &= e_0^2 - e_0^1,\\
\partial(e_1^2) &= e_0^0 - e_0^2.
\end{aligned}$$

The stabilizers of e_0^0 and e_0^1 are generated by a 4-rotation and an independent reflection, so they are both isomorphic to D_4. In turn, the stabilizer of e_0^2 is generated by two reflections, and hence it is isomorphic to D_2. As all the 1-cells lie in reflection axes, the corresponding stabilizers are isomorphic to C_2. Hence, we obtain the following Bredon complex:

$$0 \to \bigoplus_{i=1}^{2} \beta_0^i \bigoplus_{i=1}^{2} \beta_1^i \bigoplus_{i=1}^{2} \beta_2^i \to \bigoplus_{i=1}^{5} \alpha_0^i \bigoplus_{i=1}^{5} \alpha_1^i \bigoplus_{i=1}^{4} \alpha_2^i.$$

Taking account of lines 6, 7, 10, 11, 12 and 13 in Table 3, the differentials of the chain complex are given by:

$$\begin{aligned}
\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) + (\gamma \uparrow \mathrm{stab}(e_1^2)) \\
&= \beta_0^1 + \beta_0^2 + \beta_1^1 + \beta_1^2 + \beta_2^1 + \beta_2^2, \\
\Phi_1(\beta_0^1) &= (\beta_0^1 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0^1 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^3 + \alpha_1^5 - \alpha_0^1 - \alpha_0^3 - \alpha_0^5, \\
\Phi_1(\beta_0^2) &= (\beta_0^2 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0^2 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^2 + \alpha_1^4 + \alpha_1^5 - \alpha_0^2 - \alpha_0^4 - \alpha_0^5, \\
\Phi_1(\beta_1^1) &= (\beta_1^1 \uparrow \mathrm{stab}(e_0^2)) - (\beta_1^1 \uparrow \mathrm{stab}(e_0^1)) = \alpha_2^1 + \alpha_2^2 - \alpha_1^1 - \alpha_1^4 - \alpha_1^5, \\
\Phi_1(\beta_1^2) &= (\beta_1^2 \uparrow \mathrm{stab}(e_0^2)) - (\beta_1^2 \uparrow \mathrm{stab}(e_0^1)) = \alpha_2^3 + \alpha_2^4 - \alpha_1^2 - \alpha_1^3 - \alpha_1^5, \\
\Phi_1(\beta_2^1) &= (\beta_2^1 \uparrow \mathrm{stab}(e_0^0)) - (\beta_2^1 \uparrow \mathrm{stab}(e_0^2)) = \alpha_0^1 + \alpha_0^4 + \alpha_0^5 - \alpha_2^1 - \alpha_2^3, \\
\Phi_1(\beta_2^2) &= (\beta_2^2 \uparrow \mathrm{stab}(e_0^0)) - (\beta_2^2 \uparrow \mathrm{stab}(e_0^2)) = \alpha_0^2 + \alpha_0^3 + \alpha_0^5 - \alpha_2^2 - \alpha_2^4.
\end{aligned}$$

Here, the SMF of the matrices of Φ_2 and Φ_1 gives the invariant factors (1) and (1, 1, 1, 1, 1), respectively. Then, $H_2^{\mathcal{F}}(\mathbf{p4m}, R_{\mathbb{C}}) = 0$, $H_1^{\mathcal{F}}(\mathbf{p4m}, R_{\mathbb{C}}) = 0$ and $H_0^{\mathcal{F}}(\mathbf{p4m}, R_{\mathbb{C}}) = \mathbb{Z}^9$.

Again a basis of $H_0^{\mathcal{F}}(\mathbf{p4m}, R_{\mathbb{C}})$ can be extracted of the last column of the auxiliary matrix Q. Such a basis is $([\alpha_0^4], [\alpha_0^5], [\alpha_1^3], [\alpha_1^4], [\alpha_1^5], [\alpha_2^1], [\alpha_2^2], [\alpha_2^3], [\alpha_2^4])$.

3.12. The group p4g

Here our representative for the class of equivariant 2-cells will be the triangle whose vertices are the middle point of the left side of the (big) square, the center of that square and the middle point of the lower side of that square. We will denote these vertices by O, P and Q, respectively. In particular, we identify O and P with our representatives e_0^0 and e_0^1 for the two classes of 0-cells, while Q is the image of O under a counterclockwise 4-rotation t whose center is P. There are also two classes of equivariant 1-cells, whose representatives e_1^0 and e_1^1 we, respectively, identify with OP and QO. For the other side of the triangle, we have $QP = t(OP)$. The boundaries are then defined in this way:

$$\begin{aligned}
\partial(e_2) &= e_1^0 - te_1^0 + e_1^1, \\
\partial(e_1^0) &= e_0^1 - e_0^0, \\
\partial(e_1^1) &= e_0^0 - te_0^0.
\end{aligned}$$

As e_0^1 is a center of 4-rotation, its stabilizer is C_4. In turn, the stabilizer of e_0^0 is generated by a 2-rotation and a reflection, so it is isomorphic to D_2. On the other hand, the unique 1-cell with non-trivial isotropy is e_1^1, which lies in a reflection axis and then has C_2 as stabilizer. Let us write now the Bredon chain complex:

$$0 \to \mathbb{Z}\gamma \to \mathbb{Z}\beta_0 \oplus \mathbb{Z}\beta_1^1 \oplus \beta_1^2 \to \bigoplus_{i=1}^{4} \alpha_0^i \bigoplus_{i=1}^{4} \alpha_1^i \to 0.$$

Now we can compute the differentials, taking account of lines 1, 6 and 7 in Table 3 :

$$\begin{aligned}
\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) - (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) = \beta_1^1 + \beta_1^2,\\
\Phi_1(\beta_0) &= (\beta_0 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0 \uparrow \mathrm{stab}(e_0^0))\\
&= \alpha_1^1 + \alpha_1^2 + \alpha_1^3 + \alpha_1^4 - \alpha_0^1 - \alpha_0^2 - \alpha_0^3 - \alpha_0^4,\\
\Phi_1(\beta_1^1) &= (\beta_1^1 \uparrow \mathrm{stab}(e_0^0)) - (\beta_1^1 \uparrow \mathrm{stab}(e_0^0)) = \alpha_0^1 + \alpha_0^2 - \alpha_0^1 - \alpha_0^3 = \alpha_0^2 - \alpha_0^3,\\
\Phi_1(\beta_1^2) &= (\beta_1^2 \uparrow \mathrm{stab}(e_0^0)) - (\beta_1^2 \uparrow \mathrm{stab}(e_0^0)) = \alpha_0^3 + \alpha_0^4 - \alpha_0^2 - \alpha_0^4 = \alpha_0^3 - \alpha_0^2.
\end{aligned}$$

Observe that the two induced characters denoted by $(\beta_1^1 \uparrow \mathrm{stab}(e_0^0))$ in the expression of $\Phi_1(\beta_1^1)$ are not the same, because they are induced from different inclusions $\mathbb{Z}/2 \hookrightarrow D_4$. Same happens with β_1^2.

After computing the SMF of the matrices of Φ_2 and Φ_1, we obtain the invariant factors (1) and $(1,1)$, respectively. Hence, we have $H_2^{\mathcal{F}}(\mathbf{p4g}, R_{\mathbb{C}}) = 0$, $H_1^{\mathcal{F}}(\mathbf{p4g}, R_{\mathbb{C}}) = 0$ and $H_0^{\mathcal{F}}(\mathbf{p4g}, R_{\mathbb{C}}) = \mathbb{Z}^6$.

From the matrix Q obtained in the computation of the SMF of the matrix of Φ_1 we conclude that a basis for $H_0^{\mathcal{F}}(\mathbf{p4g}, R_{\mathbb{C}})$ is given by $([\alpha_0^1], [\alpha_0^2], [\alpha_0^4], [\alpha_1^1], [\alpha_1^2], [\alpha_1^4])$.

3.13. The group p3

For the group **p3** a representative e_2 for the class of equivariant 2-cells is the rhombus of the picture. We order the vertices clockwise starting from the upper left, and denote them as usual by P, Q, R and S. Then representatives e_0^0, e_0^1 and e_0^2 of the three classes of 0-cells are given by P, Q and R, and a 3-rotation t around O takes P to S. In turn, there are two classes e_1^0 and e_1^1 of 1-cells, which can be, respectively, identified with OP and PQ. Observe that $t(OP) = t(OS)$ and $t(PQ) = t(SQ)$. We have the boundaries:

$$\begin{aligned}
\partial(e_2) &= e_1^0 + e_1^1 - te_1^1 - te_1^0,\\
\partial(e_1^0) &= e_0^1 - e_0^0,\\
\partial(e_1^1) &= e_0^2 - e_0^1.
\end{aligned}$$

The three vertices are 3-rotation centers, and their stabilizers are isomorphic to C_3. On the other hand, the group acts freely in each class of equivariant 1-cells, and hence the Bredon complex has the form:

$$0 \to \mathbb{Z}\gamma \to \mathbb{Z}\beta_0 \oplus \mathbb{Z}\beta_1 \to \bigoplus_{i=1}^{3} \mathbb{Z}\alpha_0^i \bigoplus_{i=1}^{3} \mathbb{Z}\alpha_1^i \bigoplus_{i=1}^{3} \mathbb{Z}\alpha_2^i \to 0.$$

Computing the differentials of the complex, and taking into account line 1 in Table 3 we obtain:

$$\begin{aligned}
\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) - (\gamma \uparrow \mathrm{stab}(e_1^1)) - (\gamma \uparrow \mathrm{stab}(e_1^0)) = 0,\\
\Phi_1(\beta_0) &= (\beta_0 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^2 - \alpha_1^1 - \alpha_0^3 = \alpha_0^2 - \alpha_0^3,
\end{aligned}$$

$$\Phi_1(\beta_1) = (\beta_1 \uparrow \mathrm{stab}(e_0^2)) - (\beta_1 \uparrow \mathrm{stab}(e_0^1)) = \alpha_2^1 + \alpha_2^2 + \alpha_2^3 - \alpha_1^1 - \alpha_1^2 - \alpha_1^3.$$

As Φ_2 is trivial, we have $H_2^{\mathcal{F}}(\mathbf{p3}, R_{\mathbb{C}}) = \mathbb{Z}$. On the other hand, the SNF of the matrix of Φ_1 produces the invariant factors $(1, 1)$, whence $H_1^{\mathcal{F}}(\mathbf{p3}, R_{\mathbb{C}}) = 0$ and $H_0^{\mathcal{F}}(\mathbf{p3}, R_{\mathbb{C}}) = \mathbb{Z}^7$. From the auxiliary matrix of the SNF we obtain a basis $([\alpha_0^2], [\alpha_0^3], [\alpha_1^1], [\alpha_1^2], [\alpha_1^3], [\alpha_2^1], [\alpha_2^2])$ for $H_0^{\mathcal{F}}(\mathbf{p3}, R_{\mathbb{C}})$.

3.14. The group p3m1

A representative e_2 for the equivalence class of equivariant 2-cells for this action is the equilateral triangle whose vertices are the upper left vertex in the picture of Figure 2.1 and the two closest rotation centers. We name this three vertices as O, P and Q, starting from the upper left and counting clockwise. The representatives e_0^0, e_0^1 and e_0^2 of the three classes of 1-cells are identified with these three vertices in that order. There are also three classes of 1-cells, e_1^0, e_1^1 and e_1^2, which we identify, respectively, with OP, PQ and QO. Now we can describe the boundaries between the cells:

$$\begin{aligned}
\partial(e_2) &= e_1^0 + e_1^1 + e_1^2,\\
\partial(e_1^0) &= e_0^1 - e_0^0,\\
\partial(e_1^1) &= e_0^2 - e_0^1,\\
\partial(e_1^2) &= e_0^0 - e_0^2.
\end{aligned}$$

As in the previous group, the vertices are centers of 3-rotations, and moreover the sides of the triangle lie on rotation axes. Then, the stabilizers of the 0-cells are isomorphic to D_3, and the stabilizers of the 1-cells are isomorphic to C_2. So we have the complex:

$$0 \to \mathbb{Z}\gamma \to \bigoplus_{i=1}^{2} \beta_0^i \bigoplus_{i=1}^{2} \beta_1^i \bigoplus_{i=1}^{2} \beta_2^i \to \bigoplus_{i=1}^{4} \alpha_0^i \bigoplus_{i=1}^{4} \alpha_1^i \bigoplus_{i=1}^{4} \alpha_2^i \to 0.$$

Now we can get the differentials of the chain complex, recall lines 1, 8 and 9 of Table 3:

$$\begin{aligned}
\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) + (\gamma \uparrow \mathrm{stab}(e_1^2))\\
&= \beta_0^1 + \beta_0^2 + \beta_1^1 + \beta_1^2 + \beta_2^1 + \beta_2^2.\\
\Phi_1(\beta_0^1) &= (\beta_0^1 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0^1 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^3 - \alpha_0^1 - \alpha_0^3,\\
\Phi_1(\beta_0^2) &= (\beta_0^2 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0^2 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^2 + \alpha_1^3 - \alpha_0^2 - \alpha_0^3,\\
\Phi_1(\beta_1^1) &= (\beta_1^1 \uparrow \mathrm{stab}(e_0^2)) - (\beta_1^1 \uparrow \mathrm{stab}(e_0^1)) = \alpha_2^1 + \alpha_2^3 - \alpha_1^1 - \alpha_1^3,\\
\Phi_1(\beta_1^2) &= (\beta_1^2 \uparrow \mathrm{stab}(e_0^2)) - (\beta_1^2 \uparrow \mathrm{stab}(e_0^1)) = \alpha_2^2 + \alpha_2^3 - \alpha_1^2 - \alpha_1^3,\\
\Phi_1(\beta_2^1) &= (\beta_2^1 \uparrow \mathrm{stab}(e_0^0)) - (\beta_2^1 \uparrow \mathrm{stab}(e_0^2)) = \alpha_0^1 + \alpha_0^3 - \alpha_2^1 - \alpha_2^3,\\
\Phi_1(\beta_2^2) &= (\beta_2^2 \uparrow \mathrm{stab}(e_0^0)) - (\beta_2^2 \uparrow \mathrm{stab}(e_0^2)) = \alpha_0^2 + \alpha_0^3 - \alpha_2^2 - \alpha_2^3.
\end{aligned}$$

The respective SNF of the matrices of Φ_2 and Φ_1, respectively, produce invariant factors (1) and $(1,1,1,1)$. Then the Bredon homology groups are $H_2^{\mathcal{F}}(\mathbf{p3m1}, R_{\mathbb{C}}) = 0$, $H_1^{\mathcal{F}}(\mathbf{p3m1}, R_{\mathbb{C}}) = \mathbb{Z}$ and $H_0^{\mathcal{F}}(\mathbf{p3m1}, R_{\mathbb{C}}) = \mathbb{Z}^5$.

The computation of the auxiliary matrices of the SNF in particular show that a basis for $H_1^{\mathcal{F}}(\mathbf{p3m1}, R_{\mathbb{C}})$ is given by $[\beta_0^1 + \beta_1^1 + \beta_2^1]$ and a basis for $H_0^{\mathcal{F}}(\mathbf{p3m1}, R_{\mathbb{C}})$ is $([\alpha_0^3], [\alpha_1^1], [\alpha_1^2], [\alpha_1^3], [\alpha_2^3])$.

3.15. The group p31m

In the picture we see a rhombus divided in two equilateral triangles. As the representative e_2 of the class of 2-cells it can be taken the triangle T whose vertices are the center and the two lower vertices of the left triangle of the rhombus. We denote by O, P and Q the vertices of T, starting in the one in the left-hand side and counting clockwise. Now, the two classes of 0-cells for the action will have representatives e_0^0 and e_0^1 identified with P and Q, and we take account of the fact that $Q = t(O)$, being t the 3-rotation with center P. In turn, there are two classes of 1-cells, represented by e_1^0 and e_1^1, which we identify with OP and QO. Observe that $t(OP) = QP$. Now we can describe the boundaries here:

$$\begin{aligned}\partial(e_2) &= e_1^0 + te_1^0 + e_1^1,\\ \partial(e_1^0) &= e_0^1 - e_0^0,\\ \partial(e_1^1) &= e_0^0 - te_0^0.\end{aligned}$$

The stabilizer of e_0^0 is generated by a 3-rotation and reflection, and hence is isomorphic to D_3. In turn, the stabilizer of e_0^2 is generated by a 3-rotation and is then isomorphic to C_3. On the other hand, the group acts freely over the class of e_1^0, while the stabilizer of e_1^1 is C_2, as this representative lies in a reflection axis. Then the Bredon chain complex is:

$$0 \to \mathbb{Z}\gamma \to \mathbb{Z}\beta_0 \oplus \mathbb{Z}\beta_1^1 \oplus \mathbb{Z}\beta_1^2 \to \bigoplus_{i=1}^{3} \mathbb{Z}\alpha_0^i \bigoplus_{i=1}^{3} \mathbb{Z}\alpha_1^i \to 0.$$

Now let us compute the differentials of the complex, taking account of lines 1, 3, 8 and 9 of Table 3:

$$\begin{aligned}\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) - (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) = \beta_1^1 + \beta_1^2,\\ \Phi_1(\beta_0) &= (\beta_0 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^2 + \alpha_1^3 - \alpha_0^1 - \alpha_0^2 - \alpha_0^3,\\ \Phi_1(\beta_1^1) &= (\beta_1^1 \uparrow \mathrm{stab}(e_0^0)) - (\beta_1^1 \uparrow \mathrm{stab}(e_0^0)) = 0,\\ \Phi_1(\beta_1^2) &= (\beta_1^2 \uparrow \mathrm{stab}(e_0^0)) - (\beta_1^2 \uparrow \mathrm{stab}(e_0^0)) = 0.\end{aligned}$$

Observe that, unlike what happens in the case of **p4g**, here the differences $(\beta_1^i \uparrow \mathrm{stab}(e_0^0)) - (\beta_1^i \uparrow \mathrm{stab}(e_0^0)$ are trivial because in D_3 there is only one conjugation class of elements of order 2.

Now the computation of the SNF for the two differentials produce the invariant factors (1) and (1) for Φ_2 and Φ_1, respectively. Then, $H_2^{\mathcal{F}}(\mathbf{p31m}, R_{\mathbb{C}}) = 0$, $H_1^{\mathcal{F}}(\mathbf{p31m}, R_{\mathbb{C}}) = \mathbb{Z}$ and $H_0^{\mathcal{F}}(\mathbf{p31m}, R_{\mathbb{C}}) = \mathbb{Z}^5$.

The auxiliary matrices of the SNF produce the bases $[\beta_1^1]$ for $H_1^{\mathcal{F}}(\mathbf{p31m}, R_{\mathbb{C}})$ and $([\alpha_0^2], [\alpha_0^3], [\alpha_1^1], [\alpha_1^2], [\alpha_1^3])$ for $H_0^{\mathcal{F}}(\mathbf{p31m}, R_{\mathbb{C}})$.

3.16. The group p6

The representative e_2 the class of equivariant 2-cells by the action of **p6** will be exactly the same triangle as in the previous group. We keep the names of the vertices O, P and Q, and we consider another vertex R, the middle point of the lower side of the triangle. There will be three of classes of equivalence of 0-cells, whose representatives e_0^0, e_0^1 and e_0^2 are identified with O, P and R. The vertex Q is now the image of O under the 2-rotation r_1 centered in R (and also under the 3-rotation r_2 centered in P). There are also two classes of 1-cells, whose representatives e_1^0 and e_1^1 are identified, respectively, with the segments OP and QR. Observe that $r_2(OP) = QP$ and $r_1(QR) = OR$. We can now write the boundaries for the cells:

$$\begin{aligned}
\partial(e_2) &= e_1^0 + r_2 e_1^0 + e_1^1 + r_1 e_1^1,\\
\partial(e_1^0) &= e_0^1 - e_0^0,\\
\partial(e_1^1) &= e_0^2 - t e_0^0.
\end{aligned}$$

Observe that e_0^0, e_0^1 and e_0^2 are, respectively, 6-, 3- and 2- rotation centers, and their stabilizers are, respectively, isomorphic to C_6, C_3 and C_2. On the other hand, the group **p6** act freely over each of the classes of 1-cells, so we have the following Bredon chain complex:

$$0 \to \mathbb{Z}\gamma \to \mathbb{Z}\beta_0 \oplus \mathbb{Z}\beta_1 \to \bigoplus_{i=1}^{6} \alpha_0^i \bigoplus_{i=1}^{3} \alpha_1^i \bigoplus_{i=1}^{2} \alpha_2^i \to 0.$$

Now we compute the differentials, taking into account line 1 of Table 3:

$$\begin{aligned}
\Phi_2(\gamma) &= (\gamma \uparrow \operatorname{stab}(e_1^0)) - (\gamma \uparrow \operatorname{stab}(e_1^0)) + (\gamma \uparrow \operatorname{stab}(e_1^1)) - (\gamma \uparrow \operatorname{stab}(e_1^1)) = 0,\\
\Phi_1(\beta_0) &= (\beta_0 \uparrow \operatorname{stab}(e_0^1)) - (\beta_0 \uparrow \operatorname{stab}(e_0^0))\\
&= \alpha_1^1 + \alpha_1^2 + \alpha_1^3 - \alpha_0^1 - \alpha_0^2 - \alpha_0^3 - \alpha_0^4 - \alpha_0^5 - \alpha_0^6,\\
\Phi_1(\beta_1) &= (\beta_1 \uparrow \operatorname{stab}(e_0^2)) - (\beta_1 \uparrow \operatorname{stab}(e_0^0))\\
&= \alpha_2^1 + \alpha_2^2 - \alpha_0^1 - \alpha_0^2 - \alpha_0^3 - \alpha_0^4 - \alpha_0^5 - \alpha_0^6.
\end{aligned}$$

As the first homomorphism is trivial, we get $H_2^{\mathcal{F}}(\mathbf{p6}, R_{\mathbb{C}}) = \mathbb{Z}$. On the other hand, the SNF of the matrix of Φ_1 has $(1, 1)$ as invariant factors, and thus $H_1^{\mathcal{F}}(\mathbf{p6}, R_{\mathbb{C}}) = 0$ and $H_0^{\mathcal{F}}(\mathbf{p6}, R_{\mathbb{C}}) = \mathbb{Z}^9$.

Clearly a basis for $H_2^{\mathcal{F}}(\mathbf{p6}, R_{\mathbb{C}})$ is given by $[\gamma]$, while the auxiliary computations of the SNF provide the basis of $H_1^{\mathcal{F}}(\mathbf{p6}, R_{\mathbb{C}})$ given by $([\alpha_0^2], [\alpha_0^3], [\alpha_0^4], [\alpha_0^5], [\alpha_0^6], [\alpha_1^2], [\alpha_1^3], [\alpha_2^1], [\alpha_2^2])$.

3.17. The group p6m

To get our representative e_2 for the class of equivariant 2-cells in this group, we take in the previous group **p6** the triangle defined by the vertices O, P and R there. We keep the name of the vertices here, and identify them in this order with the representatives e_0^0, e_0^1 and e_0^2 of the three equivalence classes of 0-cells. We also identify the representatives e_1^0, e_1^1 and e_1^2 of the three classes of 1-cells with the segments OP, PR and RO. Now the boundaries are:

$$\begin{aligned}\partial(e_2) &= e_1^0 + e_1^1 + e_1^2,\\ \partial(e_1^0) &= e_0^1 - e_0^0,\\ \partial(e_1^1) &= e_0^2 - e_0^1,\\ \partial(e_1^2) &= e_0^0 - e_0^2.\end{aligned}$$

Again e_0^0, e_0^1 and e_0^2 are, respectively, 6-, 3- and 2-rotation centers. In addition, all of them lie in a reflection axis. Hence, the stabilizers of these vertices are, respectively, D_6, D_3 and D_2. As the representatives of the 1-cells all lie in some reflection axis, their stabilizers are all isomorphic to C_2. So we have the following Bredon chain complex:

$$0 \to \mathbb{Z}\gamma \to \bigoplus_{i=1}^{2} \beta_0^i \bigoplus_{i=1}^{2} \beta_1^i \bigoplus_{i=1}^{2} \beta_2^i \to \bigoplus_{i=1}^{6} \alpha_0^i \bigoplus_{i=1}^{3} \alpha_1^i \bigoplus_{i=1}^{4} \alpha_2^i \to 0.$$

We compute the differentials for this complex, taking into account lines 8, 9, 14, 15, 16 and 17 of Table 3:

$$\begin{aligned}\Phi_2(\gamma) &= (\gamma \uparrow \mathrm{stab}(e_1^0)) + (\gamma \uparrow \mathrm{stab}(e_1^1)) + (\gamma \uparrow \mathrm{stab}(e_1^2)\\ &= \beta_0^1 + \beta_0^2 + \beta_1^1 + \beta_1^2 + \beta_2^1 + \beta_2^2,\\ \Phi_1(\beta_0^1) &= (\beta_0^1 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0^1 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^1 + \alpha_1^3 - \alpha_0^1 - \alpha_0^3 - \alpha_0^5 - \alpha_0^6,\\ \Phi_1(\beta_0^2) &= (\beta_0^2 \uparrow \mathrm{stab}(e_0^1)) - (\beta_0^2 \uparrow \mathrm{stab}(e_0^0)) = \alpha_1^2 + \alpha_1^3 - \alpha_0^2 - \alpha_0^4 - \alpha_0^5 - \alpha_0^6,\\ \Phi_1(\beta_1^1) &= (\beta_1^1 \uparrow \mathrm{stab}(e_0^2)) - (\beta_1^1 \uparrow \mathrm{stab}(e_0^1)) = \alpha_2^1 + \alpha_2^2 - \alpha_1^1 - \alpha_1^3,\\ \Phi_1(\beta_1^2) &= (\beta_1^2 \uparrow \mathrm{stab}(e_0^2)) - (\beta_1^2 \uparrow \mathrm{stab}(e_0^1)) = \alpha_2^3 + \alpha_2^4 - \alpha_1^2 - \alpha_1^3,\\ \Phi_1(\beta_2^1) &= (\beta_2^1 \uparrow \mathrm{stab}(e_0^0)) - (\beta_2^1 \uparrow \mathrm{stab}(e_0^2)) = \alpha_0^1 + \alpha_0^4 + \alpha_0^5 + \alpha_0^6 - \alpha_2^1 - \alpha_2^3,\\ \Phi_1(\beta_2^2) &= (\beta_2^2 \uparrow \mathrm{stab}(e_0^0)) - (\beta_2^2 \uparrow \mathrm{stab}(e_0^2)) = \alpha_0^2 + \alpha_0^3 + \alpha_0^5 + \alpha_0^6 - \alpha_2^2 - \alpha_2^4.\end{aligned}$$

The calculation of the SNF for Φ_2 and Φ_1 produce the invariant factors (1) and $(1,1,1,1,1)$, respectively. Then we have $H_2^{\mathcal{F}}(\mathbf{p6m}, R_{\mathbb{C}}) = 0$, $H_1^{\mathcal{F}}(\mathbf{p6m}, R_{\mathbb{C}}) = 0$ and $H_0^{\mathcal{F}}(\mathbf{p6m}, R_{\mathbb{C}}) = \mathbb{Z}^8$.

The auxiliary matrix Q of the SNF permits to identify a basis of $H_0^{\mathcal{F}}(\mathbf{p6m}, R_{\mathbb{C}})[\beta_1^1]$, which is given by $([\alpha_0^4], [\alpha_0^5], [\alpha_0^6], [\alpha_1^1], [\alpha_1^3], [\alpha_2^1], [\alpha_2^3], [\alpha_2^4])$.

Remark 3.1. Observe that applying [18, Theorem 5.27] to the Bredon homology groups of the wallpaper groups and recalling that the Baum-Connes conjecture holds for these groups, we recover the K-theory computations of [14, Section

5]. Hence, our results can also be interpreted as a way to approach directly the left-hand side of the conjecture in the case of wallpaper groups.

TABLE 4. Bredon homology

Group	H_2	H_1	Basis H_1	H_0	Basis H_0
p1	$\mathbb{Z}$	$\mathbb{Z}^2$	$([\beta_1],[\beta_2])$	$\mathbb{Z}$	$[\alpha]$
p2	$\mathbb{Z}$	0	-	$\mathbb{Z}^5$	$([\alpha_0^1],[\alpha_0^2],[\alpha_1^2],[\alpha_2^2],[\alpha_3^2])$
pm	0	$\mathbb{Z}^3$	$([\beta_1^1],[\beta_1^2],[\beta_2^1])$	$\mathbb{Z}^3$	$([\alpha_0^2],[\alpha_1^1],[\alpha_1^2])$
pg	0	$\mathbb{Z}/2\oplus\mathbb{Z}$	$([\beta_0],[\beta_1])$	$\mathbb{Z}$	$[\alpha]$
cm	0	$\mathbb{Z}^2$	$([\beta_1^1],[\beta_1^2])$	$\mathbb{Z}^2$	$([\alpha^1],[\alpha^2])$
pmm	0	0	-	$\mathbb{Z}^9$	$([\alpha_0^3],[\alpha_1^3],[\alpha_1^4],[\alpha_2^3],[\alpha_2^4],[\alpha_3^1],[\alpha_3^2],[\alpha_3^3],[\alpha_3^4])$
pmg	0	$\mathbb{Z}$	$[\beta_1^1+\beta_2^1]$	$\mathbb{Z}^4$	$([\alpha_1^2],[\alpha_2^1],[\alpha_2^2],[\alpha_3^2])$
pgg	0	$\mathbb{Z}/2$	$[\beta_0]$	$\mathbb{Z}^3$	$([\alpha_1^2],[\alpha_2^1],[\alpha_2^2])$
cmm	0	0	-	$\mathbb{Z}^6$	$([\alpha_0^1+\alpha_0^2],[\alpha_0^3],[\alpha_0^4],[\alpha_1^1],[\alpha_1^3],[\alpha_2^2])$
p4	$\mathbb{Z}$	0	-	$\mathbb{Z}^8$	$([\alpha_0^2],[\alpha_1^2],[\alpha_1^3],[\alpha_1^4],[\alpha_2^1],[\alpha_2^2],[\alpha_2^3],[\alpha_2^4])$
p4m	0	0	-	$\mathbb{Z}^9$	$([\alpha_0^4],[\alpha_0^5],[\alpha_1^3],[\alpha_1^4],[\alpha_1^5],[\alpha_2^1],[\alpha_2^2],[\alpha_2^3],[\alpha_2^4])$
p4g	0	0	-	$\mathbb{Z}^6$	$([\alpha_0^1],[\alpha_0^2],[\alpha_0^4],[\alpha_1^1],[\alpha_1^2],[\alpha_1^4])$
p3	$\mathbb{Z}$	0	-	$\mathbb{Z}^7$	$([\alpha_0^2],[\alpha_0^3],[\alpha_1^1],[\alpha_1^2],[\alpha_1^3],[\alpha_2^1],[\alpha_2^2])$
p3m1	0	$\mathbb{Z}$	$[\beta_0^1+\beta_1^1+\beta_2^1]$	$\mathbb{Z}^5$	$([\alpha_0^3],[\alpha_1^1],[\alpha_1^2],[\alpha_1^3],[\alpha_2^3])$
p31m	0	$\mathbb{Z}$	$[\beta_1^1]$	$\mathbb{Z}^5$	$([\alpha_0^2],[\alpha_0^3],[\alpha_1^1],[\alpha_1^2],[\alpha_1^3])$
p6	$\mathbb{Z}$	0	-	$\mathbb{Z}^9$	$([\alpha_0^2],[\alpha_0^3],[\alpha_0^4],[\alpha_0^5],[\alpha_0^6],[\alpha_1^2],[\alpha_1^3],[\alpha_2^1],[\alpha_2^2])$
p6m	0	0	-	$\mathbb{Z}^8$	$([\alpha_0^4],[\alpha_0^5],[\alpha_0^6],[\alpha_1^1],[\alpha_1^3],[\alpha_2^1],[\alpha_2^3],[\alpha_2^4])$

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