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PARTIAL SUMS AND INCLUSION RELATIONS FOR STARLIKE FUNCTIONS ASSOCIATED WITH AN EVOLUTE OF A NEPHROID CURVE

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Dedicated to Prof. Ajay Kumar

Abstract. A class of normalized univalent functions f defined in an open unit disk of the complex plane is introduced and studied such that the values of the quantity $zf'(z)/f(z)$ lies inside the evolute of a nephroid curve. The inclusion relations of the newly defined class with other subclasses of starlike functions and radius problems concerning the second partial sums are investigated. All the obtained results are sharp.

1. Introduction

A nephroid [15, p. 69] is a kidney-shaped curve defined by

$$x(t) = a(3\cos t - \cos(3t)), \quad y(t) = a(3\sin t - \sin(3t)) = 4a\sin^3 t \tag{1.1}$$

with the cartesian equation $(x^2+y^2-4a^2)^3 = 108a^4y^2$ and having two cusps on the real axis. Wani and Swaminathan [30] investigated the class of analytic functions defined in an open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ associated with translated nephroid (1.1) with $a = 1/3$. Given a curve, its evolute [5, p. 281] is the locus of the centres of curvature of all points of that curve. If (X, Y) denotes the coordinates of the centre of curvature of a given curve, then

$$X = x - \frac{y_1(1+y_1^2)}{y_2} \quad \text{and} \quad Y = y + \frac{1+y_1^2}{y_2} \quad \left(\text{where } y_i = \frac{d^i y}{dx^i},\ i = 1, 2\right)$$

so that the required equation of evolute is obtained by changing $X \to x$ and $Y \to y$ in the relation obtained between X and Y. Accordingly, the evolute [15, p. 69] of the nephroid curve (1.1) is again a nephroid on half the linear scale, with its cusp at right angles to that the original and parametric equations

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given by

$$x(t) = \frac{1}{2}a(3\cos t + \cos(3t)) = 2a\cos^3 t, \quad y(t) = \frac{1}{2}a(3\sin t + \sin(3t)). \tag{1.2}$$

The present manuscript is devoted to introduce and study the class of analytic functions in $\mathbb{D}$ associated with the translated evolute (1.2) of the nephroid (1.1) for the case $a = 1/2$.

Let $\mathcal{A}$ be the class of analytic functions $f : \mathbb{D} \to \mathbb{C}$ with the normalization $f(0) = 0 = f'(0) - 1$ and let $\mathcal{S}$ be its subclass consisting of univalent functions. Using subordination, Ma and Minda [16] in 1992 gave a unified representation $\mathcal{S}^*(\phi)$ of various subclasses of starlike functions which consist of functions $f \in \mathcal{A}$ satisfying $zf'(z)/f(z) \prec \phi(z)$ for all $z \in \mathbb{D}$. Here the function ϕ is univalent with positive real part that maps $\mathbb{D}$ onto a domain symmetric with respect to real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. An example of such a function is $\phi_{ev} : \mathbb{D} \to \mathbb{C}$ defined by

$$\phi_{ev}(z) = 1 + \frac{3}{4}z + \frac{1}{4}z^3 \qquad (z \in \mathbb{D}) \tag{1.3}$$

which implies that the class $\mathcal{S}^*_{ev} := \mathcal{S}^*(\phi_{ev})$ is well-defined. Note that $\phi_{ev}(\mathbb{D})$ is bounded by a translated evolute of the form (1.2) with the parametric equations:

$$x(t) = 1 + \frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t) = 1 + \cos^3 t,\ y(t) = \frac{3}{4}\sin(t) + \frac{1}{4}\sin(3t),\ t \in [-\pi, \pi],$$

and it takes the form $(4x^2 + 4y^2 - 8x + 3)^3 = 27(x - 1)^2$ in the cartesian coordinates. The boundary curve has cusps at $\pi/2$, $-\pi/2$ which can be seen by noticing that $x'(t) = 0 = y'(t)$ at these two points and $x''(t)$, $y''(t)$ do not vanish together. An important example of a function in the class $\mathcal{S}^*_{ev}$ is

$$f_{ev}(z) = z\exp\left(\frac{3}{4}z + \frac{1}{12}z^3\right) = z + \frac{3}{4}z^2 + \frac{9}{32}z^3 + \frac{59}{384}z^4 + \cdots. \tag{1.4}$$

A function $f \in \mathcal{S}^*_{ev}$ satisfies $f(z)/z \prec f_{ev}(z)/z$ and $zf'(z)/f(z) \prec zf'_{ev}(z)/f_{ev}(z)$ in $\mathbb{D}$ by [16, Theorem 1', p. 161]. For $f \in \mathcal{S}^*_{ev}$, the growth theorem [16, Corollary 1', p. 161]: $-f_{ev}(-|z|) \leq |f(z)| \leq f_{ev}(|z|)$ holds in $\mathbb{D}$ with equality for some non-zero $z \in \mathbb{D}$ if and only if f is a rotation of f_{ev}. Also, the covering theorem [16, Corollary 2', p. 161]: $f(\mathbb{D}) \supseteq \left\{w : |w| < -f_{ev}(-1) = e^{-5/6} \approx 0.434598\right\}$ is satisfied for each $f \in \mathcal{S}^*_{ev}$. To obtain the distortion theorem for the class $\mathcal{S}^*_{ev}$, let us calculate the minimum and maximum of the expression $|\phi_{ev}(z)|$ over the circle $|z| = r$, $r \in (0, 1)$. For $t \in [0, \pi]$ and $|z| = r$, the function

$$|\phi_{ev}(re^{it})|^2 = \frac{1}{16}(r^6 + 6r^4\cos(2t) + 8r^3\cos(3t) + 9r^2 + 24r\cos t + 16)$$

attains maximum at $t = 0$ and minimum at $t = \pi$. Consequently, it follows that

$$\min_{|z|=r} |\phi_{ev}(z)| = \phi_{ev}(-r) \quad \text{and} \quad \max_{|z|=r} |\phi_{ev}(z)| = \phi_{ev}(r).$$

By [16, Theorem 2, p. 162] if $f \in \mathcal{S}^*_{ev}$ and $|z| = r < 1$, then $f'_{ev}(-r) \leq |f'(z)| \leq f'_{ev}(r)$ such that equality holds for some non-zero $z \in \mathbb{D}$ if and only if f is a rotation of f_{ev}. By [6, Theorem 2.2], the first four Taylor series coefficients of a function $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}^*_{ev}$ satisfies $|a_2| \leq 3/4$, $|a_3| \leq 3/8$, $|a_4| \leq 1/4$ and $|a_5| \leq 3/16$. These bounds are sharp for the functions given by

$$f_i(z) = z \exp\left(\int_0^z \frac{\phi_{ev}(t^{i-1}) - 1}{t} dt\right), \quad i = 2, 3, 4, 5.$$

In Section 2, a preliminary lemma is proved to find the radius of the largest disk centred at $(a, 0)$, $0 < a < 2$ that can be embedded into the image domain $\phi_{ev}(\mathbb{D})$. Similarly, problem of finding the smallest disk containing $\phi_{ev}(\mathbb{D})$ is also tackled. This result is utilized to prove the inclusion relations of the class $\mathcal{S}^*_{ev}$ with other subclasses of starlike functions in Section 3. The last section of the paper is devoted to partial sums of analytic functions. Given a function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, the polynomial $z + a_2z^2 + \cdots + a_nz^n$ is called nth partial sum of f. Radii constants are determined pertaining to the class $\mathcal{S}^*_{ev}$ for the second partial sum. All the obtained bounds are sharp.

2. Preliminary result

Suppose that ϕ_{ev} is given by (1.3). In this section, we will establish the values of r_a and R_a such that the domain $\phi_{ev}(\mathbb{D})$ contains the disk $|w - a| < r_a$ and is contained in the disk $|w - a| < R_a$, where $0 < a < 2$. It is a vital result to obtain the inclusion relations of the class $\mathcal{S}^*_{ev}$ with other well-known subclasses of starlike functions.

Theorem 2.1. *For $0 < a < 2$, we have*

$$\{w : |w - a| < r_a\} \subseteq \phi_{ev}(\mathbb{D}) \subseteq \{w : |w - a| < R_a\},$$

where r_a, R_a are given by

$$r_a = \begin{cases} a, & 0 < a \leq 5/8; \\ \sqrt{\dfrac{5}{4} - 2a + a^2}, & \dfrac{5}{8} \leq a \leq \dfrac{11}{8}; \\ 2 - a, & \dfrac{11}{8} \leq a < 2, \end{cases}$$

and

$$R_a = \begin{cases} 2 - a, & 0 < a \leq 1; \\ a, & 1 \leq a < 2. \end{cases}$$

Proof. For $0 \leq t \leq \pi$, if we let $\cos t = x$, then the square of the distance of the point $(a, 0)$ to the boundary point $\phi_{ev}(e^{it})$ of $\phi_{ev}(\mathbb{D})$ is given by

$$h(x) = \frac{1}{4}(8x^3(1 - a) + 3x^2 + 4a^2 - 8a + 5).$$

We see that $h'(0) = 0 = h'(x_1)$, where $x_1 = 1/(4(a-1))$, $a \neq 1$. Observe that $-1 \leq x_1 \leq 1$ if and only if $a \leq 3/4$ or $a \geq 5/4$. Moreover, we have

$$\begin{aligned}
h(x_1) - h(0) &= \frac{1}{64(1-a)^2} > 0, \ a \neq 1, \\
h(1) - h(x_1) &= \frac{(5-4a)^2(7-8a)}{64(a-1)^2} \geq 0 \ \text{ if and only if } \ a \leq \frac{7}{8} \text{ or } a = \frac{5}{4}, \\
h(0) - h(-1) &= \frac{5-8a}{4} \geq 0 \ \text{ if and only if } \ a \leq \frac{5}{8}, \\
h(-1) - h(x_1) &= \frac{(3-4a)^2(8a-9)}{64(a-1)^2} \geq 0 \ \text{ if and only if } \ a = \frac{3}{4} \text{ or } a \geq \frac{9}{8}, \\
h(1) - h(-1) &= 4 - 4a \geq 0 \ \text{ if and only if } \ a \leq 1, \\
h(1) - h(0) &= \frac{11-8a}{4} \geq 0 \ \text{ if and only if } \ a \leq \frac{11}{8}.
\end{aligned}$$

Consider the following three cases:

Case 1: Let $0 < a \leq 3/4$. In this case, $x_1 \in [-1,1]$ and $h(1) \geq h(x_1)$. For $0 < a \leq 5/8$, $h(1) \geq h(x_1) > h(0) \geq h(-1)$, so that

$$r_a = \sqrt{\min_{x\in[-1,1]} h(x)} = \sqrt{h(-1)} = a \quad \text{and} \quad R_a = \sqrt{\max_{x\in[-1,1]} h(x)} = \sqrt{h(1)} = 2-a.$$

For $5/8 < a \leq 3/4$, $h(1) \geq h(x_1) > h(-1) > h(0)$, which gives

$$r_a = \sqrt{h(0)} = \sqrt{\frac{5}{4} - 2a + a^2} \quad \text{and} \quad R_a = \sqrt{h(1)} = 2-a.$$

Case 2: Let $3/4 < a < 5/4$. In this case, $x_1 \notin [-1,1]$, $h(-1) > h(0)$ and $h(1) \geq 0$. If $3/4 < a \leq 1$, then $h(1) \geq h(-1) > h(0)$ and if $1 \leq a \leq 5/4$, then $h(-1) > h(1) \geq h(0)$. Thus $r_a = \sqrt{h(0)}$ and

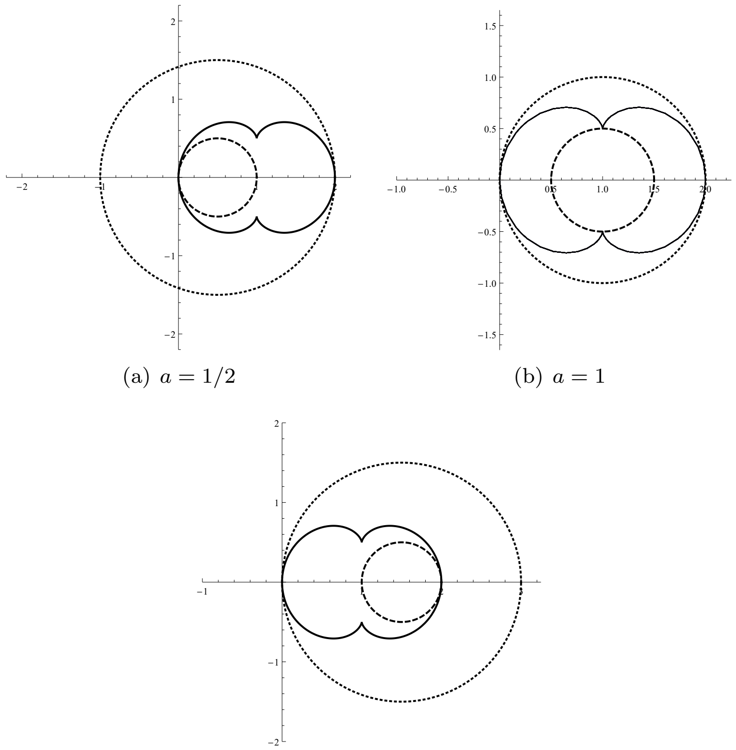
$$R_a = \begin{cases} \sqrt{h(1)} = 2-a, & 3/4 \leq a \leq 1; \\ \sqrt{h(-1)} = a, & 1 \leq a \leq 5/4. \end{cases}$$

Case 3: Let $5/4 \leq a < 2$. In this case, $x_1 \in [-1,1]$ and $h(-1) \geq h(x_1) > h(0)$. By the comparison of the values of the function h at the critical points and end points, it follows that for $5/4 \leq a \leq 11/8$, $h(-1) > h(x_1) > h(1) \geq h(0)$ and for $11/8 \leq a < 2$, we have $h(-1) > h(x_1) > h(0) > h(1)$. Hence $r_a = \sqrt{h(0)}$ for $5/4 \leq a \leq 11/8$, $r_a = \sqrt{h(1)}$ for $11/8 \leq a < 2$ and the value of R_a is $\sqrt{h(-1)}$.

The cases $a = 1/2$ ($r_a = 1/2, R_a = 3/2$), $a = 1$ ($r_a = 1/2, R_a = 1$) and $a = 3/2$ ($r_a = 1/2, R_a = 3/2$) of Theorem 2.1 are depicted in Figure 1. □

3. Inclusion relations

There are several subclasses of starlike functions which can be defined for different choices of the function ϕ in the class $\mathcal{S}^*(\phi)$ introduced by Ma and



(a) $a = 1/2$ (b) $a = 1$

(c) $a = 3/2$

FIGURE 1. Illustration of Theorem 2.1

Minda [16]. The inclusion relations of the class $\mathcal{S}^*_{ev}$ with the classes $\mathcal{S}^*[A, B] := \mathcal{S}^*(\phi_{A,B})$, $\mathcal{S}^*(\sqrt{1+cz})$, $k - \mathcal{S}^* := \mathcal{S}^*(p_k)$, where

$$p_k(z) = \frac{1}{1-k^2} \cosh\left(\frac{2}{\pi} \cos^{-1}(k) \log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right) - \frac{k^2}{1-k^2},$$

$\mathcal{S}^*_e(\alpha) := \mathcal{S}^*(\alpha + (1-\alpha)e^z)$ and $\mathcal{S}^*_L(\alpha) := \mathcal{S}^*(\alpha + (1-\alpha)\sqrt{1+z})$ are investigated in this section. Here, $\phi_{A,B}(z) = (1+Az)/(1+Bz)$ and the parameters A, B, c, k, α satisfy $-1 \leq B < A \leq 1$, $0 < c \leq 1$, $1 < k < \infty$ and $0 \leq \alpha < 1$. For $c = 1$ or $\alpha = 0$, the classes $\mathcal{S}^*(\sqrt{1+cz})$ and $\mathcal{S}^*_L(\alpha)$ reduce to the class $\mathcal{S}^*_L$. The case $\alpha = 0$ in $\mathcal{S}^*_e(\alpha)$ yields the class $\mathcal{S}^*_e$. The connection of the class $\mathcal{S}^*_{ev}$ with the subclasses of starlike functions associated with the left-half of shifted lemniscate of Bernoulli and cosine hyperbolic function is also established. These classes are defined as $\mathcal{S}^*_{RL} := \mathcal{S}^*(\phi_{RL})$ and $\mathcal{S}^*_{\varrho} := \mathcal{S}^*(\phi_{\varrho})$, where

$$\phi_{RL}(z) = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$$

and $\phi_{\varrho}(z) = \cosh\sqrt{z}$. All these classes were introduced and studied in [2, 9–12, 18–20, 27, 29].

Theorem 3.1. *The following inclusion relationships hold true for* $\mathcal{S}^*_{ev}$:

(i) $\mathcal{S}^*_L \subseteq \mathcal{S}^*_{ev}$. *In particular,* $\mathcal{S}^*(\sqrt{1+cz}) \subseteq \mathcal{S}^*_{ev}$ *for* $0 < c \leq 1$ *and* $\mathcal{S}^*_L(\alpha) \subseteq \mathcal{S}^*_{ev}$ *for* $0 \leq \alpha < 1$.
(ii) $\mathcal{S}^*_e(\alpha) \subseteq \mathcal{S}^*_{ev}$ *for* $\alpha_0 \leq \alpha < 1$, *where* $\alpha_0 \approx 0.59317$.
(iii) $k - \mathcal{S}^* \subseteq \mathcal{S}^*_{ev}$ *for* $2 \leq k < \infty$.

Proof. (i) Let $f \in \mathcal{S}^*_L$. In order to show that $\phi_L(\mathbb{D}) \subseteq \phi_{ev}(\mathbb{D})$, where $\phi_L(z) = \sqrt{1+z}$, consider the function $d_L(t)$ as the difference of the square of the distances of points on the boundary curves $\phi_{ev}(e^{it})$ and $\phi_L(e^{it})$ with the point $(1,0)$:

$$\begin{aligned} d_L(t) &= |\phi_{ev}(e^{it})-1|^2 - |\phi_L(e^{it})-1|^2 \\ &= \frac{3\cos(2t)}{8} - 2\cos\left(\frac{t}{2}\right) + 2\cos\left(\frac{t}{4}\right)\sqrt{2\cos\left(\frac{t}{2}\right)} - \frac{3}{8}. \end{aligned}$$

Figure 3(a) shows that $d_L(t)$ is always non-negative for $t \in [0,\pi]$ which gives $\phi_L(\mathbb{D}) \subseteq \phi_{ev}(\mathbb{D})$ (see Figure 5(a) as well) and hence $\mathcal{S}^*_L \subseteq \mathcal{S}^*_{ev}$.

Note that, for $c_1 \leq c_2$, if Γ_{c_1} and Γ_{c_2} are the boundaries of the image domain of the mappings $\sqrt{1+c_1 z}$ and $\sqrt{1+c_2 z}$, respectively, under $\mathbb{D}$, then $\Gamma_{c_1} \subseteq \Gamma_{c_2}$. Therefore, it follows that $\Gamma_c \subseteq \Gamma_1$ for $0 < c \leq 1$. Consequently, $\mathcal{S}^*(\sqrt{1+cz}) \subseteq \mathcal{S}^*_L \subseteq \mathcal{S}^*_{ev}$ for $0 < c \leq 1$.

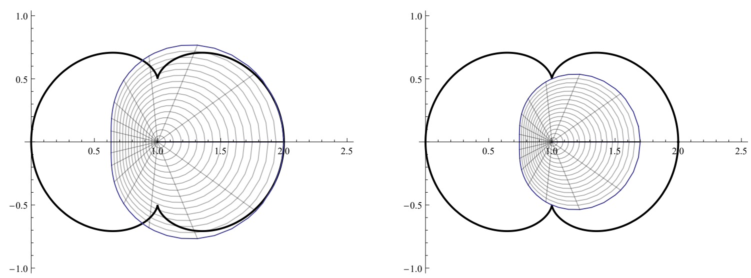
Also, observe that $\Omega_{\alpha_1} \subseteq \Omega_{\alpha_2}$ for $\alpha_1 \geq \alpha_2$, where $\Omega_{\alpha_1}, \Omega_{\alpha_2}$ are the boundaries of the image domain under the mappings $\alpha_1 + (1-\alpha_1)\sqrt{1+z}$ and $\alpha_2 + (1-\alpha_2)\sqrt{1+z}$, respectively. Thus $\Omega_\alpha \subseteq \Omega_0$ and hence $\mathcal{S}^*_L(\alpha) \subseteq \mathcal{S}^*_L \subseteq \mathcal{S}^*_{ev}$ for $0 \leq \alpha < 1$.

(ii) If $f \in \mathcal{S}^*_e(\alpha)$, then $zf'(z)/f(z) \prec \psi_\alpha(z) := \alpha + (1-\alpha)e^z$, which gives

$$\alpha + (1-\alpha)e^{-1} < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha + (1-\alpha)e.$$

The conditions $\alpha + (1-\alpha)e^{-1} \geq 0$ and $\alpha + (1-\alpha)e \leq 2$ gives $\alpha \geq 0$ and $\alpha \geq (2-e)/(1-e)$, respectively. However, the condition $\alpha \geq (2-e)/(1-e)$ is not necessary for the inclusion relation $\mathcal{S}^*_e(\alpha) \subseteq \mathcal{S}^*_{ev}$ (see Figure 2(a) which depicts the case $\alpha = (2-e)/(1-e)$). As a result, the notion of cusp will be exploited to obtain the desired bound on α. The function ϕ_{ev} has a cusp at $(1, 1/2)$. To obtain the inclusion relation, we first calculate the value of t such that $\psi_\alpha(e^{it}) = 1 + i/2$, that is

$$\operatorname{Re}(\alpha + (1-\alpha)e^{e^{it}}) = 1 \quad \text{and} \quad \operatorname{Im}(\alpha + (1-\alpha)e^{e^{it}}) = \frac{1}{2}.$$



(a) $\alpha = \frac{2-e}{1-e}$ (b) $\alpha = \alpha_0$

FIGURE 2. Inclusion relations for $\mathcal{S}_e^*(\alpha)$

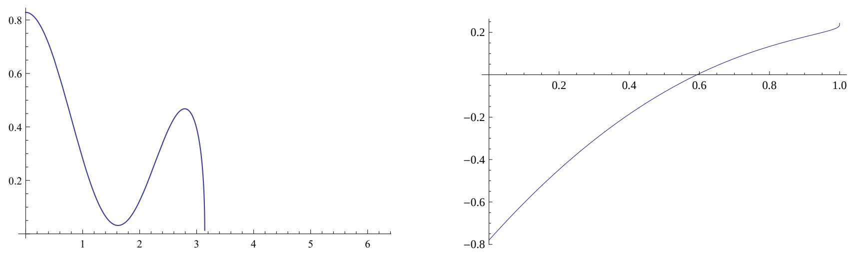
A straightforward calculation shows that $e^{2\cos t} = 1 + 1/(4(1-\alpha)^2)$ so that

$$t = \cos^{-1}\left(\frac{1}{2}\log\left(1 + \frac{1}{4(1-\alpha)^2}\right)\right) := t_0.$$

If $d_e(\alpha)$ denotes the difference of the square of the distances of points $\phi_{ev}(e^{it})$ for $t = \pi/2$ and $\psi_\alpha(e^{it})$ for $t = t_0$ on the boundary curves with the point $(1, 0)$, then

$$\begin{aligned} d_e(\alpha) &= |\phi_{ev}(e^{i\pi/2}) - 1|^2 - |\psi_\alpha(e^{it_0}) - 1|^2 \\ &= \frac{1}{4} - (1-\alpha^2)(1 + e^{2\cos t_0} - 2e^{\cos t_0}\cos(\sin t_0)). \end{aligned}$$

For $\psi_\alpha(\mathbb{D})$ to be a subset of $\phi_{ev}(\mathbb{D})$, it is necessary that $d_e(\alpha) \geq 0$ which gives $\alpha \geq \alpha_0$, where $\alpha_0 \approx 0.59317$ is a root of the equation $d_e(\alpha) = 0$ in $(0, 1)$. The Figure 3(b) depicts the graph of $d_e(\alpha)$ for $\alpha \in [0, 1)$. In fact, $\psi_{\alpha_0}(\mathbb{D}) \subseteq \phi_{ev}(\mathbb{D})$ as seen by Figure 2(b).



(a) $d_L(t)$ (b) $d_e(\alpha)$

FIGURE 3. Graph of distance functions

Also, for every $\alpha_1 \leq \alpha_2$, if $\Lambda_{\alpha_1}, \Lambda_{\alpha_2}$ are the boundaries of the image domain of $\mathbb{D}$ under the mappings $\alpha_1 + (1-\alpha_1)e^z$ and $\alpha_2 + (1-\alpha_2)e^z$, respectively, then $\Lambda_{\alpha_2} \subseteq \Lambda_{\alpha_1}$. Thus $\mathcal{S}^*_e(\alpha) \subseteq \mathcal{S}^*_e(\alpha_0) \subseteq \mathcal{S}^*_{ev}$ for $\alpha_0 \leq \alpha < 1$.

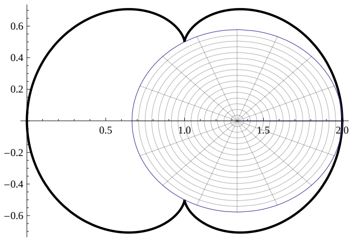


FIGURE 4. Inclusion relation for $2 - \mathcal{S}^*$

(iii) Let $f \in k - \mathcal{S}^*$. Then $w = zf'(z)/f(z)$ lies in the domain $\{w \in \mathbb{C} : \operatorname{Re} w > k|w-1|\}$ which represents is an elliptic region ($k > 1$) whose boundary curve Υ_k is represented by the equation

$$\frac{(x-\lambda)^2}{a^2} + \frac{y^2}{b^2} = 1$$

for $\lambda = k^2/(k^2-1)$, $a = k/(k^2-1)$ and $b = 1/\sqrt{k^2-1}$. A necessary condition for $\Upsilon_k \subseteq \phi_{ev}(\mathbb{D})$ is that $\lambda - a \geq 0$ and $\lambda + a \leq 2$, which gives $k \geq 2$. Moreover, $\Upsilon_2 \subseteq \phi_{ev}(\mathbb{D})$ (see Figure 4). Also, for $k_1 \leq k_2$ we observe that $\Upsilon_{k_1} \supseteq \Upsilon_{k_2}$. Thus, $k - \mathcal{S}^* \subseteq 2 - \mathcal{S}^* \subseteq \mathcal{S}^*_{ev}$ for $2 \leq k < \infty$. □

Remark 3.2. By Figure 5(b)-(c), it is easily seen that the image domains $\phi_{RL}(\mathbb{D})$ and $\phi_{\varrho}(\mathbb{D})$ are contained in the evolute domain. Consequently, it follows that $\mathcal{S}^*_{RL} \subseteq \mathcal{S}^*_{ev}$ and $\mathcal{S}^*_{\varrho} \subseteq \mathcal{S}^*_{ev}$.

The next theorem determines the conditions on A and B so that the Janowski class of starlike functions is a subfamily of $\mathcal{S}^*_{ev}$.

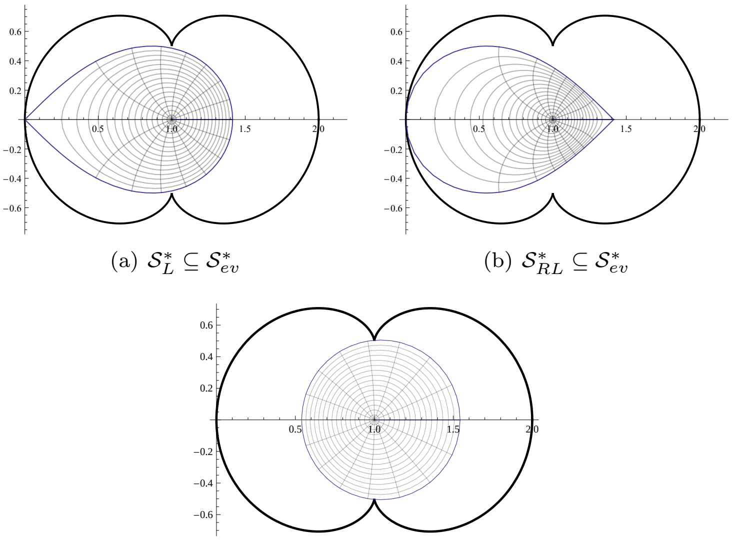
Theorem 3.3. *Let* $-1 < B < A \leq 1$. *Then* $\mathcal{S}^*[A,B] \subseteq \mathcal{S}^*_{ev}$ *if one of the following holds*:

(i) $-1 < B \leq (-3/5)$ *and* $A \leq (1+2B)$;
(ii) $(-3/5) \leq B \leq (3/5)$ *and* $2A \leq 2B + \sqrt{1-B^2}$;
(iii) $(3/5) \leq B < 1$.

Proof. If $f \in \mathcal{S}^*[A,B]$, then by [24, Lemma 2.1, p. 267],

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2}\right| \leq \frac{A-B}{1-B^2}, \tag{3.1}$$

which represents a disk with radius $r := (A-B)/(1-B^2)$ and centre $a := (1-AB)/(1-B^2)$. In accordance with Theorem 2.1, we discuss the following cases so that the disk (3.1) lies in $\phi_{ev}(\mathbb{D})$.



(a) $\mathcal{S}^*_L \subseteq \mathcal{S}^*_{ev}$ (b) $\mathcal{S}^*_{RL} \subseteq \mathcal{S}^*_{ev}$

(c) $\mathcal{S}^*_{\varrho} \subseteq \mathcal{S}^*_{ev}$

FIGURE 5. Inclusion relations for $\mathcal{S}^*_L$, $\mathcal{S}^*_{RL}$ and $\mathcal{S}^*_{\varrho}$

Case 1: Let $0 < a \leq 5/8$. By Theorem 2.1, the disk (3.1) lies in $\phi_{ev}(\mathbb{D})$ provided $r \leq a$. Both the conditions can be rewritten as

$$0 < \frac{1-AB}{1-B^2} \leq \frac{5}{8} \quad \text{and} \quad \frac{A-B}{1-B^2} \leq \frac{1-AB}{1-B^2}. \tag{3.2}$$

Clearly, $B \neq 0$ and the second inequality of (3.2) is always satisfied. Firstly let us assume that $B > 0$. From the first inequality of (3.2), $(3+5B^2)/(8B) \leq A \leq 1$ which gives $3/5 \leq B < 1$.

The case $B < 0$ gives $A < (3+5B^2)/(8B) < 0$ which can rewritten as $5B^2 - 8AB + 3 < 0$. This gives $A < -\sqrt{15}/4$ and

$$\frac{4A}{5} - \frac{\sqrt{16A^2-15}}{5} < B < \frac{4A}{5} + \frac{\sqrt{16A^2-15}}{5}$$

which is not possible as it leads to

$$A < \frac{4A}{5} - \frac{\sqrt{16A^2-15}}{5} < B.$$

Therefore the case $B < 0$ is unattainable. Thus the inequalities (3.2) holds for

$$\frac{3}{5} \leq B < 1 \quad \text{and} \quad \frac{3+5B^2}{8B} \leq A. \tag{3.3}$$

Case 2: For $5/8 \leq a \leq 11/8$, Theorem 2.1 infers $r \leq \sqrt{(5/4) - 2a + a^2}$. This gives

$$5(1 - B^2) \leq 8(1 - AB) \leq 11(1 - B^2) \tag{3.4}$$

and

$$\frac{5}{4} - 2\left(\frac{1 - AB}{1 - B^2}\right) + \left(\frac{1 - AB}{1 - B^2}\right)^2 - \left(\frac{A - B}{1 - B^2}\right)^2 \geq 0. \tag{3.5}$$

The inequality (3.5) can be further implied to

$$\frac{1 - 4A^2 + 8AB - 5B^2}{4 - 4B^2} \geq 0,$$

which holds for $3/5 \leq B < 1$, whatever A may be. Also, it is satisfied if

$$B < A \leq B + \frac{\sqrt{1 - B^2}}{2} \quad \text{and} - 1 < B \leq \frac{3}{5}.$$

We now consider the following subcases:

Subcase 1: Let $B > 0$. The first part of inequality (3.4) gives $8AB \leq (3 + 5B^2)$. This holds for $3/5 \leq B < 1$ and $B < A \leq (3 + 5B^2)/(8B)$. The second part of inequality (3.4) holds true for every $B > 0$.

Subcase 2: For $B < 0$, the first part of inequality (3.4) is always satisfied, while its second part holds if $-1 < B \leq -3/11$ and $B < A \leq (-3{+}11B^2)/(8B)$. Moreover, the condition $-3/11 \leq B < 0$ assures that the second inequality is satisfied, whatever A may be.

Combining the above two subcases and by making use of the observation that

$$\min\left\{\frac{-3 + 11B^2}{8B}, B + \frac{\sqrt{1 - B^2}}{2}\right\} = \frac{-3 + 11B^2}{8B}, \quad -1 < B \leq -\frac{3}{5},$$

we conclude that the inequalities (3.4) and (3.5) holds for the following constraints:

$$-1 < B \leq -\frac{3}{5}, \quad B < A \leq \frac{-3 + 11B^2}{8B}; \tag{3.6}$$

$$-\frac{3}{5} \leq B \leq \frac{3}{5}, \quad B < A \leq B + \frac{\sqrt{1 - B^2}}{2} \tag{3.7}$$

and

$$\frac{3}{5} \leq B < 1, \quad B < A \leq \frac{3 + 5B^2}{8B}. \tag{3.8}$$

Case 3: For $11/8 \leq a < 2$, Theorem 2.1 provides $r \leq 2 - a$ and substituting the values of a and r, we obtain

$$11(1 - B^2) \leq 8(1 - AB) \leq 16(1 - B^2), \tag{3.9}$$

$$\frac{A - B}{1 - B^2} \leq 2 - \frac{1 - AB}{1 - B^2}. \tag{3.10}$$

From the first part of inequality (3.9), $-1 < B \leq -3/11$ and $(-3+11B^2)/(8B) \leq A \leq 1$ and the second part is satisfied provided $-1/2 \leq B < 1$ (for all A) or $B < A \leq (-1+2B^2)/B$ with $-1 < B \leq -1/2$. The inequality (3.10) holds if $B \geq 0$ (for all A) or $A \leq 1+2B$ with $B \leq 0$. As a result, (3.9) and (3.10) are not simultaneously satisfied if $B \geq 0$.

Note that $\max\{B, (-3+11B^2)/(8B)\} = (-3+11B^2)/(8B)$ and $\min\{1, (-1+2B^2)/B, 1+2B\} = 1+2B$ for $B < 0$. Also, the condition $(-3+11B^2)/(8B) < 1+2B$ is satisfied only if $-1 < B \leq -3/5$ or $B > 0$. These observations lead us to conclude that the conditions (3.9) and (3.10) are collectively satisfied if

$$-1 < B \leq -\frac{3}{5}, \quad \frac{-3+11B^2}{8B} \leq A \leq 1+2B. \tag{3.11}$$

The conditions (3.6) and (3.11) combine together to deduce part (i) of the theorem. The part (ii) is exactly (3.7). Similarly, the part (iii) follows by amalgamating (3.3) and (3.8). This completes the proof of the theorem. □

The subclass $\mathcal{S}^*[A, B]$ of starlike functions has been widely studied for different choices of A and B. For instance, the classes $\mathcal{S}^*[1, -(M-1)/M]$ $(M > 1/2)$, $\mathcal{S}^*[1-\alpha, 0]$ $(0 \leq \alpha < 1)$ and $\mathcal{S}^*[\alpha, -\alpha]$ $(0 < \alpha \leq 1)$ were introduced by Janowski [9], Singh [26] and Padmanabhan [21], respectively. We shall now discuss their inclusion relations with $\mathcal{S}^*_{ev}$.

Corollary 3.4. *The following inclusion relations hold for* $\mathcal{S}^*_{ev}$:

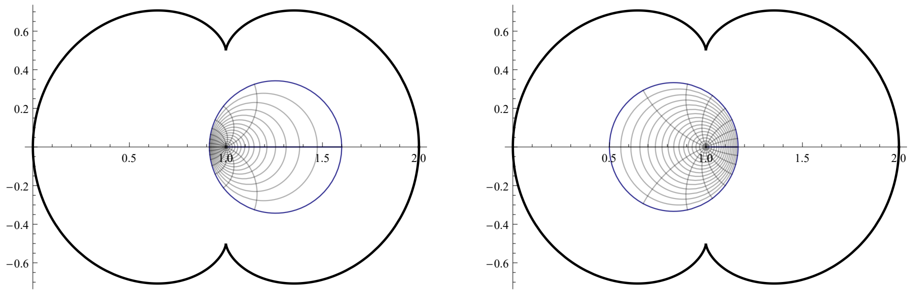
(i) $\mathcal{S}^*[1, -(M-1)/M] \subseteq \mathcal{S}^*_{ev}$ *for* $1/2 < M \leq M_0$, *where* $M_0 = 5/8$ *and* $\mathcal{S}^*_{ev} \subseteq \mathcal{S}^*[1, 0]$.
(ii) $\mathcal{S}^*[1-\alpha, 0] \subseteq \mathcal{S}^*_{ev}$ *for* $1/2 \leq \alpha < 1$.
(iii) $\mathcal{S}^*[\alpha, -\alpha] \subseteq \mathcal{S}^*_{ev}$ *for* $0 < \alpha \leq \alpha_0$, *where* $\alpha_0 = 1/\sqrt{17} \approx 0.242536$.

Proof. For (i), let $f \in \mathcal{S}^*[1, -(M-1)/M]$. On taking $A = 1$ and $B = -(M-1)/M$ in Theorem 3.3(ii) we obtain the desired result. Also, the second inclusion of Theorem 2.1 gives $\mathcal{S}^*_{ev} \subseteq \mathcal{S}^*[1, 0]$. The proof of rest of the two parts also follows by Theorem 3.3(ii) for particular choices of A and B. □

To illustrate parts (i) and (iii) of Theorem 3.3, observe that $\mathcal{S}^*[-3/5, -3/4]$ and $\mathcal{S}^*[3/4, 1/2]$ are subclasses of $\mathcal{S}^*_{ev}$, which is depicted geometrically in Figure 6.

4. Application to partial sums

This section is devoted to discuss the radius problems for the second partial sum of a function belonging to several well-known subclasses of starlike functions. With $\phi_{\leftmoon}(z) = z + \sqrt{1+z^2}$, $\phi_R(z) = 1 + (kz+z^2)/(k^2-kz)$, where $k = \sqrt{2}+1$, $\phi_C(z) = 1 + 4z/3 + 2z^2/3$, $\phi_{lim}(z) = 1 + \sqrt{2}z + z^2/2$, $\phi_s(z) = 1+\sin(z)$, $\phi_{SG}(z) = 2/(1+e^{-z})$, $\phi_{Ne}(z) = 1+z-z^3/3$, $\phi_{\wp}(z) = 1+ze^z$, $\phi_{car}(z) = 1+z+z^2/2$ and $\phi_{sh}(z) = 1+\sinh^{-1}(z)$, suppose that their corresponding Ma and Minda classes are denoted by $\mathcal{S}^*_{\leftmoon} := \mathcal{S}^*(\phi_{\leftmoon})$, $\mathcal{S}^*_R := \mathcal{S}^*(\phi_R)$,



(a) $A = -3/5,\ B = -3/4$ (b) $A = 3/4,\ B = 1/2$

FIGURE 6. Illustration of parts (i) and (iii) of Theorem 3.3

$\mathcal{S}^*_C := \mathcal{S}^*(\phi_C)$, $\mathcal{S}^*_{lim} := \mathcal{S}^*(\phi_{lim})$, $\mathcal{S}^*_s := \mathcal{S}^*(\phi_s)$, $\mathcal{S}^*_{SG} := \mathcal{S}^*(\phi_{SG})$, $\mathcal{S}^*_{Ne} := \mathcal{S}^*(\phi_{Ne})$, $\mathcal{S}^*_\wp := \mathcal{S}^*(\phi_\wp)$, $\mathcal{S}^*_{car} := \mathcal{S}^*(\phi_{car})$ and $\mathcal{S}^*_{sh} := \mathcal{S}^*(\phi_{sh})$, respectively. These classes were studied in [3, 4, 7, 8, 13, 14, 22, 25, 30, 32].

Before stating the results for the second partial sum, we will determine the conditions under which the functions $z + a_n z^n$ $(n = 2, 3, 4, \ldots)$ and $z/(1 - Az)^2$ belong to $\mathcal{S}^*_{ev}$.

Lemma 4.1. (i) *A function* $f_n(z) = z + a_n z^n$ $(n = 2, 3, 4, \ldots)$ *belongs to the class* $\mathcal{S}^*_{ev}$ *if and only if* $|a_n| \le \sqrt{1/(5 - 8n + 4n^2)}$ *for* $n = 2, 3, 4, \ldots$.

(ii) *The function* $g(z) = z/(1 - Az)^2 \in \mathcal{S}^*_{ev}$ *if* $|A| \le 1/\sqrt{17}$.

Proof. (i) The values of $w = zf_n'(z)/f_n(z)$ lies in the disk

$$\left| w - \frac{1 - n|a_n|^2}{1 - |a_n|^2} \right| < \frac{(n-1)|a_n|}{1 - |a_n|^2}, \tag{4.1}$$

with centre $a = (1 - n|a_n|^2)/(1 - |a_n|^2)$ and radius $r = ((n-1)|a_n|)/(1 - |a_n|^2)$. Also, as $f_n \in \mathcal{S}^*$ therefore $|a_n| \le 1/n$ which gives $a \le 1$. The case $0 < a \le 5/8$ and $r \le a$ leads to an erroneous condition. Therefore we shall consider the case $5/8 \le a \le 1$ and $r \le \sqrt{(5/4) - 2a + a^2}$ by Theorem 2.1. Substituting the values of a and r, we obtain

$$5(1 - |a_n|^2) \le 8(1 - n|a_n|^2) \le 8(1 - |a_n|^2) \tag{4.2}$$

and

$$\frac{(n-1)|a_n|}{1 - |a_n|^2} \le \frac{1}{2}\sqrt{\frac{1 - 2|a_n|^2 + |a_n|^4(4n^2 - 8n + 5)}{(1 - |a_n|^2)^2}}. \tag{4.3}$$

The inequality (4.2) is always satisfied. However, (4.3) leads to

$$|a_n| \le \sqrt{1/(5 - 8n + 4n^2)} \quad \text{for } n = 2, 3, 4, \ldots.$$

(ii) Let $g(z) = z/(1-Az)^2$. For $A = 1$, clearly the Koebe function $z/(1-z)^2$ does not belong to $\mathcal{S}^*_{ev}$. For $A \neq 1$, consider the disk

$$\left|\frac{zg'(z)}{g(z)} - \frac{1+|A|^2}{1-|A|^2}\right| < \frac{2|A|}{1-|A|^2} \tag{4.4}$$

with centre $a = (1+|A|^2)/(1-|A|^2)$ and radius $r = (2|A|)/(1-|A|^2)$. It can be easily seen that $a > 1$. Since the case $11/8 \leq a < 2$ is not possible, therefore in view of Theorem 2.1, the disk (4.4) lies in $\phi_{ev}(\mathbb{D})$ if $1 < a \leq 11/8$ and $r \leq \sqrt{5/4 - 2a + a^2}$ which simplifies to $|A| \leq 1/\sqrt{17}$. □

The following result investigates the radius problems for the second partial sum of a function $f \in \mathcal{S}^*_{ev}$. The function f_{ev} defined by (1.4) verifies the sharpness of the derived radius constants. Denote its second partial sum by ς, which is given by $\varsigma(z) = z + 3z^2/4$.

Theorem 4.2. *Let $f(z) = z + b_2z^2 + b_3z^3 + \cdots \in \mathcal{S}^*_{ev}$ and its second partial sum be given by $f_2(z) = z + b_2z^2$. Then f_2 is starlike in $|z| < 2/3$ and convex in $|z| < 1/3$. Moreover, $f_2(\rho z)/\rho \in \mathfrak{F}$ for $0 < \rho \leq R$ for the subclasses $\mathfrak{F} \subseteq \mathcal{S}^*$ and radii R mentioned in Table 1. All bounds are best possible.*

Proof. Since $f \in \mathcal{S}^*_{ev}$, $|b_2| \leq 3/4$ and

$$\operatorname{Re}\left(\frac{zf_2'(z)}{f_2(z)}\right) = \operatorname{Re}\left(1 + \frac{b_2z}{1+b_2z}\right) \geq 1 - \frac{|b_2||z|}{1-|b_2||z|} \geq \frac{4-6|z|}{4-3|z|}$$

which is positive for $|z| < 2/3$. Also, a function $f(z)$ is convex in $|z| < r$ if and only if $zf'(z)$ is starlike in $|z| < r$. This gives f_2 is convex in $|z| < 1/3$. For sharpness, consider the function f_{ev} whose second partial sum ς satisfies

$$\left.\frac{z\varsigma'(z)}{\varsigma(z)}\right|_{z=-2/3} = 0 \quad \text{and} \quad 1 + \left.\frac{z\varsigma''(z)}{\varsigma'(z)}\right|_{z=-1/3} = 0.$$

This verifies the sharpness of the constants $2/3$ and $1/3$, respectively.

For proving (a), note that the disk (4.1) lies inside $\phi_L(\mathbb{D})$ if

$$\frac{|a_2|}{1-|a_2|^2} \leq \left(\sqrt{1 - \left(\frac{1-2|a_2|^2}{1-|a_2|^2}\right)^2} - \left(1 - \left(\frac{1-2|a_2|^2}{1-|a_2|^2}\right)^2\right)\right)^{1/2}$$

by [1, Lemma 2.2, p. 6559]. This proves that the function $z + a_2z^2$ belongs to the class $\mathcal{S}^*_L$ if and only if $|a_2| \leq 1/\sqrt{6}$. Accordingly if $\rho \leq 4/(3\sqrt{6}) := R_1$, then $\rho|b_2| \leq 1/\sqrt{6}$ and hence $f_2(\rho z)/\rho = z + \rho b_2z^2 \in \mathcal{S}^*_L$. For sharpness, observe that

$$\left(\frac{z\varsigma'(z)}{\varsigma(z)}\right)^2 - 1 = \frac{3z(8+9z)}{(4+3z)^2}.$$

For $z = re^{it}$, we have

$$\left|\left(\frac{z\varsigma'(z)}{\varsigma(z)}\right)^2 - 1\right|^2 = \frac{9r^2(64 + 81r^2 + 144r\cos t)}{(16 + 9r^2 + 24r\cos t)^2}.$$

TABLE 1. Radii constants concerning the second partial sum

S.No	*Class* $\mathcal{S}^*(\phi)$	*Radius*
(a)	$\mathcal{S}^*_L$	$\dfrac{4}{3\sqrt{6}} \approx 0.544331$
(b)	$\mathcal{S}^*_{RL}$	$\dfrac{4}{3}\sqrt{\dfrac{21\sqrt{2}-29-\sqrt{18\sqrt{2}-25}}{2(22\sqrt{2}-31)}} \approx 0.428897$
(c)	$\mathcal{S}^*_e$	$\dfrac{4(e-1)}{3(2e-1)} \approx 0.5164$
(d)	$\mathcal{S}^*_{\leftmoon}$	$\dfrac{4(2-\sqrt{2})}{3(3-\sqrt{2})} \approx 0.492531$
(e)	$\mathcal{S}^*_R$	$\dfrac{2(3-2\sqrt{2})}{3(2-\sqrt{2})} \approx 0.195262$
(f)	$\mathcal{S}^*_C$	$\dfrac{8}{15} \approx 0.533333$
(g)	$\mathcal{S}^*_{lim}$	$\dfrac{4(9-4\sqrt{2})}{21} \approx 0.63679$
(h)	$\mathcal{S}^*_\varrho$	$\dfrac{4(\cos(1)-1)}{3(\cos(1)-2)} \approx 0.419902$
(i)	$\mathcal{S}^*_{SG}$	$\dfrac{2(e-1)}{3e} \approx 0.421414$
(j)	$\mathcal{S}^*_{Ne}$	$\dfrac{8}{15} \approx 0.533333$
(k)	$\mathcal{S}^*_\wp$	$\dfrac{4}{3(1+e)} \approx 0.358589$
(l)	$\mathcal{S}^*_{car}$	$\dfrac{4}{9} \approx 0.444444$
(m)	$\mathcal{S}^*_{sh}$	$\dfrac{4\sinh^{-1}(1)}{3(1+\sinh^{-1}(1))} \approx 0.624631$
(n)	$\mathcal{S}^*_s$	$\dfrac{4\sin(1)}{3+3\sin(1)} \approx 0.609274$

Set $r = R_1$ and choose $t_0 \in [0, 2\pi]$ such that $\cos t_0 = -\sqrt{2/3} \approx -0.816497$, then a simple calculation shows that

$$\left| \left(\frac{R_1 e^{it_0} \varsigma'(R_1 e^{it_0})}{\varsigma(R_1 e^{it_0})} \right)^2 - 1 \right|^2 = \frac{9r^2(64 + 81r^2 + 144r\cos t_0)}{(16 + 9r^2 + 24r\cos t_0)^2} = 1$$

which proves that the obtained bound is sharp.

For part (b), note that $z + a_2z^2 \in \mathcal{S}^*_{RL}$ if and only if

$$|a_2| \leq \sqrt{\frac{(21\sqrt{2} - 29 - \sqrt{18\sqrt{2} - 25}}{2(22\sqrt{2} - 31)}} := \delta$$

by [18, Theorem 2.3, p. 7]. Consequently, it follows that if $\rho \leq 4\delta/3 := R_2$, then $\rho|b_2| \leq \delta$ so that the second partial sum $f_2(\rho z)/\rho$ belongs to the class $\mathcal{S}^*_{RL}$. Its sharpness can be deduced by a long and tedious calculation as carried out in part (a) and therefore its details are omitted.

To prove part (c), [19, Theorem 2.6(i), p. 374] shows that $z + a_2z^2 \in \mathcal{S}^*_e$ if and only if $|a_2| \leq (e-1)/(2e-1)$. Thus for $\rho \leq 4(e-1)/(6e-3) := R_3$, as $\rho|b_2| \leq (e-1)/(2e-1)$ we must have $f_2(\rho z)/\rho \in \mathcal{S}^*_e$. Also we see that the radius R_3 is sharp as $z\varsigma'(z)/\varsigma(z)$ equals $\phi_e(-1) = 1/e$ at $z = -R_3$.

In part (d), by [22, Corollary 2.9, p. 978] for $n = 2$, $z + a_2z^2$ belongs to the class $\mathcal{S}^*_{\leftmoon}$ if and only if $|a_2| \leq (2-\sqrt{2})/(3-\sqrt{2})$. Therefore, it is easy to deduce that $f_2(\rho z)/\rho \in \mathcal{S}^*_{\leftmoon}$ for $\rho \leq 4(2-\sqrt{2})/(9-3\sqrt{2}) := R_4$. The computation $-R_4\varsigma'(-R_4)/\varsigma(-R_4) = \phi_{\leftmoon}(-1) = \sqrt{2}-1$ verifies the sharpness.

Since the function $z+a_2z^2 \in \mathcal{S}^*_R$ if and only if $|a_2| \leq (3-2\sqrt{2})/(4-2\sqrt{2})$ by [14, Theorem 2.3(i), p. 203], it follows that if $\rho \leq 4(3-2\sqrt{2})/(12-6\sqrt{2}) := R_5$, then $f_2(\rho z)/\rho \in \mathcal{S}^*_R$. This bound is best possible as $z\varsigma'(z)/\varsigma(z) = \phi_R(-1) = 2(\sqrt{2}-1)$ at $z = -R_5$. This substantiates the proof for part (e).

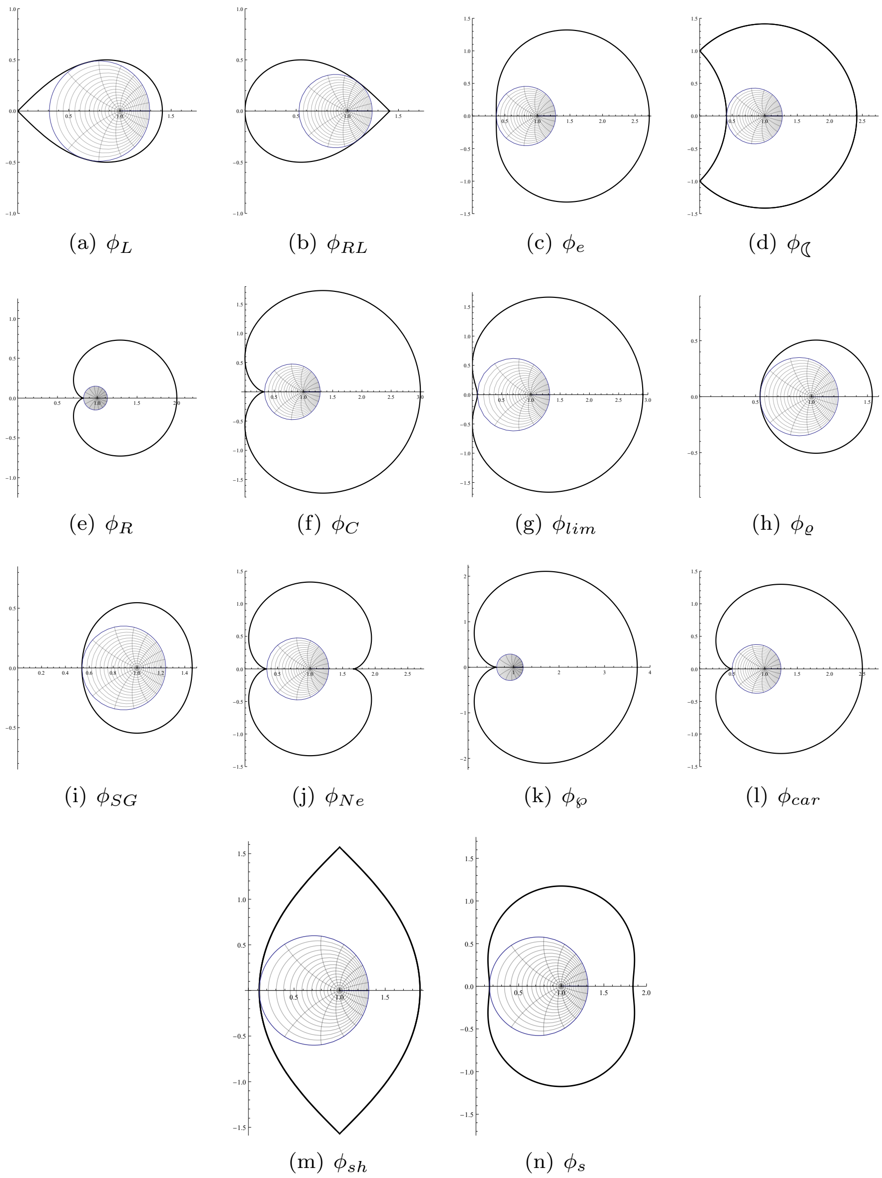
The proof for part (f) is a consequence of [25, Theorem 2.4(i), p. 926], wherein for $n = 2$ the function $z + a_2z^2 \in \mathcal{S}^*_C$ if and only if $|a_2| \leq 2/5$. This infers that the second partial sum $f_2(\rho z)/\rho \in \mathcal{S}^*_C$, where $\rho \leq 8/15 := R_6$ and since $z\varsigma'(z)/\varsigma(z) = \phi_C(-1) = 1/3$ at $z = -R_6$, the bound cannot be improved.

For (g), in view of [17, Proposition 1, p. 9] for $s = 1/\sqrt{2}$ and $n = 2$, the function $z + a_2z^2 \in \mathcal{S}^*_{lim}$ if and only if $|a_2| \leq (9-4\sqrt{2})/7$ which gives $f_2(\rho z)/\rho \in \mathcal{S}^*_{lim}$ for $\rho \leq 4(9-4\sqrt{2})/21 := R_7$. The result is sharp for the function ς, as $z\varsigma'(z)/\varsigma(z) = \phi_{lim}(-1) = (3-2\sqrt{2})/2$ at $z = -R_7$.

For (h), we shall employ [20, Example 2(i)] which states that $z + a_2z^2 \in \mathcal{S}^*_\varrho$ if and only if $|a_2| \leq (\cos(1)-1)/(\cos(1)-2)$. The desired radius in this case is $\rho \leq 4(\cos(1)-1)/(3(\cos(1)-2)) := R_8$ which is sharp as depicted by the computation $z\varsigma'(z)/\varsigma(z) = \phi_\varrho(-1) = \cos 1$ at $z = -R_8$.

The part (i) can be deduced from [7, Example 4.1(i), p. 967]. However, there has been a minor typographical misprint in their statement and its correct version may be stated as "the function $z + a_2z^2 \in \mathcal{S}^*_{SG}$ if and only if $|a_2| \leq (e-1)/2e$". Thus $f_2(\rho z)/\rho \in \mathcal{S}^*_{SG}$ for $\rho \leq 2(e-1)/(3e) := R_9$. The calculation $z\varsigma'(z)/\varsigma(z) = \phi_{SG}(-1) = 2/(1+e)$ at $z = -R_9$ establishes the sharpness.

The part (j) follows on similar lines as that of part (f) with the same radius $R_{10} = 8/15$ since the function $z + a_2z^2 \in \mathcal{S}^*_{Ne}$ if and only if $|a_2| \leq 2/5$ [31, Theorem 4.3(i), p. 93].



(a) ϕ_L (b) ϕ_{RL} (c) ϕ_e (d) $\phi_{☾}$

(e) ϕ_R (f) ϕ_C (g) ϕ_{lim} (h) ϕ_{ϱ}

(i) ϕ_{SG} (j) ϕ_{Ne} (k) $\phi_{\wp}$ (l) ϕ_{car}

(m) ϕ_{sh} (n) ϕ_s

FIGURE 7. Sharpness of Radii Constants in Theorem 3.1

For part (k), by [13, Example 4.1(ii), p. 24] $z + a_2z^2 \in \mathcal{S}^*_{\wp}$ if and only if $|a_2| \leq 1/(e+1)$ which proves that $f_2(\rho z)/\rho \in \mathcal{S}^*_{\wp}$ for $\rho \leq 4/(3+3e) := R_{11}$.

The value $z\varsigma'(z)/\varsigma(z) = \phi_{\wp}(-1) = 1-(1/e)$ at $z = -R_{11}$ asserts that the result is sharp.

As $z + a_2 z^2 \in \mathcal{S}^*_{car}$ if and only if $|a_2| \leq 1/3$ by [8, Corollary 3.3(ii), p. 1155], the part (l) is satisfied for $\rho \leq 4/9 := R_{12}$. Since $z\varsigma'(z)/\varsigma(z) = \phi_{car}(-1) = 1/2$ at $z = -R_{12}$, the bound is best possible.

By making use of [3, Lemma 2.6, p. 997], the disk (4.1) lies inside $\phi_{sh}(\mathbb{D})$ for $n = 2$, if

$$\frac{|a_2|}{1-|a_2|^2} \leq \frac{1-2|a_2|^2}{1-|a_2|^2} - (1 - \sinh^{-1} 1)$$

which gives $z + a_2 z^2 \in \mathcal{S}^*_{sh}$ if and only if $|a_2| \leq (\sinh^{-1} 1)/(1 + \sinh^{-1} 1)$. The conclusion of part (m) holds provided $\rho \leq 4(\sinh^{-1} 1)/(3 + 3\sinh^{-1} 1) := R_{13}$. This bound is sharp as the function $z\varsigma'(z)/\varsigma(z)$ assumes the value $\phi_{sh}(-1) = 1 - \sinh^{-1} 1$ at $z = -R_{13}$.

For the last part (n), according to [4, Example 3.4(a), p. 220], $z + a_2 z^2 \in \mathcal{S}^*_s$ if and only if $|a_2| \leq (\sin 1)/(1 + \sin 1)$. As a result, for $\rho \leq 4(\sin 1)/(3 + 3\sin 1) := R_{14}$, the function $f_2(\rho z)/\rho \in \mathcal{S}^*_s$. For sharpness, note that the value of $z\varsigma'(z)/\varsigma(z)$ reduces to $\phi_s(-1) = 1 - \sin 1$ at $z = -R_{14}$.

The sharpness in each part of this theorem is illustrated in Figure 7 wherein the shaded image domains depict the plot of $z\varsigma'(z)/\varsigma(z)$ under the subdisk $|z| < R_i$, $i = 1, 2, \ldots, 14$. These image domains lie inside $\phi(\mathbb{D})$ associated with the Ma and Minda classes $\mathcal{S}^*(\phi)$ taken in each part of the theorem. □

For different choices of a starlike function f, the last result of this section makes use of Lemma 4.1 to determine the sharp radii for its second partial sum f_2 to belong to the class $\mathcal{S}^*_{ev}$.

Theorem 4.3. *Let $f \in \mathfrak{F} \subseteq \mathcal{S}^*$ and $f_2(z) = z + a_2 z^2$ be the second partial sum of f.*

(i) *If $\mathfrak{F} = \mathcal{S}^*_e$, $\mathcal{S}^*_{Ne}$, $\mathcal{S}^*_s$, $\mathcal{S}^*_{\wp}$, $\mathcal{S}^*_{sh}$, $\mathcal{S}^*_{\mathbb{C}}$ or $\mathcal{S}^*_{car}$, then $f_2(\rho z)/\rho \in \mathcal{S}^*_{ev}$ for $\rho \leq 1/\sqrt{5} \approx 0.447214$.*

(ii) *If $\mathfrak{F} = \mathcal{S}^*_L$, $\mathcal{S}^*_{SG}$ or $\mathcal{S}^*_{\varrho}$, then $f_2(\rho z)/\rho \in \mathcal{S}^*_{ev}$ for $\rho \leq 2/\sqrt{5} \approx 0.894427$.*

(iii) *If $\mathfrak{F} = \mathcal{S}^*_{RL}$ or $\mathcal{S}^*_R$, then $f_2 \in \mathcal{S}^*_{ev}$.*

(iv) *If $\mathfrak{F} = \mathcal{S}^*_C$, then $f_2(\rho z)/\rho \in \mathcal{S}^*_{ev}$ for $\rho \leq 3/(4\sqrt{5}) \approx 0.3341$.*

(v) *If $\mathfrak{F} = \mathcal{S}^*_{lim}$, then $f_2(\rho z)/\rho \in \mathcal{S}^*_{ev}$ for $\rho \leq 1/\sqrt{10} \approx 0.316228$.*

(vi) *If $\mathfrak{F} = \mathcal{S}^*_{ev}$, then $f_2(\rho z)/\rho \in \mathcal{S}^*_{ev}$ for $\rho \leq 4/(3\sqrt{5}) \approx 0.596284$.*

All the radii are sharp.

Proof. (i) If $f \in \mathfrak{F}$, then $|a_2| \leq 1$ by [19, Theorem 2.3, p. 372], [31, Theorem 4.4, p. 97], [4, p. 215], [13, Theorem 4.2, p. 26], [3], [23, Theorem 2.1, p. 1429] and [8, p. 1148]. If $\rho \leq 1/\sqrt{5}$, then $\rho|a_2| \leq 1/\sqrt{5}$ so that $f_2(\rho z)/\rho \in \mathcal{S}^*_{ev}$ by Lemma 4.1. For sharpness, note that the second partial sum of the extremal functions in each class $\mathfrak{F}$ is given by $h_1(z) = z + z^2$. The point $z_0 = -(1/5) + 2i/5$ lies on the circle $|z| = 1/\sqrt{5}$ and

$$\left.\frac{zh_1'(z)}{h_1(z)}\right|_{z=z_0} = 1 + \frac{i}{2}$$

which corresponds to the cusp and hence the bound $1/\sqrt{5}$ is best possible.

The part (ii) follows on similar lines by using the bounds for the absolute value of the second coefficient of the functions in the specified subclasses of starlike functions. In this case, if $f \in \mathcal{S}_L^*$, $\mathcal{S}_{SG}^*$ or $\mathcal{S}_\varrho^*$, then $|a_2| \leq 1/2$ by [28, Theorem 2, p. 352], [7, Theorem 4.1, p. 966] and [20], respectively, so that $f_2(\rho z)/\rho \in \mathcal{S}_{ev}^*$ for $\rho \leq 2/\sqrt{5}$. The extremal function for the mentioned classes generates the second partial sum $h_2(z) = z+(1/2)z^2$ and the expression $zh_2'(z)/h_2(z) = 2(1+z)/(2+z)$ equals $1+(i/2)$ at the point $z_0 = -(2/5)+4i/5$ which establishes the sharpness.

In part (iii) for $f \in \mathcal{S}_{RL}^*$, $|a_2| \leq (5-3\sqrt{2})/2$ by [18, Theorem 2.2(vi), p. 6] and for $f \in \mathcal{S}_R^*$, $|a_2| \leq \sqrt{2}-1$ by [14, p. 201]. This shows that $|a_2| \leq 1/\sqrt{5}$ in both the cases and thus $f \in \mathcal{S}_{ev}^*$ by Lemma 4.1.

For the other classes listed in parts (iv), (v) and (vi), $|a_2|$ is bounded by the numbers 4/3 [25, p. 924], $\sqrt{2}$ [32, Theorem 6, p. 7] and 3/4 for f belonging to the classes $\mathcal{S}_C^*$, $\mathcal{S}_{lim}^*$ and $\mathcal{S}_{ev}^*$, respectively. Here, $h_3(z) = z + (4/3)z^2$, $h_4(z) = z + \sqrt{2}z^2$ and $h_5(z) = z + (3/4)z^2$ are the second partial sums of the extremal functions for the classes $\mathcal{S}_C^*$, $\mathcal{S}_{lim}^*$ and $\mathcal{S}_{ev}^*$, respectively. The quantities

$$\frac{zh_3'(z)}{h_3(z)} = \frac{3+8z}{3+4z}, \quad \frac{zh_4'(z)}{h_4(z)} = \frac{1+2\sqrt{2}z}{1+\sqrt{2}z} \quad \text{and} \quad \frac{zh_5'(z)}{h_5(z)} = \frac{4+6z}{4+3z}$$

assume the value $1+(i/2)$ at the points $3(-1+2i)/20$, $(-1+2i)/(5\sqrt{2})$ and $4(-1+2i)/15$ that lie on the circle with radius $3/(4\sqrt{5})$, $1/\sqrt{10}$ and $4/(3\sqrt{5})$ respectively which show that the bounds are sharp. □

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