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ON A GENERALIZATION OF UNIT REGULAR RINGS

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Abstract. In this paper, we introduce a class of rings generalizing unit regular rings and being a subclass of semipotent rings, which is called idempotent unit regular. We call a ring R an idempotent unit regular ring if for all $r \in R - J(R)$, there exist a non-zero idempotent e and a unit element u in R such that $er = eu$, where this condition is left and right symmetric. Thus, we have also that there exist a non-zero idempotent e and a unit u such that $ere = eue$ for all $r \in R - J(R)$. Various basic characterizations and properties of this class of rings are proved and it is given the relationships between this class of rings and some well-known classes of rings such as semiperfect, clean, exchange and semipotent. Moreover, we obtain some results about when the endomorphism ring of a module in a class of left R-modules X is idempotent unit regular.

1. Introduction

Throughout this paper, R means an associative ring with identity. We write $J(R)$ (or J) for the Jacobson radical of R and $Nil(R)$ for the set of nilpotent elements of R. A ring R is called a unit regular ring if for all $r \in R$ there exists a unit element u such that $rur = r$ which is introduced by Ehrlich in [9] and the set of all unit regular elements is denoted by $ur(R)$. If $R = ur(R)$, then it is called a unit regular ring. The properties of unit regular rings have been extensively studied in the literature (see [3], [9] and [10]). It is easily proved that R is unit regular if and only if every element of R is the product of an idempotent and an invertible element and hence, we see that if a ring R is unit regular, then $J(R) = 0$ and in addition, a unit regular ring is an idempotent unit regular ring, but the converse may not be satisfied. This paper is organized as follows:

Firstly, in Section 2, we give some basic properties and examples of idempotent unit regular rings and in addition, we obtain the relationships between these rings and some very well-known rings such as semiperfect, J-clean, nil clean, clean, exchange and semipotent rings. In Section 3, we investigate the

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rings of upper(lower) triangular matrices and any $n \times n$ matrices over idempotent unit regular rings. Finally, in Section 4, we prove that if $\mathrm{End}(M_R)$ is the ring of endomorphisms on a semisimple module $0 \neq M_R$, then we prove that $\mathrm{End}(M_R)$ is an idempotent unit regular ring and in addition, for any projective R-module P_R over a right perfect ring R, the endomorphism ring $\mathrm{End}(P_R)$ is idempotent unit regular. In addition, we give some conditions to be that $\mathrm{End}(M_R)$ is an idempotent unit regular ring, where M_R is a nonsingular injective (continuous, discrete or quasi injective) module.

If every element not in $J(R)$ of a ring R is a sum of a unit and an idempotent which is not identity, then we call it a clean ring with idempotent not identity, which is introduced in [1]. Any idempotent unit regular ring may not be a clean ring. We can directly see that every clean ring with idempotent not identity is idempotent unit regular. But, we don't know whether or not any clean ring is an idempotent unit regular ring and also a clean ring with idempotent not identity.

Let's recall some well-known definitions. A ring R is semi-simple if ${}_RR$ is a semi-simple module. A ring R is semi-perfect if and only if $R/J(R)$ is semi-simple and idempotents lift modulo $J(R)$. A ring R is an exchange ring if, for every $r \in R$, there exist an idempotent $e \in rR$ such that $(1-e) \in (1-r)R$ (see [15]). For example, every regular ring (that is, a ring, such that for all $a \in R$, there exists $b \in R$ satisfying $a = aba$) is exchange. Clean rings were defined by Nicholson as a subclass of exchange rings in [15] and have been studied by many authors. A ring is called clean if every element is the sum of an idempotent and a unit. Every semi-perfect and unit regular ring are clean (see [3] and [11] for more details). A semi-potent ring is that there exist $0 \neq e = e^2$ and $r \in R$ such that $ar = e$ for all $a \in R - J(R)$ and this ring property is both left and right symmetric which was introduced by Nicholson in [14]. We can see that every idempotent unit regular ring is semi-potent since $ea = eu$ implies that $u^{-1}ea = u^{-1}eu$.

We denote $M_n(R)$ and $U_n(R)$ for the ring of all $n \times n$ matrices and the ring of all $n \times n$ upper triangular matrices over the ring R, respectively. Let E_{ij} be an $n \times n$ matrix with ij-th entry 1, otherwise 0. The ring of integers modulo n is denoted by $\mathbb{Z}_n$. We denote the polynomial ring over R by $R[x]$ and the formal power series over R by $R[[x]]$ with elements $a_0 + a_1x + \cdots + a_ix^i + \cdots$, where $a_i \in R$ for $i \in \mathbb{N}$.

2. Preliminary results

Definition 2.1. Let R be an associative ring with identity. $r \in R - J(R)$ is called an idempotent unit regular element if there exist a nonzero idempotent element e and a unit element u such that $er = eu$. We denote the set of idempotent unit regular elements by $iur(R)$. If $R = iur(R) \cup J(R)$, then it is called an idempotent unit regular ring. Thus, the class of idempotent unit regular rings is between the class of unit regular rings and the class of semipotent rings.

Lemma 2.2. *Idempotent unit regular property is left and right symmetric.*

Proof. Let $ea = eu$, where $0 \neq e$ idempotent and u unit. If we take as $e_1 = u^{-1}eu$ and $e_2 = au^{-1}e$, then we have that $ae_1 = e_2u$. Let $e_3 = u^{-1}e_2u$ and then $ae_1 = ue_3$. Since $e_1e_3 = e_1$ and $e_3e_1 = e_3$ and $(e_1 - e_3)^2 = 0$, we have that $ae_1 = ue_3 = u(1-e_1+e_3)(1+e_1-e_3)e_3 = u(1-e_1+e_3)e_1$, where $u(1-e_1+e_3)$ unit and $0 \neq e_1$ idempotent, as required. □

Example 2.3. (i) Every abelian regular ring is unit regular and hence, it is idempotent unit regular (see [9]).

(ii) $\mathbb{Z}_4$ is an idempotent unit regular ring (and also a clean ring with idempotent not identity), but it is not a unit regular ring.

(iii) Now, we give an example which is weakly clean, exchange and hence, semi-potent but not an idempotent unit regular. Let F be a field and $M_{\mathbb{N}}(F)$ be a ring of all infinite matrices over F with finite columns. Let $R = \{A = [a_{ij}] \in M_{\mathbb{N}}(F)$: there exists $n_A \in \mathbb{N}$ such that $a_{ij} = a_{(i+1)(j+1)}$ for every $i \geq n_A$ and $j \geq 1\}$. By [18, Example 3.1], we see that every idempotent and unit in R are upper triangular if we ignore the first finitely many rows. Then the product of a nonzero idempotent and a matrix (that are nonzero below the main diagonal ignoring the first finitely many rows) cannot be written as a product of a nonzero idempotent and a unit in R. Thus, R is not an idempotent unit regular ring. But it is a weakly clean ring and hence exchange and then semi-potent ring (see [18, Remark 2.4]).

(iv) Let $V = \oplus_{i=0}^{\infty}\mathbb{Q}$ and $S = \{(b, b, \ldots) : b \in \mathbb{Z}\}$. Let R be the subring of $\prod_{i=0}^{\infty}\mathbb{Q}$ generated by V and S. Then R is an idempotent unit regular abelian ring, but it is not exchange and hence it is not clean and not unit regular (see [19]). In general, let S be a subring of a ring T and $R(T, S) = \{(d_1, \ldots, d_n, c, c, \ldots) : n \geq 1, d_i \in T, c \in S\}$ which is a ring under componentwise addition and multiplication. Then $R(T, S)$ is idempotent unit regular if and only if so is T. Using also [3] we understand that the class of unit regular rings $\subset$ the class of clean rings with idempotent not identity $\subset$ the class of idempotent unit regular rings.

R is called a J (nil)-clean ring, where $J = J(R)$ in case for every element $a \in R$, there exist an idempotent e and an element $b \in J$ ($b \in Nil(R)$) such that $a = e + b$ (see [6] and [8]). We see that every J (nil)-clean ring is clean. We will prove in the following that such clean rings are also idempotent unit regular.

Lemma 2.4. *J-clean, nil-clean abelian and uniquely clean rings are clean rings with idempotent not identity and so, idempotent unit regular rings.*

Proof. Let R be a J-clean ring and $a \notin J$. Then we have an idempotent element e and an element $b \in J$ such that $a = e + b$. Thus, $a = 1 - e + (-1 + 2e + b)$, where $(-1+2e+b)$ is unit since $(-1+2e)^2=1$ and $e\neq 0$. Thus, R is a clean ring with idempotent not identity and so it is idempotent unit regular.

Let R be a nil-clean ring and $a\notin J$. Since R is nil-clean, there exist an idempotent e and $b\in N(R)$ such that $a=e+b$. We have that $N(R)\subseteq J(R)$ by [8], and hence R is J-clean which yields the required.

Let R be a uniquely clean ring. By Corollary 3.4 in [7] there exists a central idempotent $e\in R$ such that $e-a\in J(R)$ for any $a\in R$. If $a\notin J(R)$, then $a-e\in J(R)$ with $e\neq 0$ and hence R is J-clean which yields the required. □

Corollary 2.5. *Let R/J be a boolean ring and idempotents lift modulo J. Then R is an idempotent unit regular ring.*

Proof. Let $a+J$ be a non-zero element of R/J. Then $a^2-a\in J$ and hence there exists an idempotent $e\in R$ such that $a-e\in J$. Thus, we have an element $b\in J$ such that $a=e+b$ which says that R is J-clean. By Lemma 2.4, R is an idempotent unit regular ring. □

A ring R is called I-finite if it has no infinite orthogonal family of idempotents. It is proved in [14] that every primitive idempotent is local in a semi-potent ring and I-finite semi-potent ring is semi-perfect. We give the following lemma without proof.

Lemma 2.6. *The following are satisfied*:

(i) *Let $(R_i)_{i\in I}$ be a family of some rings. All the R_i are idempotent unit regular if and only if $\prod_{i\in I} R_i$ is idempotent unit regular.*

(ii) *Let $f: R\to S$ be a ring isomorphism with $f(1_R)=1_S$. If R is an idempotent unit regular ring, then so is S.*

(iii) *Every primitive idempotent in an idempotent unit regular ring is local.*

(iv) *Every I-finite idempotent unit regular ring is semi-perfect and hence, it is clean.*

Lemma 2.7. *If R is an idempotent regular ring, then so is R/J. The converse is satisfied if idempotents lift modulo J. Moreover, if I is a nil ideal and R/I is an idempotent unit regular, then so is R.*

Proof. Since non-zero idempotents are not in J, the first part is obvious. For the converse, if $a\notin J$, then there exist a non-zero idempotent $e\in R$ and a unit $u+J$ with $(e+J)(a+J)=(e+J)(u+J)$. Then we have that $ea-eu\in J$ and hence there exists $u_1\in R$ with $(u+ea-eu)u_1=1$. Thus, $(eu+ea-eu)u_1=e$ gives that $ea=eu_1^{-1}$.

We know that if I is a nil ideal, then idempotents lift modulo I. We can prove similar to the above proof that R is idempotent unit regular when R/I is idempotent unit regular. □

Proposition 2.8. *The following rings are clean rings with idempotent not identity and so, idempotent unit regular.*

(i) *Semi-simple rings,*

(ii) *Semi-perfect rings,*
(iii) *Artinian rings,*
(iv) *Finite rings.*

Proof. (i) A division ring D is unit regular. If R is a semi-simple ring, then $R \cong \prod_{i=1}^{n} M_{k_i}(D_i)$ by Wedderburn-Artin Theorem. If $f \in M_{k_i}(D_i)$, then it is equivalent to a diagonal matrix with idempotent entries by the Gaussian Elimination Method and so, f is unit regular. Thus, every semi-simple ring is a clean ring with idempotent not identity by [3] and also, idempotent unit regular.

(ii) If R is semi-perfect, then $R/J(R)$ is semi-simple and idempotents lift modulo $J(R)$. It follows from the part (i) that $R/J(R)$ is a clean ring with idempotent not identity. If $a \notin J(R)$, then we have that $a + J(R) = (e + u) + J(R)$, where e is idempotent not identity and u is unit. Then $a = e + u + j$, where $j \in J$, and hence R is a clean ring with idempotent not identity and so, idempotent unit regular.

(iii) and (iv) can be understood from the fact that finite rings and artinian rings are semi-perfect. $\square$

Proposition 2.9. *Let R be idempotent unit regular. If idempotents lift modulo $J(R)$, then we have the following*:

(i) $R[u]/J(R[u]) \cong R/J(R)$, *where* $R[x]/\langle x^{n+1}\rangle = R[u] = R + Ru + \cdots + Ru^n$, *where* $u = x + \langle x^{n+1}\rangle$ *such that* $u^{n+1} = 0$.
(ii) *Idempotents lift modulo* $J(R[u])$.
(iii) $R[x]/\langle x^{n+1}\rangle$ *is idempotent unit regular for* $n \geq 1$.

Proof. (i) We know that $J(R[u]) = J + \langle u\rangle$, where $\langle u\rangle$ is an ideal of $R[u]$ generated by u and hence, we have that $R[u]/J(R[u]) \cong R/J(R)$.

(ii) Let $e + J(R[u]) = a_0 + J(R[u])$ be idempotent in $R[u]/J(R[u])$, where $e = a_0 + a_1u + \cdots + a_nu^n$. Then $a_0^2 - a_0 \in J(R[u])$ and so $a_0^2 - a_0 \in J(R)$. Since idempotents lift modulo $J(R)$, there exists an idempotent $f \in R$ such that $a_0 - f \in J(R)$ and then $e - f \in J(R[u])$. Thus, idempotents lift modulo $J(R[u])$.

(iii) Since $R/J(R)$ is idempotent unit regular, so is $R[u]/J(R[u])$ by the part (i). Then $R[u]$ is idempotent unit regular by the part (ii) and Lemma 2.7. $\square$

Proposition 2.10. *R is idempotent unit regular if and only if the formal power series $R[[x]]$ is idempotent unit regular.*

Proof. Let $f = a + a_1x + a_2x^2 + \cdots \notin J(R[[x]]) = J + R[[x]]x$. Since R is an idempotent unit regular, there exist a non-zero idempotent e and a unit u with $ea = eu$. Then we have that $ef = e(u + a_1x + a_2x^2 + \cdots)$ with a unit $u + a_1x + a_2x^2 + \cdots$. Conversely, let $R[[x]]$ be an idempotent unit regular ring and $a \notin J$, then $a \notin J(R[[x]])$. Thus, there exist a non-zero idempotent $f^2 = f = b_0 + b_1x + b_2x^2 + \cdots$ and a unit $u = c_0 + c_1x + c_2x^2 + \cdots$ such that $fa = fu$ and hence, $b_0a = b_0c_0$, where $0 \neq b_0 = b_0^2$ and c_0 unit, as required. $\square$

Proposition 2.11. *Let R be a ring. Then the following are equivalent:*

(i) *R is idempotent unit regular.*
(ii) *For all $a \notin J$, $uav \in iur(R)$, where $u, v \in U(R)$.*
(iii) *R^{op} is idempotent unit regular.*

Proof. (i)⇒(ii) Since R is idempotent unit regular, for all $a \notin J$, $ea = eu_1$, where e is non-zero idempotent and u_1 is unit. Then $eu^{-1}ua = eu^{-1}uu_1$ and hence $ueu^{-1}uav = ueu^{-1}uu_1v$ and then $uav \in iur(R)$.

(ii)⇒(i) Let we have a non-zero idempotent e and a unit u_1 with $euav = eu_1$ for $a \notin J$. Thus, we hold such that $u^{-1}euav = u^{-1}euu^{-1}u_1$ and hence, $u^{-1}eua = u^{-1}euu^{-1}u_1v^{-1}$.

(i)⇔(iii) It follows from Lemma 2.2. □

3. Matrices over idempotent unit regular rings

In this section, we investigate the rings of upper(lower) triangular matrices and any $n \times n$ matrices over idempotent unit regular rings.

Proposition 3.1. *Let R and S be any two rings and M be an (R, S)-bimodule. Let $E = \left[\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right]$ be the formal triangular matrix ring. Then R and S are idempotent unit regular rings if and only if E is an idempotent unit regular ring.*

Proof. Let $I = \left[\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right]$, where we see that I is a nil ideal of E. Then we have that $E/I \cong T$, where $T = \left[\begin{smallmatrix} R & 0 \\ 0 & S \end{smallmatrix}\right]$ and $J(T) = \left[\begin{smallmatrix} J(R) & 0 \\ 0 & J(S) \end{smallmatrix}\right]$. We can understand easily that T is idempotent unit regular. Then E is idempotent unit regular by Lemma 2.7, too.

Conversely, let E be an idempotent unit regular ring and $a \in R - J(R)$. Then we hold that $\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right] \notin J(E)$ and hence, we have that a non-zero idempotent $\left[\begin{smallmatrix} e_1 & m \\ 0 & e_2 \end{smallmatrix}\right]$ and a unit $\left[\begin{smallmatrix} u_1 & m_1 \\ 0 & u_2 \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} e_1 & m \\ 0 & e_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} e_1 & m \\ 0 & e_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} u_1 & m_1 \\ 0 & u_2 \end{smallmatrix}\right]$ and we can see that $e_1 \neq 0$. Thus, $e_1a = e_1u_1$, where e_1 is a non-zero idempotent and u_1 is unit and hence, R is idempotent unit regular. Similarly, we can prove that so is S. □

Proposition 3.2. *Let R and S be any two rings, M and N be (R, S) and (S, R)-bimodules, respectively. Let $Y = \left[\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right]$ be the ring of Morita context (see [11]). If R and S are idempotent unit regular and $Y_0 = \left[\begin{smallmatrix} R & M_0 \\ N_0 & S \end{smallmatrix}\right]$, then so is the subring Y_0 of the Morita context ring Y, where $M_0 = \{x \in M : xN \subseteq J(R)\}$ and $N_0 = \{x \in N : xM \subseteq J(S)\}$.*

Proof. Let $A = \left[\begin{smallmatrix} a & m \\ n & b \end{smallmatrix}\right] \notin J(Y_0) = \left[\begin{smallmatrix} J(R) & M_0 \\ N_0 & J(S) \end{smallmatrix}\right]$. First, we can suppose that $a \notin J(R)$. Then we have that there exist a non-zero idempotent e and a unit u such that $ea = eu$. Thus, we hold that $\left[\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & m \\ n & b \end{smallmatrix}\right] = \left[\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} u & m \\ 0 & -1 \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} u & m \\ 0 & -1 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} u^{-1} & u^{-1}m \\ 0 & -1 \end{smallmatrix}\right]$.

Now, let $b \notin J(S)$. Then we have that there exist a non-zero idempotent e and a unit u such that $eb = eu$. Thus, we have that $\left[\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right]\left[\begin{smallmatrix} a & m \\ n & b \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right]\left[\begin{smallmatrix} -1 & 0 \\ n & u \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} -1 & 0 \\ n & u \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} -1 & 0 \\ u^{-1}n & u^{-1} \end{smallmatrix}\right]$. □

Using the induction on n, we can prove the following lemma.

Lemma 3.3. *The following are satisfied*:

(i) $U=[u_{ij}]\in U_n(R)$ *is invertible if and only if all the* u_{ii} *are invertible.*

(ii) *Let* $E=[e_{ij}]\in U_n(R)$ *be a nonzero idempotent. Then there exists at least one* $i\in\{1,\dots,n\}$ *such that* e_{ii} *is a nonzero idempotent.*

Proposition 3.4. *R is idempotent unit regular if and only if $U_n(R)$ is idempotent unit regular.*

Proof. Let $A=[a_{ij}]\notin J(U_n(R))$ and then we have that $a_{rr}\notin J(R)$ for some $r\in\{1,\dots,n\}$. Since R is idempotent unit regular, there exist a nonzero idempotent e and a unit u such that $ea_{rr}=eu$. Then $eE_{rr}A=eE_{rr}U$, where $U=[u_{ij}]$ with $u_{ii}=-1$ except $i=r$, $u_{rr}=u$, $u_{r(r+t)}=a_{r(r+t)}$, where $r+1\le r+2\le\cdots\le r+t\le n$ and otherwise $u_{ij}=0$. Then U has an inverse in $U_n(R)$ by Lemma 3.3.

Conversely, let $U_n(R)$ be idempotent unit regular and $a\notin J(R)$. Let $A=[a_{ij}]$, where $a_{ii}=a$ for $i\in\{1,\dots,n\}$, otherwise 0. Then $A\notin J(U_n(R))$ and there exist $E^2=E\neq 0$ and a unit matrix $U=[u_{ij}]$ such that $EA=EU$. Since $E=[e_{ij}]\neq 0$, there exists at least one i such that $e_{ii}\neq 0$ by Lemma 3.3. Thus, $e_{ii}a=e_{ii}u_{ii}$, where $0\neq e_{ii}^2=e_{ii}$ and u_{ii} a unit in R and hence, we are done. □

Lemma 3.5. *Let R be an idempotent unit regular ring. Then $M_2(R)$ is also an idempotent unit regular ring.*

Proof. Let $A=\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\notin J(M_2(R))=M_2(J(R))$. Then we have the following cases:

Case 1: Let $a\notin J(R)$. Then there exist $0\neq e=e^2$ and a unit $u\in U_n(R)$ such that $ea=eu$. Thus, $\left[\begin{smallmatrix} e & 0\\ 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]=\left[\begin{smallmatrix} e & 0\\ 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix} u & b\\ 0 & 1\end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} u & b\\ 0 & 1\end{smallmatrix}\right]^{-1}=\left[\begin{smallmatrix} u^{-1} & -u^{-1}b\\ 0 & 1\end{smallmatrix}\right]$.

Case 2: Let $b\notin J(R)$. Then we have $0\neq e=e^2$ and a unit $u\in U_n(R)$ with $eb=eu$. Thus, $\left[\begin{smallmatrix} e & 0\\ 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]=\left[\begin{smallmatrix} e & 0\\ 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix} a & u\\ 1 & 0\end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} a & u\\ 1 & 0\end{smallmatrix}\right]^{-1}=\left[\begin{smallmatrix} 0 & 1\\ u^{-1} & -u^{-1}a\end{smallmatrix}\right]$.

Case 3: Let $c\notin J(R)$. Similarly, we have that $\left[\begin{smallmatrix} 0 & 0\\ 0 & e\end{smallmatrix}\right]\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]=\left[\begin{smallmatrix} 0 & 0\\ 0 & e\end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1\\ u & d\end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} 0 & 1\\ u & d\end{smallmatrix}\right]^{-1}=\left[\begin{smallmatrix} -u^{-1}d & u^{-1}\\ 1 & 0\end{smallmatrix}\right]$.

Case 4: Let $d\notin J(R)$. Then we have that $\left[\begin{smallmatrix} 0 & 0\\ 0 & e\end{smallmatrix}\right]\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]=\left[\begin{smallmatrix} 0 & 0\\ 0 & e\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0\\ c & u\end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} 1 & 0\\ c & u\end{smallmatrix}\right]^{-1}=\left[\begin{smallmatrix} 1 & 0\\ -u^{-1}c & u^{-1}\end{smallmatrix}\right]$.

By these cases, $M_2(R)$ is idempotent unit regular. □

Lemma 3.6. *Let R and $M_{n-1}(R)$ be idempotent unit regular rings and $B=\left[\begin{smallmatrix} A & X\\ Y & a_{nn}\end{smallmatrix}\right]\in M_n(R)$, where $A\in M_{n-1}(R)$, $X=\begin{bmatrix} x_1\\ \vdots\\ x_{n-1}\end{bmatrix}$ and $Y=\left[\begin{smallmatrix} y_1 & \cdots & y_{n-1}\end{smallmatrix}\right]$. If $A\notin J(M_{n-1}(R))$ or $a_{nn}\notin J(R)$, then B is idempotent unit regular.*

Proof. Let $A\notin J(M_{n-1}(R))$. Since $M_{n-1}(R)$ is idempotent unit regular, there exist a non-zero idempotent E and a unit U such that $EA=EU$. Thus, we have that $\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} A & X \\ Y & a_{nn} \end{smallmatrix}\right] = \left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} U & X \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} U & X \\ 0 & 1 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} U^{-1} & -U^{-1}X \\ 0 & 1 \end{smallmatrix}\right]$. Now, let $a_{nn} \notin J(R)$. Then there exist a non-zero idempotent e and a unit u such that $ea_{nn} = eu$ and hence, we can write that $\left[\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right]\left[\begin{smallmatrix} A & X \\ Y & a_{nn} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right]\left[\begin{smallmatrix} I_{n-1} & 0 \\ Y & u \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} I_{n-1} & 0 \\ Y & u \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I_{n-1} & 0 \\ -U^{-1}Y & U^{-1} \end{smallmatrix}\right]$. □

Lemma 3.7. *Let R be idempotent unit regular and $B = \left[\begin{smallmatrix} A & X \\ Y & a_{nn} \end{smallmatrix}\right] \in M_n(R)$, where $A \in M_{n-1}(R)$, $X = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{(n-1)n} \end{bmatrix}$ and $Y = \left[\begin{smallmatrix} y_1 & \cdots & y_{(n-1)} \end{smallmatrix}\right] = \left[\begin{smallmatrix} a_{n1} & \cdots & a_{n(n-1)} \end{smallmatrix}\right]$. If there exists at least one $x_i \notin J(R)$ for $i = 1, \dots, n-1$, then B is idempotent unit regular.*

Proof. Since $x_i \notin J(R)$ and R is an idempotent unit regular ring, there exist a non-zero idempotent e and a unit u with $ex_i = eu$. Let $E = [e_{st}] \in M_n(R)$ such that $e_{ii} = e$ and otherwise 0. Then we have a non-zero idempotent E and a unit C such that $EB = EC$, where the i-th row $R_i(C) = \left[\begin{smallmatrix} a_{i1} & \cdots & a_{i(n-1)} & u \end{smallmatrix}\right]$ and for $1 \le j < i$, $R_j(C) = \left[\begin{smallmatrix} d_{j1} & \cdots & d_{jn} \end{smallmatrix}\right]$ with $d_{jj} = 1$, $d_{j1} = \cdots = d_{j(j-1)} = d_{j(j+1)} = \cdots = d_{jn} = 0$ and in addition; for $i < j \le n$, $R_j(C) = \left[\begin{smallmatrix} d_{j1} & \cdots & d_{jn} \end{smallmatrix}\right]$ with $d_{j(j-1)} = 1$, otherwise 0. In this case, $R_n(C^{-1}) = \left[\begin{smallmatrix} -u^{-1}a_{i1} & \cdots & -u^{-1}a_{i(i-1)} & u^{-1} & -u^{-1}a_{ii} & \cdots & -u^{-1}a_{i(n-1)} \end{smallmatrix}\right]$ and for $1 \le j < i$, $R_j(C^{-1}) = \left[\begin{smallmatrix} d_{j1} & \cdots & d_{jj} & \cdots & d_{jn} \end{smallmatrix}\right]$ with $d_{jj} = 1$, otherwise 0 and, for $i \le j < n$, $R_j(C^{-1}) = \left[\begin{smallmatrix} d_{j1} & \cdots & d_{j(j-1)} & d_{jj} & \cdots & d_{jn} \end{smallmatrix}\right]$ with $d_{j(j+1)} = 1$, otherwise 0. □

Lemma 3.8. *Let R be idempotent unit regular and $B = \left[\begin{smallmatrix} A & X \\ Y & a_{nn} \end{smallmatrix}\right] \in M_n(R)$, where $A \in M_{n-1}(R)$, $X = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{(n-1)n} \end{bmatrix}$ and $Y = \left[\begin{smallmatrix} y_1 & \cdots & y_{n-1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} a_{n1} & \cdots & a_{n(n-1)} \end{smallmatrix}\right]$. If there exists at least one $y_i \notin J(R)$ for $i = 1, \dots, n-1$, then B is idempotent unit regular.*

Proof. Since $y_i \notin J(R)$ and R is an idempotent unit regular ring, there exist a non-zero idempotent e and a unit u with $ey_i = eu$. Let $E = [E_{st}] \in M_n(R)$ such that $E_{nn} = e$ and otherwise 0. Then we have a non-zero idempotent E and a unit C such that $EB = EC$, where the n-th row $R_n(C) = \left[\begin{smallmatrix} a_{n1} & \cdots & a_{n(i-1)} & u & a_{n(i+1)} & \cdots & a_{nn} \end{smallmatrix}\right]$ for $1 \le j < i$, $R_j(C) = \left[\begin{smallmatrix} b_{j1} & \cdots & b_{jn} \end{smallmatrix}\right]$ with $b_{jj} = 1$, $b_{j1} = \cdots = b_{j(j-1)} = b_{j(j+1)} = \cdots = b_{jn} = 0$ and for $i \le j < n$, $R_j(C) = \left[\begin{smallmatrix} b_{j1} & \cdots & b_{jn} \end{smallmatrix}\right]$ with $b_{j(j+1)} = 1$, otherwise 0. Then

$$R_i(C^{-1}) = \begin{bmatrix} -u^{-1}a_{n1} & \cdots & -u^{-1}a_{n(i-1)} & -u^{-1}a_{n(i+1)} & \cdots & -u^{-1}a_{nn} & u^{-1} \end{bmatrix}$$

and for $1 \le j < i$, $R_j(C^{-1}) = \left[\begin{smallmatrix} d_{j1} & \cdots & d_{jj} & \cdots & d_{jn} \end{smallmatrix}\right]$ with $d_{jj} = 1$, otherwise 0 and, for $i < j \le n$, $R_j(C^{-1}) = \left[\begin{smallmatrix} d_{j1} & \cdots & d_{j(j-1)} & d_{jj} & \cdots & d_{jn} \end{smallmatrix}\right]$ with $d_{j(j-1)} = 1$, otherwise 0. □

Theorem 3.9. *If R is idempotent unit regular, then so is $M_n(R)$.*

Proof. Let $B = [a_{ij}] = \left[\begin{smallmatrix} A & X \\ Y & a_{nn} \end{smallmatrix}\right] \notin J(M_n(R)) = M_n(J(R))$, where $A = [a_{ij}] \in M_{n-1}(R)$, $X = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{(n-1)n} \end{bmatrix}$ and $Y = \left[\begin{smallmatrix} y_1 & \cdots & y_{n-1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} a_{n1} & \cdots & a_{n(n-1)} \end{smallmatrix}\right]$. We will use an induction on n. If $n = 2$, then it is correct by Lemma 3.5. Now, we suppose that the induction is true for $n-1$. Since $B \notin M_n(J(R))$, we have the following cases:

Case 1: If $A \notin M_{n-1}(J(R))$, then B is idempotent unit regular by Lemma 3.6.

Case 2: If $a_{nn} \notin J(R)$, then B is idempotent unit regular by Lemma 3.6.

Case 3: If there exists at least one $x_i \notin J(R)$, then B is idempotent unit regular by Lemma 3.7.

Case 4: If there exists at least one $y_i \notin J(R)$, then B is idempotent unit regular by Lemma 3.8.

By these cases, we complete the proof. □

4. On the ring of endomorphisms of a module in a class X

In this section, we will study when endomorphism ring of a module is an idempotent unit regular ring.

Theorem 4.1. *Let M be any left (or right) R-module. Then $T'_\alpha = \{(K,\beta) : \alpha(K) \subseteq K, \alpha|_K - \beta$ is unit and $\beta^2 = \beta \in End(K)\}$ has a maximal element (W'_α, η') by the ordering $(K', \beta') \leq (K, \beta)$ if and only if $K' \subseteq K$ and $\beta' = \beta|_{K'}$, where $\alpha \in End(M)$. If W'_α is a direct summand of M for all $\alpha \in End(M)$ and every submodule of M that is isomorphic to a summand is a summand, then $End(M)$ is a clean ring.*

Proof. It follows from the proof of Lemma 4 in [4]. □

Let M be any left (or right) R-module, $\alpha \in End(M) - J(End(M))$ and let there exists a submodule N_α such that $\alpha(N_\alpha) \subseteq N_\alpha$ and $\alpha|_{N_\alpha} - e$ is unit, where $1 \neq e = e^2 \in End(N_\alpha)$ is idempotent. Thus, using Zorn's Lemma we can say that $T_\alpha = \{(K, \beta) : N_\alpha \leq K, \alpha(K) \subseteq K, \alpha|_K - \beta$ is unit, $\beta^2 = \beta \in End(K)$ and $\beta|_{N_\alpha} = e\}$ has a maximal element (W_α, η) such that $\eta|_{N_\alpha} = e \neq 1$ by the ordering on T_α which is $(K', \beta') \leq (K, \beta)$ if and only if $K' \subseteq K$ and $\beta' = \beta|_{K'}$.

Let's have the following conditions in the class X:

(W_α) W_α (where (W_α, η) such that $\eta \neq 1$ is a maximal element of T_α above) is a direct summand of the module M in X.

(C2) Every submodule of a module in the class X that is isomorphic to a direct summand is a direct summand.

The proof of the following theorem is the same as the proof of Lemma 4 in [4]. But we give again some necessary parts.

Theorem 4.2. *Let M be any module in a class X. If there exists a submodule N_α such that $\alpha(N_\alpha) \subseteq N_\alpha$ and $\alpha|_{N_\alpha} - e$ is unit, where $1 \neq e = e^2 \in End(N_\alpha)$ and in addition, the conditions (W_α) and (C2) (which are described above) are* *satisfied for all $\alpha \in End(M) - J(End(M))$, then $End(M)$ is a clean ring with idempotent not identity and hence an idempotent unit regular ring.*

Proof. Let $\alpha \in End(M) - J(End(M))$. Since W_α is a direct summand of M, we have a submodule K_α such that $M = W_\alpha \oplus K_\alpha$. Let π_{W_α} and π_{K_α} be the projections of M onto W_α and K_α and $\phi = \pi_{K_\alpha}\alpha|_{K_\alpha} \in End(K_\alpha)$.

Claim 2: Suppose $X \leq K_\alpha$ satisfies $\phi(X) \subseteq X$ and $\phi - \epsilon$ is a unit of $End(X)$ for an idempotent ϵ. Then $X = 0$. Claim 2 is proved in Lemma 4 in [4].

Now, take $X = Ker(\phi)$ and $\epsilon = 1_X$. By Claim 2 we have that $X = 0$ and hence ϕ is monic. Thus, $\phi(K_\alpha)$ is a direct summand by the condition (C2). Then $K_\alpha = Y \oplus \phi(K_\alpha)$ and hence $\phi(K_\alpha) = \phi(Y) \oplus \phi^2(K_\alpha)$ which gives $K_\alpha = Y \oplus \phi(Y) \oplus \phi^2(K_\alpha)$. Then we have $\{Y, \phi(Y), \phi^2(Y), \ldots\}$ which is an independent family of submodules of K_α. Let $V = \oplus_{n\in\mathbb{N}}\phi^n(Y)$ and so, $\phi(V) \subseteq V$. Following an argument of Ó Searcóid in [17] if we take as $\lambda(y) = y$, $\lambda(\phi(y)) = \phi^2(y) - y, \ldots, \lambda(\phi^{2n}(y)) = \phi^{2n}(y)$, $\lambda(\phi^{2n+1}(y)) = \phi^{2n+2}(y) - \phi^{2n}(y), \ldots$; then λ is idempotent on V. Then $(\phi|_V - \lambda)(y) = \phi(y) - y$, $(\phi|_V - \lambda)(\phi(y)) = y, \ldots, (\alpha|_V - \lambda)(\phi^{2n}(y)) = \phi^{2n+1}(y) - \phi^{2n}(y)$, $(\phi|_V - \lambda)(\phi^{2n+1}(y)) = \phi^{2n}(y), \ldots$ which has an inverse $\phi|_V - \lambda + I$ since $(\phi|_V - \lambda)^2 = I - \phi|_V + \lambda$. If we take $X = V$ and $\epsilon = \lambda$ in Claim 2, then $V = 0$ and then we have that $K_\alpha = \phi(K_\alpha)$, that is, $\phi \in End(K_\alpha)$ is unit. Taking $X = K_\alpha$ and $\epsilon = 0$ in Claim 2, we can find that $K_\alpha = 0$ and so $W_\alpha = M$. Then α is a clean element with idempotent not identity and hence it is idempotent unit regular. Therefore, $End(M)$ is an idempotent unit regular ring. $\square$

Theorem 4.3. *Let X be a class of semi-simple modules. If M is in X, then End(M) is a clean ring with idempotent not identity and hence it is an idempotent unit regular ring .*

Proof. Let $0 \neq \alpha \in End(M)$. Then there exists a simple submodule Rx of M such that $\alpha(x) \neq 0$. Thus, we have that either $Rx = R\alpha(x)$ or $Rx \oplus R\alpha(x)$. If $Rx = R\alpha(x)$, then we can take as $N = Rx$ and $e^2 = e = 0 \neq 1$ in Theorem 4.2. Thus, $\alpha|N - e$ is unit and hence, α is idempotent unit regular by Theorem 4.2.

Now, let $\alpha^n(x) \in N = Rx \oplus R\alpha(x) \oplus \cdots \oplus R\alpha^{n-1}(x)$, where $n \geq 2$. Then $\alpha|_N$ is on finitely many direct sum of simple modules and hence it is isomorphic to ring of matrices over a division ring which is equivalent to a diagonal matrix with idempotent entries by Theorem 2.7 in [12]. Therefore, it is unit regular and hence it is written as $\alpha|_N - e$ is unit and $e^2 = e \neq 1$ by [3].

Now, let $N = Rx \oplus R\alpha(x) \oplus \cdots \oplus R\alpha^n(x) \oplus \cdots$. Following an argument of Ó Searcóid in [17] if we take as $e(x) = x$, $e(\alpha(x)) = \alpha^2(x) - x, \ldots, e(\alpha^{2n}(x)) = \alpha^{2n}(x)$, $e(\alpha^{2n+1}(x)) = \alpha^{2n+2}(x) - \alpha^{2n}(x), \ldots$; then e is idempotent on N. Then $(\alpha|_N - e)(x) = \alpha(x) - x$, $(\alpha|_N - e)(\alpha(x)) = x, \ldots, (\alpha|_N - e)(\alpha^{2n}(x)) = \alpha^{2n+1}(x) - \alpha^{2n}(x)$, $(\alpha|_N - e)(\alpha^{2n+1}(x)) = \alpha^{2n}(x), \ldots$ which has an inverse $\alpha|_N - e + I$ since $(\alpha|_N - e)^2 = I - \alpha|_N + e$. By Theorem 4.2 we are done. $\square$

The following result follows directly from the previous theorem.

Corollary 4.4. *If M is a vector space over a division ring D, then $End(M)$ is a clean ring with idempotent not identity and hence it is an idempotent unit regular ring.*

Using [16] we can give the following proposition.

Proposition 4.5. (i) *Let $R = M_n(D)$ be the matrix ring over a division ring D, where $n > 0$ is an integer and let F_R be a free module. Then $End(F_R)$ is idempotent unit regular.*

(ii) *Let $R = M_n(D)$ be the matrix ring over a division ring D, where $n > 0$ is an integer and M_R is idempotent unit regular.*

(iii) *Let R be a semi-simple artinian ring and M_R be a module. Then $End(M_R)$ is idempotent unit regular.*

(iv) *For any projective R-module P_R over a right perfect ring R, the endomorphism ring $End(P_R)$ is idempotent unit regular.*

Theorem 4.6. *If M is a nonsingular injective module such that $S = End(M)$ is regular, then S is a clean ring with idempotent not identity and hence it is idempotent unit regular. In addition, if M is a continuous or discrete module, then S is idempotent unit regular.*

Proof. Let $0 \neq \alpha \in S$. Since S is regular, we have that $S = r_S(\alpha) \oplus K$, where $r_S(\alpha) = (1-e)S$, $K = eS$ and e is idempotent. Put $e\alpha e = \phi$ and $eSe = S_1$. If $r_{S_1}(\phi) = 0$, then ϕ is $1-1$ since S_1 is regular. Thus, $eM = Y \oplus \phi(eM)$. Following an argument of Ó Searcóid in [17] the same as in Theorem 4.2, we can obtain that a submodule $V = \oplus_{n\in\mathbb{N}}\phi^n(Y)$ and an idempotent λ not identity such that $\phi|_V - \lambda$ is unit. By the Peirce Decomposition we have that $\alpha = \left[\begin{smallmatrix} e\alpha e & 0 \\ (1-e)\alpha e & 0 \end{smallmatrix}\right]$ which implies that $\alpha|_V$ is a clean element with idempotent not identity. Then we can take as $N_\alpha = V$ in Theorem 4.2.

Now, let $0 \neq x \in r_{S_1}(\phi)$. Since $\alpha = e\alpha e + (1-e)\alpha e$, $\alpha x \in (1-e)S$ and hence we can get that $X = \alpha x(M) \oplus x(M)$. Let $g : X \to X$ be a morphism such that $g(x) = g(\alpha x) = x$. Then $\alpha|_X - g$ is unit, where $g \neq 1$ is idempotent. So we can take as $N_\alpha = X$ in Theorem 4.2. Thus, a nonsingular injective module satisfies the conditions (W_α) and (C2) by the proof of Lemma 4 in [4] and so, S is idempotent unit regular by Theorem 4.2.

If M is continuous, then $S/J(S) = T_1 \times T_2$ by Theorem 3.11 and Corollary 3.13 in [13], where T_1 is a regular (so nonsingular), left self injective ring and T_2 is a reduced regular ring. Then T_1 is idempotent unit regular. Since T_2 is a reduced regular ring, it is unit regular and hence it is idempotent unit regular. Thus, $S/J(S)$ is idempotent unit regular, too. Since idempotents lift modulo $J(S)$ by Proposition 3.5 and Lemma 3.7 in [13], S is idempotent unit regular. If M is discrete, then $S/J(S)$ is left continuous by Theorem 2.3 in [2]. Then it follows from the above discussion that $S/J(S)$ is idempotent unit regular. Thus, S is idempotent unit regular by Lemma 5.3 and Theorem 5.4 in [13]. □

Theorem 4.7. *Let M be a quasi-injective right R-module and $S = End_R(M)$. If $(\alpha + 1)S$ is a directly finite, then $\alpha \in S - J(S)$ is idempotent unit regular.*

Proof. By Theorem 7 in [5] there exist an idempotent e and a unit u such that $\alpha+1=e+u$, where $(1-(\alpha+1))S\cap(1-e)S\subseteq J(S)$. Thus, $e\neq 0$ because of $\alpha\in S-J(S)$ and so $e\alpha=eu$. $\square$

Corollary 4.8. *Let R be a regular right self injective ring. If $(\alpha+1)S$ is directly finite, then $\alpha\in S-J(S)$ is idempotent unit regular.*

Proof. It follows directly from the proof of Theorem 7 in [5] and Theorem 4.7. $\square$

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References

[1] R. M. S. Alfaqi, *On a special case of clean rings*, Thesis (Master), Bolu Abant Izzet Baysal University, 2021. https://tez.yok.gov.tr/UlusalTezMerkezi/giris.jsp
[2] M. H. Ali and J. M. Zelmanowitz, *Discrete implies continuous*, J. Algebra **183** (1996), no. 1, 186–192. https://doi.org/10.1006/jabr.1996.0213
[3] V. P. Camillo and D. Khurana, *A characterization of unit regular rings*, Comm. Algebra **29** (2001), no. 5, 2293–2295. https://doi.org/10.1081/AGB-100002185
[4] V. P. Camillo, D. Khurana, T. Y. Lam, W. K. Nicholson, and Y. Zhou, *A short proof that continuous modules are clean*, in Contemporary ring theory 2011, 165–169, World Sci. Publ., Hackensack, NJ, 2012. https://doi.org/10.1142/9789814397681_0015
[5] H. Chen, *Extensions of unit-regular rings*, Comm. Algebra **32** (2004), no. 6, 2359–2364. https://doi.org/10.1081/AGB-120037225
[6] H. Chen, *On strongly J-clean rings*, Comm. Algebra **38** (2010), no. 10, 3790–3804. https://doi.org/10.1080/00927870903286835
[7] H. Chen, *On uniquely clean rings*, Comm. Algebra **39** (2011), no. 1, 189–198. https://doi.org/10.1080/00927870903451959
[8] A. J. Diesl, *Nil clean rings*, J. Algebra **383** (2013), 197–211. https://doi.org/10.1016/j.jalgebra.2013.02.020
[9] G. Ehrlich, *Unit-regular rings*, Portugal. Math. **27** (1968), 209–212.
[10] K. R. Goodearl, *von Neumann Regular Rings*, second edition, Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1991.
[11] N. A. Immormino, *Clean rings & clean group rings*, Thesis (Ph.D.), Bowling Green State University, 2013.
[12] T.-M. Lok and K. P. Shum, *On matrix rings over unit-regular rings*, Beiträge Algebra Geom. **44** (2003), no. 1, 179–188.
[13] S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Mathematical Society Lecture Note Series, 147, Cambridge Univ. Press, Cambridge, 1990. https://doi.org/10.1017/CBO9780511600692
[14] W. K. Nicholson, *I-rings*, Trans. Amer. Math. Soc. **207** (1975), 361–373. https://doi.org/10.2307/1997182
[15] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278. https://doi.org/10.2307/1998510
[16] W. K. Nicholson, K. Varadarajan, and Y. Zhou, *Clean endomorphism rings*, Arch. Math. (Basel) **83** (2004), no. 4, 340–343. https://doi.org/10.1007/s00013-003-4787-9
[17] M. Ó Searcóid, *Perturbation of linear operators by idempotents*, Irish Math. Soc. Bull. No. 39 (1997), 10–13.
[18] J. Šter, *Corner rings of a clean ring need not be clean*, Comm. Algebra **40** (2012), no. 5, 1595–1604. https://doi.org/10.1080/00927872.2011.551901

[19] A. A. Tuganbaev, *Rings Close to Regular*, Mathematics and its Applications, 545, Kluwer Acad. Publ., Dordrecht, 2002. https://doi.org/10.1007/978-94-015-9878-1

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