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SOME ONE-DIMENSIONAL NOETHERIAN DOMAINS AND G-PROJECTIVE MODULES

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Abstract. Let R be a one-dimensional Noetherian domain with quotient field K and T be the integral closure of R in K. In this note we prove that if the conductor ideal $(R :_K T)$ is a nonzero prime ideal, then every finitely generated reflexive (and hence finitely generated G-projective) R-module is isomorphic to a direct sum of some ideals.

1. Introduction

Throughout this note, all rings are commutative with identity and all modules are unitary. Recall that an R-module M is said to be *Gorenstein projective* (G-projective for short) in [5] if there exists an exact sequence $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ of projective R-modules with $M = \ker(P^0 \longrightarrow P^1)$ such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is a projective R-module. The concept of the Gorenstein Dedekind domain was put forward in [4]. A domain R is a Gorenstein Dedekind domain (G-Dedekind domain for short) if and only if any submodule of any free R-module is G-projective, if and only if R is a Noetherian ring and any nonzero ideal of R is a divisorial ideal. For more descriptions of the G-Dedekind domain, one can refer to [18, Theorem 11.7.7]. Recall that a module M is said to be *torsionless* if the evaluation map $\Phi_M : M \longrightarrow M^{**}$ is injective and to be *reflexive* if Φ_M is an isomorphism. We say that a domain is a *reflexive domain* if every torsionless module of finite rank is reflexive. It can be seen from [18, Theorem 11.7.7](5) that G-Dedekind domains are exactly Noetherian reflexive domains. It was also shown in [18, Lemma 11.6.9] that every finitely generated G-projective module is reflexive.

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Following the terminology of Matlis in [12], a domain is said to have *property* FD if every finitely generated torsion-free module is isomorphic to a direct sum of some ideals. Recall that a domain R is called an *NWF domain* if every ideal of R can be generated by two elements (see [18, Definition 11.7.10]). An integral domain R is called a *Bass domain* if R is an NWF domain with module-finite integral closure. Bass proved in [1] that every Bass domain has property FD. Matlis proved in [12] that every local NWF domain also has property FD. About twenty years later, Rush proved in [17] that every NWF domain has property FD. A nonzero ideal of an integral domain is said to be *stable* if it is projective over its ring of endomorphisms. Following [2], an integral domain R is called a *Warfield domain* if, given any R-submodule A of the quotient field of R, all A-torsionless $\mathrm{End}_R(A)$-modules of finite rank are A-reflexive. Olberding proved in [15, Theorem 5.3] that a domain R is a Warfield domain if and only if every torsionless R-module is isomorphic to a direct sum of stable ideals. Inspired by these works, we investigate domains over which every finitely generated reflexive module is isomorphic to a direct sum of some ideals. We get that if R is a one-dimensional Noetherian domain such that the conductor ideal is nonzero and prime, then every finitely generated reflexive R-module is isomorphic to a direct sum of some ideals.

For unexplained concepts and notations, one can refer to [6, 11, 14, 16].

2. Decomposition of finitely generated reflexive modules

For any ideal I, we denote by $V(I)$ the set of prime ideals which contain I. For an R-module M, we also denote by $\mathrm{tr}_R(M)$ the trace ideal of M. We begin with the following observation about one-dimensional Noetherian domains.

Theorem 2.1. *Let R be a one-dimensional Noetherian domain and M be a finitely generated projective R-module. Then M is isomorphic to a direct sum of some ideals, namely, $M \cong R \oplus \cdots \oplus R \oplus J$, where this direct sum is finite, and if it is a direct sum of n ideals, then the number of R's is at least $n-1$.*

Proof. Since M is projective, we have that $\mathrm{tr}_R(M) = R$. So by [17, Lemma 4.2], we get that $M \cong R \oplus N$ for some submodule N. Since M is projective, we get that N is also projective. Thus the result can be obtained by induction. □

A complete projective resolution of the form

$$\mathbf{P} := \cdots \longrightarrow P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \longrightarrow \cdots$$

is called a *strongly complete projective resolution* and denoted by $(\mathbf{P}, f)$. Following [3, Definition 2.1], an R-module M is said to be *strongly Gorenstein projective* (SG-projective for short) if $M \cong \ker(f)$ for some strongly complete projective resolution $(\mathbf{P}, f)$.

It was proved in [17, Theorem 4.3] that every finitely generated torsion-free module over an NWF domain is isomorphic to a direct sum of some ideals. We have the following characterization for NWF domains.

Theorem 2.2. *Let R be a Noetherian domain. Then R is an NWF domain if and only if every maximal ideal of R is SG-projective.*

Proof. First we prove the necessity part. If R is an NWF domain, then every ideal (hence every maximal ideal) of R is SG-projective by [8, Theorem 3.12].

Conversely, we assume that every maximal ideal M of R is SG-projective. Then M_M is an SG-projective R_M-module. So R_M is a local NWF domain by [10, Theorem 2.25] for every maximal ideal M of R. Therefore R is an NWF domain by [10, Theorem 2.27]. □

Let R be an integral domain with quotient field K. A fractional ideal I of R is just an R-submodule of K such that $rI \subseteq R$ for some nonzero $r \in R$. Then a fractional ideal I of R is said to be *invertible* if $IJ = R$ for some fractional ideal J of R. For a nonzero fractional ideal I of R, set $I^{-1} := \{\alpha \in K \mid \alpha I \subseteq R\}$. Then it is well known that a fractional ideal I of R is invertible if and only if $II^{-1} = R$.

We also have the following observation about the conductor ideal.

Lemma 2.3. *Let R be a one-dimensional Noetherian domain with quotient field K and T be the integral closure of R in K such that the conductor ideal $C := (R :_K T)$ is not zero. Let P be a prime ideal of R. Then P is projective if and only if $P \notin V(C)$.*

Proof. If $P \notin V(C)$, then $P + C = R$ and P is invertible (hence projective) by [10, Lemma 1.8].

If $P \in V(C)$, then we have $C \subset P$ and $C_P = (R_P : T_P) \subset P_P \neq R_P$. This means that R_P is not integrally closed (since $R_P \neq T_P$). So P is not invertible. □

It was proved in [10, Theorem 1.10] that a one-dimensional Noetherian domain is a G-Dedekind domain if and only if every prime ideal which contains the conductor ideal is G-projective. A step further, we have the following corollary.

Corollary 2.4. *Let R be a one-dimensional Noetherian domain with quotient field K and its integral closure T in K. Denote the conductor ideal $(R :_K T)$ by C. Then R is an NWF domain if and only if any prime ideal P of R which contains C is SG-projective.*

Proof. Notice that every invertible ideal is projective, and hence SG-projective. Thus the result follows from Theorem 2.2 and Lemma 2.3. □

In what follows, we investigate the decomposition of finitely generated reflexive modules over one-dimensional Noetherain domains.

Theorem 2.5. *Let R be a one-dimensional Noetherian domain with quotient field K and the integral closure T. If for any prime ideal P which contains the conductor ideal $C := (R :_K T)$, $P^{-1} = (P : P)$ is an NWF domain, then every finitely generated reflexive R-module is isomorphic to a direct sum of* *some ideals. Furthermore, if R is a G-Dedekind domain, then R is also an NWF domain.*

Proof. Let M be a finitely generated reflexive R-module. Since R is Noetherian, M has a maximal projective direct summand Q. That is to say, $M = Q \oplus N$ for some N which has no projective direct summand. If the rank of N is one, then we have done since the projective module Q is decomposable by Theorem 2.1. So we assume that $\mathrm{rank}(N) \geqslant 2$. Denote the trace ideal $\mathrm{tr}_R(N)$ by J. We claim that $V(J) \cap V(C) \neq \emptyset$. Otherwise, $J + C$ will be contained in no prime ideals, and so must be R. This means that J is invertible, and so must be R by [10, Lemma 1.8] (the case that $C = 0$ is obvious). Since N is also reflexive, by [17, Theorem 4.2], N will have a direct summand which is isomorphic to R. This contradicts the maximality of Q. Let $J \subset P$ for some $P \in V(C)$. Since $P \in V(C)$, we have P is not invertible and $P^{-1} = (P : P)$ is a ring. Also notice that by [17, Theorem 4.1], J^{-1} is also a ring and N is a J^{-1}-module. Obviously $P^{-1} \subset J^{-1}$. So J^{-1} is also an NWF domain as an overring of the NWF domain P^{-1}. Thus, as a finitely generated torsion-free module over J^{-1}, N is isomorphic to a direct sum of some ideals. Since both Q and N are decomposable, M is isomorphic to a direct sum of some ideals.

Under the above condition, if R is a G-Dedekind domain, then every finitely generated torsion-free module is G-projective, and hence reflexive and, by the result of the above paragraph, is isomorphic to a direct sum of some ideals. Therefore every ideal of R can be generated by two elements. So R is an NWF domain. □

Corollary 2.6. *Let R be a one-dimensional Noetherian domain with quotient field K and its integral closure T in K. If $P^{m+1} \subset (R :_K T) \subset P^m$ for some nonzero prime ideal P of R and some positive integer m such that P^{-1} is an NWF domain, then every finitely generated reflexive R-module is isomorphic to a direct sum of some ideals.*

Proof. Just notice that P is the only prime ideal which contains the conductor ideal. □

Example 2.7. The polynomial domain $R = \mathbb{Q}[X^3, X^5, X^7]$ is not a G-Dedekind domain. But every finitely generated reflexive R-module is isomorphic to a direct sum of some ideals.

Proof. First, we note that the quotient field K of R is $\mathbb{Q}(X)$ and the integral closure T of R is $\mathbb{Q}[X]$. It is routine to check that the conductor ideal $(R :_K T) = X^5\mathbb{Q}[X]$. Let P be the ideal generated by the set $\{X^3, X^5, X^7\}$. Then P is maximal. It can be seen that $P^2 \subset X^5\mathbb{Q}[X] \subset P$. Some calculation shows that $P^{-1} = \mathbb{Q} + X^2\mathbb{Q}[X]$. This is an NWF domain since every ideal is 2-generated by [10, Theorem 2.9]. Therefore every finitely generated reflexive R-module is isomorphic to a direct sum of some ideals by Corollary 2.6. Since $P^{-1} = R + RX^2 + RX^4$ can not be generated by two elements as an R-module, R is not reflexive, and hence not a G-Dedekind domain by [13, Theorem 40]. □

Corollary 2.8. *Let R be a one-dimensional Noetherian domain with quotient field K and its integral closure T in K. If $(R :_K T) = P$ for some nonzero prime ideal P of R, then every finitely generated reflexive R-module (hence also every finitely generated G-projective R-module) is isomorphic to a direct sum of some ideals.*

Proof. Just notice that P is the only prime ideal which contains the conductor ideal and $P^{-1} = T$ is a Dedekind domain. □

Example 2.9. The polynomial domain $R = \mathbb{Q} + X^n\mathbb{Q}[X]$ $(n \geqslant 3)$ is not a G-Dedekind domain. But every finitely generated reflexive R-module is isomorphic to a direct sum of some ideals.

Proof. Just notice that the conductor ideal $X^n\mathbb{Q}[\mathbb{X}]$ is prime and can not be generated by two elements. The result comes from [9, Theorem 3.10] and Corollary 2.8. □

Let R be a one-dimensional Noetherian domain with quotient field K and integral closure T. We first consider the case when $(R :_K T)$ is a maximal ideal.

Lemma 2.10. *Let R, K, T be rings as mentioned above and $M = (R :_K T)$ be a maximal ideal of R. Then the following statements are equivalent:*

(1) *T is a G-projective R-module;*
(2) *T_M is a G-projective R_M-module;*
(3) *R is a G-Dedekind domain.*

Proof. (1)⇒(2) This is obvious since R is Noetherian.

(2)⇒(3) Notice that $M_M = (R :_K T)_M = (R_M :_K T_M)$, i.e., $M_M = (T_M)^{-1}$. Since $(R_M :_K T_M) \neq 0$, we have T_M is a finitely generated R_M-module by [10, Proposition 1.9]. So we have $(\mathrm{Ext}^1_R(M, R))_M = \mathrm{Ext}^1_{R_M}(M_M, R_M) = \mathrm{Ext}^1_{R_M}(T_M^{-1}, R_M) = 0$ by [18, Lemma 11.6.9]. For those maximal ideals other than M, say, $P \neq M$, we have P is projective. So R_P is a Dedekind domain and $(\mathrm{Ext}^1_R(M, R))_P = \mathrm{Ext}^1_{R_P}(M_P, R_P) = 0$ also. Since $\mathrm{Ext}^1_R(M, R)$ is locally zero, we have $\mathrm{Ext}^1_R(M, R) = 0$. So R is a G-Dedekind domain by [9, Theorem 3.10].

(3)⇒(1) Just notice that T is a finitely generated torsion-free R-module. □

Theorem 2.11. *Let R, K, T be rings as mentioned above. If T is not a G-projective R-module, then every finitely generated G-projective R-module is projective.*

Proof. Let M be a finitely generated G-projective R-module. Then M is reflexive by [18, Lemma 11.6.9]. So $M \cong I_1 \oplus I_2 \oplus \cdots \oplus I_k$ for some ideals. Since M is G-projective, these ideals are G-projective. So we can assume that M is just a G-projective ideal. If $\mathrm{Im}(f) \nsubseteq (R :_K T)$ for some $f \in M^* = \mathrm{Hom}_R(M, R)$, then $\mathrm{Im}(f) + (R :_K T) = R$, which implies that $\mathrm{Im}(f)$ is invertible by [10, Lemma

1.8]. So the sequence $M \longrightarrow \mathrm{Im}(f) \longrightarrow 0$ will split. This implies that M must be isomorphic to $\mathrm{Im}(f)$ and projective also. If $\mathrm{Im}(f) \subset (R :_K T)$ for every $f \in M^*$, then tm $(t \in T, m \in M)$ is corresponding to an element of M^{**} by defining $tm(f) = tf(m)$. Since $M = M^{**}$, we have $tm \in M$, which means that M is also a T-ideal. Since T is one-dimensional Notherian, there are only finitely many prime T-ideals containing M. So T_M is a semi-local Dedekind domain. Hence T_M is a principal ideal domain. Thus we have $T_M \cong M_M$ is also a G-projective R_M-ideal. But, by Lemma 2.10, we will have that T is also a G-projective R-module, which contradicts the condition. This contradiction shows that there exists some $f \in M^*$ with $\mathrm{Im}(f) \nsubseteq (R :_K T)$, and this means that M is projective. □

Example 2.12. The polynomial domain $R = \mathbb{Q} + X^n\mathbb{Q}[X] \subset \mathbb{Q}[X]$ $(n \geqslant 3)$ is not a G-Dedekind domain. But every finitely generated G-projective R-module is projective.

Proof. Just notice that the integral closure of R in its quotient field is $\mathbb{Q}[X]$ and the conductor ideal $X^n\mathbb{Q}[X]$ is maximal. The result comes from the fact that R is not a G-Dedekind domain and Lemma 2.10 and Theorem 2.11. □

3. Divisorial ideals of the domain $\mathbb{Q} + X^n\mathbb{Q}[X]$

We begin this section with a characterization of G-Dedekind domains by divisorial ideals. To do this, first we introduce some terminology and notations. For a nonzero fractional ideal I of an integral domain R, set $I_v := (I^{-1})^{-1}$. Then a fractional ideal I is said to be *divisorial* (or a v-ideal) if $I_v = I$. Over a general ring R, Enochs and Jenda defined in [5] the *Gorenstein projective dimension*, denoted by $\mathrm{Gpd}_R(-)$, for arbitrary (non-finite) modules via resolutions with Gorenstein projective modules. The *Gorenstein global dimension* of a ring R, denoted G-gldim(R), is the supremum of the Gorenstein projective dimensions of all R-modules ([18, Definition 11.4.1]). Denote by FPD the classical *finitistic projective dimension* of R. The (left) *finitistic Gorenstein projective dimension* of R, denoted by $\mathrm{FGPD}(R)$, is defined as

$$\mathrm{FGPD}(R) = \sup\left\{\mathrm{Gpd}_R(M) \;\middle|\; \begin{array}{l} M \text{ is a left } R\text{-module with finite} \\ \text{Gorenstein projective dimension} \end{array}\right\}.$$

Theorem 3.1. *Let R be a one-dimensional Noetherian domain with quotient field K. Then the following are equivalent:*

(1) *every divisorial ideal of R is G-projective;*
(2) $\mathrm{Gpd}_R(I) < \infty$ *for any divisorial ideal I of R;*
(3) $\mathrm{Gpd}_R(P) < \infty$ *for any prime ideal P of R;*
(4) *R is a G-Dedekind domain.*

Proof. (1)⇒ (2) This is obvious.

(2)⇒ (3) This comes from the fact that any nonzero prime ideal of a one-dimensional Noetherian domain is divisorial by [13, Theorem 37].

(3)⇒ (4) By [18, Theorem 4.3.21], $\mathrm{FPD}(R) = \dim(R) = 1$. Also notice that $\mathrm{FGPD}(R) = \mathrm{FPD}(R)$ by [7, Theorem 2.28]. By hypothesis we get that $\mathrm{Gpd}_R(P) \leqslant 1$ for any prime ideal P of R. Thus, as in the proof of [10, Proposition 1.7], we get that $\mathrm{id}_R(K/R) \leqslant 1$, which means that R is Gorenstein. Since R is a one-dimensional Noetherian domain, we get that R is in fact a G-Dedekind domain.

(4)⇒ (1) This is obvious since $\text{G-gldim}(R) \leqslant 1$. □

Example 3.2. Set $R := \mathbb{Q} + X^3\mathbb{Q}[X]$. Then the ideal $I := (X^3, X^4, X^5)$ is divisorial. Since I is not G-projective by [10, Example 1.3], R is not a G-Dedekind domain.

Next we consider the divisorial ideals of the domain $\mathbb{Q} + X^n\mathbb{Q}[X]$ $(n \geqslant 2)$. We will show that any two non-projective divisorial ideals of R are isomorphic to each other.

Proposition 3.3. *Let R be a one-dimensional Noetherian domain with quotient field K and integral closure T such that $(R :_K T)$ is maximal and T is a PID. If I is a divisorial ideal of R, then either I is projective or $I \cong T$ as R-modules.*

Proof. As in the proof of Theorem 2.11, if I is not projective, then $\mathrm{Im}(f) \subset (R :_K T)$ for every $f \in I^* = \mathrm{Hom}_R(I, R)$. This means that I is also an ideal of T. Because T is a principal ideal domain, we get that $I = aT$ for some $a \in I$. Thus $I \cong T$ as R-modules. □

Since the domain $R = \mathbb{Q} + X^n\mathbb{Q}[X]$ $(n \geqslant 2)$ satisfies the condition of Proposition 3.3, we get that the divisorial ideals of R can be classified into two classes: one is a class of projective ideals and the other is a class of ideals which are isomorphic to $\mathbb{Q}[X]$. Next we study some properties of invertible ideals of R.

Proposition 3.4. *Let R be a one-dimensional Noetherian domain with quotient field K and integral closure T. If α, β are two elements of T prime to each other, then $(R :_K T) \subset R\alpha + R\beta$.*

Proof. Since α is prime to β, there exist $u, v \in T$ such that $u\alpha + v\beta = 1$. Thus, for any $x \in (R :_K T)$, we have $x = ux\alpha + vx\beta \in R\alpha + R\beta$. □

Theorem 3.5. *Let R be a one-dimensional Noetherian domain with quotient field K and integral closure T such that the conductor ideal $(R :_K T)$ is maximal. If $\alpha \in T$ and $\beta \in (R :_K T)$ such that some power of α is contained in $(R :_K T)$, then the fractional R-ideal $I = (1 + \alpha, \beta)$ is invertible.*

Proof. Assume that $\alpha^n \in (R :_K T)$ for some positive integer n. Let $J = (1 + \sum_{i=1}^{n-1}(-\alpha)^i, \beta)$ be another fractional R-ideal. Then $IJ = (1 + (-1)^{n+1}\alpha^n, \beta + \beta\alpha, \beta + \beta(\sum_{i=1}^{n-1}(-\alpha)^i), \beta^2)$ is an ideal of R such that $IJ + (R :_K T) = R$ since $1 + (-1)^{n+1}\alpha^n$ is not inside the maximal ideal $(R :_K T)$. Therefore IJ, and hence I is invertible by [10, Lemma 1.8]. □

Example 3.6. Set $R := \mathbb{Q} + X^n\mathbb{Q}[X] \subset \mathbb{Q}[X]$ $(n \geqslant 2)$. Then the following fractional R-ideals $(1+X, X^n)$ and $(1+2X+X^2, X^n+X^{n+1})$ are invertible.

Proof. Just notice that the conductor ideal $X^n\mathbb{Q}[X]$ is maximal. □

Let R be a domain with quotient field K. We denote the group of invertible ideals of R by $\operatorname{Inv}(R)$. It contains the group $\operatorname{Prin}(R)$ of fractional principal ideals aR, $a \in K \setminus \{0\}$. The factor group $\operatorname{Pic}(R) = \operatorname{Inv}(R)/\operatorname{Prin}(R)$ is called the Picard group of R [14, Definition 12.5].

Theorem 3.7. *Let $R = \mathbb{Q} + X^n\mathbb{Q}[X]$ $(n \geqslant 2)$. Then the Picard group of R is infinite.*

Proof. Let $I = (1+X^{n-1}, X^n)$. Then I is invertible by Theorem 3.5. We do some calculations to show that any power of I is not principal. Notice that $I^2 = (1+2X^{n-1}+X^{2n}, X^n+X^{2n-1}, X^{2n})$ and $1+2X^{n-1}+X^{2n}$ is prime to X^{2n}. Then we have $X^n\mathbb{Q}[X] \subset I^2$ by Proposition 3.4. Thus we have $I^2 = (1+2X^{n-1}, X^n)$ and inductively $I^k = (1+kX^{n-1}, X^n)$. □

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