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WEAK FACTORIZATIONS OF $H^1(\mathbb{R}^n)$ IN TERMS OF MULTILINEAR FRACTIONAL INTEGRAL OPERATOR ON VARIABLE LEBESGUE SPACES

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Abstract. This paper provides a constructive proof of the weak factorizations of the classical Hardy space $H^1(\mathbb{R}^n)$ in terms of multilinear fractional integral operator on the variable Lebesgue spaces, which the result is new even in the linear case. As a direct application, we obtain a new proof of the characterization of $\mathrm{BMO}(\mathbb{R}^n)$ via the boundedness of commutators of the multilinear fractional integral operator on the variable Lebesgue spaces.

1. Introduction and main results

The theory of Hardy spaces is vast and complicated, it has been systematically developed and plays an important role in harmonic analysis and PDEs. A well-known result of Coifman, Rochberg and Weiss [1] provided a constructive proof of the weak factorizations of the classical Hardy space $H^1(\mathbb{R}^n)$ in terms of Riesz transforms. The result depends upon the duality between $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ and upon a new result linking $\mathrm{BMO}(\mathbb{R}^n)$ and the $L^p(\mathbb{R}^n)$ boundedness of certain commutator operators. Later on, Li and Wick [6] obtained the same results in the multilinear setting. Subsequently, Wang and Zhu [9] proved the factorization theorem for Hardy space via the multilinear fractional integral operator on the weighted Lebesgue spaces.

On the other hand, function spaces with variable exponent arouse strong interest not only in harmonic analysis but also in applied mathematics. The theory of function spaces with variable exponent has made great progress since some elementary properties were given by Kováčik and Rákosník [5] in 1991. Later, Cruz-Uribe, Fiorenza, Martell and Pérez in [2] proved that many classical operators in harmonic analysis, such as maximal operators, singular integrals, commutators and fractional integrals are bounded on the variable Lebesgue

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space. Recently, Tan et al. [7] studied some multilinear operators are bounded on the variable Lebesgue spaces. In [8], Wang proved the weak factorization of Hardy spaces in terms of the Calderón-Zygmund operator on the variable Lebesgue spaces.

Motivated by the works above, we shall show the factorization theorem for Hardy space via the multilinear fractional integral operator on the variable Lebesgue spaces. In particular, the result is new even in the linear case. As a direct corollary, we establish a full characterization of $\mathrm{BMO}(\mathbb{R}^n)$ via commutators of the multilinear fractional integral operator. The key point of the present paper is first: to obtain the factorization theorem for the classical Hardy space in terms of the multilinear fractional integral operator, it needs some tedious calculations in applications; and second: to establish the factorization theorem on the variable Lebesgue spaces without individual conditions on the variable exponent.

Firstly, we recall some standard notations in variable L^p spaces. Given a measurable function $p(\cdot):\mathbb{R}^n\to[1,\infty)$, $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda>0$,

$$\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx<\infty.$$

This is a Banach space with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}}=\inf\left\{\lambda>0:\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx\le 1\right\}.$$

The variable Lebesgue spaces are a special case of Musielak-Orlicz spaces. Define $\mathcal{P}(\mathbb{R}^n)$ to be the set of $p(\cdot):\mathbb{R}^n\to[1,\infty)$ such that

$$p_-=\operatorname*{ess\,inf}_{x\in\mathbb{R}^n}p(x)>1,\ p_+=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}p(x)<\infty.$$

Denote $p'(x)=p(x)/(p(x)-1)$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, see [3].

In variable Lebesgue spaces there are some important lemmas as follows.

Lemma 1.1 ([3]). *If $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and satisfies*

$$|p(x)-p(y)|\le\frac{C}{-\log(|x-y|)},\quad |x-y|\le 1/2,\tag{1.1}$$

and

$$|p(x)-p_\infty|\le\frac{C}{\log(e+|x|)},\quad |y|\ge|x|,\tag{1.2}$$

then $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$.

Lemma 1.2 ([5]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where $r_p = 1 + 1/p_- - 1/p_+$.

This inequality is named the generalized Hölder's inequality with respect to the variable Lebesgue spaces.

Lemma 1.3 ([4]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions* (1.1) *and* (1.2) *in Lemma 1.1. Then*

$$\|\chi_Q\|_{L^{p(\cdot)}} \approx \begin{cases} |Q|^{1/p(x)} & \text{if } |Q| \leq 2^n \text{and } x \in Q, \\ |Q|^{1/p_\infty} & \text{if } |Q| \geq 1 \end{cases}$$

for every cube (*or ball*) *$Q \subset \mathbb{R}^n$, where $p_\infty = \lim_{x\to\infty} p(x)$.*

We now recall the definition of multilinear fractional integral operator. Let $0 < \alpha < mn$, $m \in \mathbb{N}$, the multilinear fractional integral operator is defined by

$$I_\alpha(f_1, f_2, \ldots, f_m)(x) = \int_{\mathbb{R}^{mn}} \frac{\prod_{j=1}^m f_j(y_j)}{|(x - y_1, \ldots, x - y_m)|^{mn-\alpha}} dy_1 \cdots dy_m.$$

For convenience, we also denote $K(x, y_1, \ldots, y_m) = |(x-y_1, \ldots, x-y_m)|^{\alpha-mn}$. For $l = 1, 2, \ldots, m$, we define the multilinear "multiplication" operators Π_l as follows.

$$\begin{aligned}\Pi_l(g, h_1, \ldots, h_m)(x) := h_l(I_\alpha^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x) \\ - gI_\alpha(h_1, \ldots, h_m)(x),\end{aligned}$$

where $(I_\alpha^*)_l$ is the l-th partial adjoint of I_α.

Our main result is the following factorization result for $H^1(\mathbb{R}^n)$ in terms of the multilinear operator Π_l. The result is new even in the linear.

Theorem 1.1. *Let $1 \leq l \leq m$, $0 < \alpha < mn$, $q(\cdot), p_1(\cdot), \ldots, p_m(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\frac{1}{p_1(x)} + \cdots + \frac{1}{p_m(x)} - \frac{\alpha}{n} = \frac{1}{q(x)}$, $x \in \mathbb{R}^n$. Then for any $f \in H^1(\mathbb{R}^n)$, there exist sequences $\{\lambda_s^k\} \in \ell^1$ and functions $g_s^k \in L^{q'(\cdot)}(\mathbb{R}^n), h_{s,1}^k \in L^{p_1(\cdot)}(\mathbb{R}^n), \ldots, h_{s,m}^k \in L^{p_m(\cdot)}(\mathbb{R}^n)$ such that*

$$f = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_s^k \Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k) \tag{1.3}$$

in the sense of $H^1(\mathbb{R}^n)$. Moreover,

$$\|f\|_{H^1(\mathbb{R}^n)} \approx \inf \left\{ \sum_{k=1}^\infty \sum_{s=1}^\infty |\lambda_s^k| \|g_s^k\|_{L^{q'(\cdot)}} \|h_{s,1}^k\|_{L^{p_1(\cdot)}} \cdots \|h_{s,m}^k\|_{L^{p_m(\cdot)}} \right\},$$

where the infimum above is taken over all possible representations of that satisfy (1.3).

As a direct application, we will give the characterization of BMO via commutators of the multilinear fractional integral operator on variable Lebesgue spaces. In analogy with the linear case, we define the l-th possible multilinear commutators of the m-th multilinear fractional integral operator I_α as follows.

$$[b, I_\alpha]_l(f_1, \dots, f_m)(x) := I_\alpha(f_1, \dots, bf_l, \dots, f_m)(x) - bI_\alpha(f_1, \dots, f_m)(x).$$

It was first shown in [7] that given $0 < \alpha < mn$, $q(\cdot), p_1(\cdot), \dots, p_m(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\frac{1}{p_1(x)} + \cdots + \frac{1}{p_m(x)} - \frac{\alpha}{n} = \frac{1}{q(x)}$, $x \in \mathbb{R}^n$, then

$$[b, I_\alpha]_l : L^{p_1(\cdot)} \times \cdots \times L^{p_m(\cdot)} \to L^{q(\cdot)}.$$

Theorem 1.2. *Let $1 \leq l \leq m$, $0 < \alpha < mn$, $q(\cdot), p_1(\cdot), \dots, p_m(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\frac{1}{p_1(x)} + \cdots + \frac{1}{p_m(x)} - \frac{\alpha}{n} = \frac{1}{q(x)}$, $x \in \mathbb{R}^n$. The commutator $[b, I_\alpha]_l$ is bounded from $L^{p_1(\cdot)} \times \cdots \times L^{p_m(\cdot)}$ to $L^{q(\cdot)}$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

Throughout this paper, the letter C denotes constants which are independent of the main variables and may change from one occurrence to another. We use the symbol $A \lesssim B$ denote that there exists a constant $C > 0$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \approx B$.

2. Auxiliary lemmas and proof of theorems

In this section we turn to proving our main results. We collect some facts that will be useful in proving the main result. We first provide a technical lemma about certain $H^1(\mathbb{R}^n)$ functions.

Lemma 2.1. *Let f be a function satisfying the following estimates*:

(i) $\int_{\mathbb{R}^n} f(x)dx = 0$;

(ii) *there exist balls $B_1 = B(x_1, r)$ and $B_2 = B(x_2, r)$ for some $x_1, x_2 \in \mathbb{R}^n$ and $r > 0$ such that*

$$|f(x)| \leq u_1(x)\chi_{B_1}(x) + u_2(x)\chi_{B_2}(x),$$

where $\|u_i\|_{L^\infty} \leq Cr^{-n}$, $i = 1, 2$;

(iii) $|x_1 - x_2| \geq 4r$.

Then there exists a positive constant C independent of x_1, x_2, r such that

$$\|f\|_{H^1} \leq C \log \frac{|x_1 - x_2|}{r}.$$

Proof. Assume that $f := f_1 + f_2$, where $|f_i| \leq u_i$ and $\operatorname{supp} f_i \subset B_i$, $i = 1, 2$. We will show that f has the following atomic decomposition

$$f = \sum_{i=1}^{2} \sum_{j=1}^{J_0+1} \lambda_i^j a_i^j, \tag{2.1}$$

where J_0 is the smallest integer larger than $\log \frac{|x_1 - x_2|}{r}$ and for each j, a_i^j is a $(1, \infty)$-atom and λ_i^j a real number satisfying that

$$|\lambda_i^j| \leq C. \tag{2.2}$$

To this end, for $i = 1, 2$, we write

$$f_i = \big[f_i(x) - \tilde{\lambda}_i^1 \chi_{B_i}\big] + \tilde{\lambda}_i^1 \chi_{B_i} = f_i^1(x) + \tilde{\lambda}_i^1 \chi_{B_i},$$

where

$$\tilde{\lambda}_i^1 := \frac{1}{|2B_i|} \int_{B_i} f_i(x)dx.$$

Let $\lambda_i^1 := \|f_i^1\|_{L^\infty} |2B_i|$ and $a_i^1 := f_i^1/\lambda_i^1$. From the fact that

$$\|a_i^1\|_{L^\infty} = \frac{\|f_i^1\|_{L^\infty}}{\lambda_i^1} = \frac{1}{|2B_i|},$$

we know that a_i^1 is a $(1, \infty)$-atom supported on $2B_i$ and λ_i^1 satisfies (2.2). For $i = 1, 2$, we further write

$$\tilde{\lambda}_i^1 \chi_{2B_i} = \tilde{\lambda}_i^1 \chi_{2B_i} - \tilde{\lambda}_i^2 \chi_{4B_i} + \tilde{\lambda}_i^2 \chi_{4B_i} =: f_i^2 + \tilde{\lambda}_i^2 \chi_{4B_i},$$

where

$$\tilde{\lambda}_i^2 := \frac{1}{|4B_i|} \int_{B_i} f_i(x)dx.$$

Let $\lambda_i^2 := \|f_i^2\|_{L^\infty} |4B_i|$ and $a_i^2 := f_i^2/\lambda_i^2$. Then we see that a_i^2 is a $(1, \infty)$-atom supported on $4B_i$ and

$$|\lambda_i^2| \le |\tilde{\lambda}_i^1| |4B_i| \le \frac{|4B_i|}{|2B_i|} \|f_i\|_\infty |B_i| \le C.$$

Continuing in this process we see that for $j \in \{1, 2, \dots, J_0\}$,

$$f = \sum_{i=1}^2 \left[\sum_{j=1}^{J_0} f_i^j\right] + \sum_{i=1}^2 \tilde{\lambda}_i^{J_0} \chi_{2^{J_0}B_i} = \sum_{i=1}^2 \left[\sum_{j=1}^{J_0} \lambda_i^j a_i^j\right] + \sum_{i=1}^2 \tilde{\lambda}_i^{J_0} \chi_{2^{J_0}B_i},$$

where for $j \in \{2, 3, \dots, J_0\}$,

$$\begin{aligned} \tilde{\lambda}_i^j &:= \frac{1}{|2^j B_i|} \int_{B_i} f_i(x)dx, \\ f_i^j &:= \tilde{\lambda}_i^{j-1} \chi_{2^{j-1}B_i} - \tilde{\lambda}_i^j \chi_{2^j B_i}, \\ \lambda_i^j &:= \|f_i^j\|_{L^\infty} |2^j B_i|, \quad \text{and} \quad a_i^j := f_i^j/\lambda_i^j. \end{aligned}$$

Moreover, for each i and j, a_i^j is a $(1, \infty)$-atom and $|\lambda_i^j| \le C$.

For $\sum_{i=1}^2 \tilde{\lambda}_i^{J_0} \chi_{2^{J_0}B_i}$, we set

$$\begin{aligned} \tilde{\lambda}^{J_0} &:= \frac{1}{|B(\frac{x_1+x_2}{2}, 2^{J_0+1}r)|} \int_{B(x_1,r)} f_1(x)dx \\ &:= -\frac{1}{|B(\frac{x_1+x_2}{2}, 2^{J_0+1}r)|} \int_{B(x_2,r)} f_2(x)dx. \end{aligned}$$

This shows that

$$\sum_{i=1}^2 \tilde{\lambda}_i^{J_0} \chi_{B(x_i, 2^{J_0}r)} = \big[\tilde{\lambda}_1^{J_0} \chi_{B(x_1, 2^{J_0}r)} - \tilde{\lambda}^{J_0} \chi_{B(\frac{x_1+x_2}{2}, 2^{J_0+1}r)}\big]$$

$$
\begin{aligned}
&+ \big[\tilde{\lambda}^{J_0}\chi_{B(\frac{x_1+x_2}{2},2^{J_0+1}r)} + \tilde{\lambda}_2^{J_0}\chi_{B(x_2,2^{J_0}r)}\big] \\
=: &\sum_{i=1}^{2} f_i^{J_0+1}.
\end{aligned}
$$

For $i = 1, 2$, let

$$\lambda_i^{J_0+1} := \|f_i^{J_0+1}\|_{L^\infty}|B(\tfrac{x_1+x_2}{2}, 2^{J_0+1}r)|$$

and

$$a_i^{J_0+1} := f_i^{J_0+1}/\lambda_i^{J_0+1}.$$

Then we see that $a_i^{J_0+1}$ is a $(1,\infty)$-atom and $\lambda_i^{J_0+1}$ satisfies (2.2). Thus, we have (2.1) holds, which implies that $f \in H^1(\mathbb{R}^n)$ with

$$\|f\|_{H^1} \le \sum_{i=1}^{2}\sum_{j=1}^{J_0+1} |\lambda_i^j| \le C \log\frac{|x_1 - x_2|}{r}.$$

This finishes the proof of Lemma 2.1. □

Lemma 2.2. *Suppose* $1 \le l \le m$. *Let* $0 < \alpha < mn$, $q(\cdot), p_1(\cdot), \dots, p_m(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *with*

$$\frac{1}{p_1(x)} + \cdots + \frac{1}{p_m(x)} - \frac{\alpha}{n} = \frac{1}{q(x)},\ x \in \mathbb{R}^n.$$

There exists a positive constant C *such that for any* $g \in L^{q'(\cdot)}(\mathbb{R}^n)$ *and* $h_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i = 1, 2, \dots, m$,

$$\|\Pi_l(g, h_1, \dots, h_m)\|_{H^1(\mathbb{R}^n)} \le C\|g\|_{L^{q'(\cdot)}}\|h_1\|_{L^{p_1(\cdot)}} \cdots \|h_m\|_{L^{p_m(\cdot)}}.$$

Proof. Note that for any $g \in L^{q'(\cdot)}(\mathbb{R}^n)$ and $h_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i = 1, 2, \dots, m$, by Lemma 1.2, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} |g(x)I_\alpha(h_1, \dots, h_m)(x)|dx &\le C\|g\|_{L^{q'(\cdot)}}\|I_\alpha(h_1, \dots, h_m)\|_{L^{q(\cdot)}} \\
&\le C\|g\|_{L^{q'(\cdot)}}\prod_{i=1}^{m}\|h_i\|_{L^{p_i(\cdot)}}.
\end{aligned}
$$

On the other hand, the directly calculation gives that

$$\frac{1}{p_l'(x)} = \sum_{j\neq l}\frac{1}{p_j(x)} + \frac{1}{q'(x)} - \frac{\alpha}{n},$$

form which follows that

$$I_\alpha : L^{p_1(\cdot)} \times \cdots \times L^{p_{l-1}(\cdot)} \times L^{q'(\cdot)} \times L^{p_{l+1}(\cdot)} \times \cdots \times L^{p_m(\cdot)} \to L^{p_l'(\cdot)}.$$

This implies that $\Pi_l(g, h_1, \dots, h_m)(x) \in L^1(\mathbb{R}^n)$ by Hölder duality. Moreover,

$$\int_{\mathbb{R}^n} \Pi_l(g, h_1, \dots, h_m)(x)dx = 0.$$

Hence, for $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} b(x)\Pi_l(g,h_1,\ldots,h_m)(x)dx\right| &= \left|\int_{\mathbb{R}^n} g(x)[b,I_\alpha]_l(h_1,\ldots,h_m)(x)dx\right| \\ &\le C\|g\|_{L^{q'(\cdot)}}\|[b,I_\alpha]_l(h_1,\ldots,h_m)\|_{L^{q(\cdot)}} \\ &\le C\|g\|_{L^{q'(\cdot)}}\|h_1\|_{L^{p_1(\cdot)}}\cdots\|h_m\|_{L^{p_m(\cdot)}}\|b\|_{BMO}.\end{aligned}$$

Therefore, $\Pi_l(g,h_1,\ldots,h_m)$ is in $H^1(\mathbb{R}^n)$ with

$$\|\Pi_l(g,h_1,\ldots,h_m)\|_{H^1(\mathbb{R}^n)} \le C\|g\|_{L^{q'(\cdot)}}\|h_1\|_{L^{p_1(\cdot)}}\cdots\|h_m\|_{L^{p_m(\cdot)}}.$$

The proof of Lemma 2.2 is completed. □

Lemma 2.3. *Let $1 \le l \le m$, $0 < \alpha < mn$, $q(\cdot), p_1(\cdot),\ldots,p_m(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\frac{1}{p_1(x)}+\cdots+\frac{1}{p_m(x)}-\frac{\alpha}{n}=\frac{1}{q(x)}$, $x\in\mathbb{R}^n$. For every $H^1(\mathbb{R}^n)$-atom $a(x)$ and for all $\varepsilon>0$, there exist $g \in L^{q'(\cdot)}(\mathbb{R}^n)$ and $h_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i=1,2,\ldots,m$, and a large positive number M (depending only on ε) such that*

$$\|a-\Pi_l(g,h_1,\ldots,h_m)\|_{H^1(\mathbb{R}^n)} < \varepsilon$$

and that $\|g\|_{L^{q'(\cdot)}}\|h_1\|_{L^{p_1(\cdot)}}\cdots\|h_m\|_{L^{p_m(\cdot)}} \le CM^{mn-\alpha}$, where C is an absolute positive constant.

Proof. Let $a(x)$ be an $H^1(\mathbb{R}^n)$-atom, supported in $B(x_0,r)$, satisfying that

$$\int_{\mathbb{R}^n} a(x)dx = 0 \quad \text{and} \quad \|a\|_{L^\infty(\mathbb{R}^n)} \le |B(x_0,r)|^{-1}.$$

Fix $1 \le l \le m$. Now select $y_l \in \mathbb{R}^n$ so that $y_{l,i}-x_{0,i}=\frac{Mr}{\sqrt{n}}$, where $x_{0,i}$ (resp. $y_{l,i}$) is the i-th coordinate of x_0 (resp. y_l) for $i=1,2,\ldots,n$. Note that for this y_l, we have $|x_0-y_l|=Mr$. Similar to the relation of x_0 and y_l, we choose y_1 such that y_0 and y_1 satisfies the same relationship as x_0 and y_l do. Then by induction we choose $y_2,\ldots,y_{l-1},y_{l+1},\ldots,y_m$. We write $B_i=B(y_i,r)$ and set

$$\begin{aligned} g(x) &:= \chi_{B_l}(x), \\ g(B_l) &:= \int_{B(y_l,r)} g(z_l)dz_l, \\ h_j(x) &:= \chi_{B_j}(x),\ j\ne l, \\ h_l(x) &:= \frac{a(x)}{(I_\alpha^*)_l(h_1,\ldots,h_{l-1},g,h_{l+1},\ldots,h_m)(x_0)}\frac{g(B_l)}{|B_l|}\chi_{B_l}(x). \end{aligned}$$

For the specific choice of the functions $h_1,\ldots,h_{l-1},g,h_{l+1},\ldots,h_m$ as above, we have that there exists a positive constant C such that

$$\begin{aligned} &|(I_\alpha^*)_l(h_1,\ldots,h_{l-1},g,h_{l+1},\ldots,h_m)(x_0)| \\ &\ge C\int_{B_1\times\cdots\times B_m} \frac{g(z_l)\prod_{j\ne l}h_j(z_j)}{(|x_0-z_1|+\cdots+|x_0-z_m|)^{mn-\alpha}}dz_1\cdots dz_m \end{aligned}$$

$$\geq C(Mr)^{\alpha-mn} g(B_l) \prod_{j\neq l} |B_j|.$$

The definition of the functions g and h_j gives that $\operatorname{supp} g = B(y_l, r)$ and $\operatorname{supp} h_j = B(y_j, r)$. Moreover,

$$\|g\|_{L^{q'(\cdot)}} = \|\chi_{B_l}\|_{L^{q'(\cdot)}} \quad \text{and} \quad \|h_j\|_{L^{p_j(\cdot)}} = \|\chi_{B_j}\|_{L^{p_j(\cdot)}}$$

for $j = 1, \dots, l-1, l+1, \dots, m$. Also we have

$$\begin{aligned}\|h_l\|_{L^{p_l(\cdot)}} &\leq \frac{\|a\|_{L^\infty}}{|(I_\alpha^*)_l(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0)|} \frac{g(B_l)}{|B_l|} \|\chi_{B_l}\|_{L^{p_l(\cdot)}} \\ &\leq C(Mr)^{mn-\alpha} \prod_{j=1}^m |B_j|^{-1} r^{-n} \|\chi_{B_l}\|_{L^{p_l(\cdot)}}.\end{aligned}$$

Combining the estimates above, if $|B(y_l, r)| = |B(y_j, r)| > 1$, by Lemma 1.3, we have

$$\begin{aligned}&\|g\|_{L^{q'(\cdot)}} \|h_1\|_{L^{p_1(\cdot)}} \cdots \|h_m\|_{L^{p_m(\cdot)}} \\ &\leq C\|\chi_{B_l}\|_{L^{q'(\cdot)}} \prod_{j\neq l} \|\chi_{B_j}\|_{L^{p_j(\cdot)}} (Mr)^{mn-\alpha} \prod_{j=1}^m |B_j|^{-1} r^{-n} \|\chi_{B_l}\|_{L^{p_l(\cdot)}} \\ &\leq C|B_l|^{\frac{1}{q'_\infty}} \prod_{j=1}^m |B_j|^{\frac{1}{(p_j)_\infty}} (Mr)^{mn-\alpha} r^{-mn-n} \\ &\leq CM^{mn-\alpha},\end{aligned}$$

where

$$q'_\infty = \lim_{x\to\infty} q'(x) = \lim_{x\to\infty} \left(1 + \frac{1}{q(x)-1}\right) = 1 + \frac{1}{q_\infty - 1}.$$

If $|B(y_l, r)| = |B(y_j, r)| < 2^n$, Lemma 1.3 implies that

$$\begin{aligned}&\|g\|_{L^{q'(\cdot)}} \|h_1\|_{L^{p_1(\cdot)}} \cdots \|h_m\|_{L^{p_m(\cdot)}} \\ &\leq C\|\chi_{B_l}\|_{L^{q'(\cdot)}} \prod_{j\neq l} \|\chi_{B_j}\|_{L^{p_j(\cdot)}} (Mr)^{mn-\alpha} \prod_{j=1}^m |B_j|^{-1} r^{-n} \|\chi_{B_l}\|_{L^{p_l(\cdot)}} \\ &\leq C|B_l|^{\frac{1}{q'(y_l)}} \prod_{j=1}^m |B_j|^{\frac{1}{p_j(y_j)}} (Mr)^{mn-\alpha} r^{-mn-n} \\ &\leq Cr^{\left(\frac{1}{p_1(y_1)} - \frac{1}{p_1(y_l)} + \cdots + \frac{1}{p_m(y_m)} - \frac{1}{p_m(y_l)} + 1 - \frac{\alpha}{n}\right)n} (Mr)^{mn-\alpha} r^{-mn-n} \\ &\leq CM^{mn-\alpha},\end{aligned}$$

where in the last inequality we use the fact that

$$\frac{1}{p_j(y_j)} - \frac{1}{p_j(y_l)} = \frac{p_j(y_l) - p_j(y_j)}{p_j(y_j)p_j(y_l)} \leq \frac{C}{((p_j)_-)^2}$$

and $|B(y_l, r)| = |B(y_j, r)| = r^n < 2^n$.

Next, we have

$$\begin{aligned}
&a(x) - \Pi_l(g, h_1, \ldots, h_m)(x) \\
&= a(x) - \Big(h_l (I_\alpha^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x) - g I_\alpha(h_1, \ldots, h_m)(x)\Big) \\
&= a(x)\frac{(I_\alpha^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0) - (I_\alpha^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x)}{(I_\alpha^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)} \\
&\quad + g(x) I_\alpha(h_1, \ldots, h_m)(x) \\
&=: W_1(x) + W_2(x).
\end{aligned}$$

By definition, it is obvious that $W_1(x)$ is supported on $B(x_0, r)$ and $W_2(x)$ is supported on $B(y_0, r)$.

We first estimate $W_1(x)$. For $x \in B(x_0, r)$, we have

$$\begin{aligned}
&|W_1(x)| \\
&= |a(x)| \frac{|(I_\alpha^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0) - (I_\alpha^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x)|}{|(I_\alpha^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)|} \\
&\le \frac{C\|a\|_{L^\infty}}{(Mr)^{\alpha - mn} g(B_l) \prod_{j \ne l} |B_j|} \int_{\prod_{j=1}^m B(y_j, r)} \frac{|x - x_0| g(z_l) \prod_{j \ne l} h_j(z_j)}{(\sum_{i=1, i \ne l}^m |z_l - z_i| + |z_l - x_0|)^{mn - \alpha + 1}} dz_1 \cdots dz_m \\
&\le \frac{C r^{-n}}{(Mr)^{\alpha - mn} g(B_l) \prod_{j \ne l} |B_j|} \frac{r g(B_l) \prod_{j \ne l} |B_j|}{(Mr)^{mn - \alpha + 1}} \\
&\le \frac{C}{M r^n}.
\end{aligned}$$

Hence we obtain that

$$|W_1(x)| \le \frac{C}{M r^n} \chi_{B(x_0, r)}(x).$$

Next we estimate $W_2(x)$. From the definition of $g(x)$ and $h_j(x)$, we have

$$\begin{aligned}
&|I_\alpha(h_1, \ldots, h_m)(x)| \\
&= \frac{1}{|(I_\alpha^*)_l(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)} \\
&\quad \times \bigg| \int_{\prod_{j \ne l} B(y_j, r) \times B(x_0, r)} \big(K(z_1, \ldots, z_{l-1}, x_0, z_{l+1}, \ldots, z_m) \\
&\qquad - K(z_1, \ldots, z_{l-1}, x, z_{l+1}, \ldots, z_m)\big) a(z_l) \prod_{j \ne l} h_j(z_j) dz_1 \cdots dz_m \bigg| \\
&\le \frac{C(Mr)^{mn - \alpha} \|a\|_{L^\infty}}{g(B_l) \prod_{j \ne l} |B_j|} \int_{\prod_{j \ne l} B(y_j, r) \times B(x_0, r)} \frac{|x - x_0| \chi_{B_l}(z_l) \prod_{j \ne l} |h_j(z_j)|}{(\sum_{i=1, i \ne l}^m |z_l - z_i| + |z_l - x_0|)^{mn - \alpha + 1}} dz_1 \cdots dz_m \\
&\le \frac{C}{M r^n},
\end{aligned}$$

where in the first equality we use the cancellation property of the atom $a(z_l)$. It follows that

$$W_2(x) \le \frac{C}{Mr^n}\chi_{B(y_l,r)}(x).$$

Combining the estimates of $W_1(x)$ and $W_2(x)$, we obtain that

$$|a(x) - \Pi_l(g, h_1, \dots, h_m)(x)| \le \frac{C}{Mr^n}(\chi_{B(x_0,r)}(x) + \chi_{B(y_l,r)}(x)). \tag{2.3}$$

Notice that

$$\int_{\mathbb{R}^n} [a(x) - \Pi_l(g, h_1, \dots, h_m)(x)]dx = 0, \tag{2.4}$$

because the atom $a(x)$ has cancellation and the second integral equals 0 just by the definitions of Π_l. Then the size estimate (2.3) and the cancellation (2.4), together with Lemma 2.1, show that

$$\|a - \Pi_l(g, h_1, \dots, h_m)\|_{H^1(\mathbb{R}^n)} < C\frac{\log M}{M}.$$

For M sufficiently large such that

$$\frac{C\log M}{M} < \varepsilon.$$

Thus, the result follows from here. □

With this approximation result above, we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.2, it is obvious that

$$\|\Pi_l(g, h_1, \dots, h_m)\|_{H^1(\mathbb{R}^n)} \le C\|g\|_{L^{q'(\cdot)}}\|h_1\|_{L^{p_1(\cdot)}} \cdots \|h_m\|_{L^{p_m(\cdot)}}.$$

It is immediate that for any representation of f as in (1.3), i.e.,

$$f = \sum_{k=1}^{\infty}\sum_{s=1}^{\infty} \lambda_s^k \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k),$$

with

$$\|f\|_{H^1(\mathbb{R}^n)} \le C\bigg\{\sum_{k=1}^{\infty}\sum_{s=1}^{\infty} |\lambda_s^k|\|g_s^k\|_{L^{q'(\cdot)}}\|h_{s,1}^k\|_{L^{p_1(\cdot)}} \cdots \|h_{s,m}^k\|_{L^{p_m(\cdot)}}\bigg\},$$

where the infimum above is taken over all possible representations of that satisfy (1.3).

We turn to show that the other inequality holds and that it is possible to obtain such a decomposition for any $f \in H^1(\mathbb{R}^n)$. Applying the atomic decomposition, for any $f \in H^1(\mathbb{R}^n)$ we can find a sequence $\{\lambda_s^1\} \in \ell^1$ and a sequence of $H^1(\mathbb{R}^n)$-atom $\{a_s^1\}$ so that $f = \sum_{s=1}^{\infty} \lambda_s^1 a_s^1$ and $\sum_{s=1}^{\infty} |\lambda_s^1| \le C\|f\|_{H^1}$.

Fix $\varepsilon > 0$ so that $C\varepsilon < 1$. We apply Lemma 2.3 to each atom a_s^1, then there exist $g_s^1 \in L^{q'(\cdot)}(\mathbb{R}^n)$, $h_{s,1}^1 \in L^{p_1(\cdot)}(\mathbb{R}^n), \dots, h_{s,m}^1 \in L^{p_m(\cdot)}(\mathbb{R}^n)$ with

$$\|a_s^1 - \Pi_{j,l}(g_s^1, h_{s,1}^1, \dots, h_{s,m}^1)\|_{H^1} < \varepsilon, \quad \forall s$$

and

$$\|g_s^1\|_{L^{q'(\cdot)}}\|h_{s,1}^1\|_{L^{p_1(\cdot)}}\cdots\|h_{s,m}^1\|_{L^{p_m(\cdot)}}\le C(\varepsilon,\alpha),$$

where $C(\varepsilon,\alpha)=CM^{mn-\alpha}$ is a constant depending on ε and α. Notice that

$$\begin{aligned}f&=\sum_{s=1}^{\infty}\lambda_s^1a_s^1\\&=\sum_{s=1}^{\infty}\lambda_s^1\Pi_l(g_s^1,h_{s,1}^1,\dots,h_{s,m}^1)+\sum_{s=1}^{\infty}\lambda_s^1(a_s^1-\Pi_l(g_s^1,h_{s,1}^1,\dots,h_{s,m}^1))\\&=:M_1+E_1.\end{aligned}$$

Moreover,

$$\|E_1\|_{H^1}\le\sum_{s=1}^{\infty}|\lambda_s^1|\|a_s^1-\Pi_l(g_s^1,h_{s,1}^1,\dots,h_{s,m}^1)\|_{H^1}\le\varepsilon\sum_{s=1}^{\infty}|\lambda_s^1|\le C\varepsilon\|f\|_{H^1}.$$

In addition, since $E_1\in H^1(\mathbb{R}^n)$, we can also find a sequence $\{\lambda_s^2\}\in\ell^1$ and a sequence of $H^1(\mathbb{R}^n)$-atom $\{a_s^2\}$ so that $E_1=\sum_{s=1}^{\infty}\lambda_s^2a_s^2$ and

$$\sum_{s=1}^{\infty}|\lambda_s^2|\le C\|E_1\|_{H^1}\le C^2\varepsilon\|f\|_{H^1}.$$

Again, applying Lemma 2.2 to each atom a_s^2, then there exist $g_s^2\in L^{q'(\cdot)}(\mathbb{R}^n)$, $h_{s,1}^2\in L^{p_1(\cdot)}(\mathbb{R}^n),\dots,h_{s,m}^2\in L^{p_m(\cdot)}(\mathbb{R}^n)$ with

$$\|a_s^2-\Pi_{j,l}(g_s^2,h_{s,1}^2,\dots,h_{s,m}^2)\|_{H^1}<\varepsilon,\qquad\forall s.$$

We then have

$$\begin{aligned}E_1&=\sum_{s=1}^{\infty}\lambda_s^2a_s^2\\&=\sum_{s=1}^{\infty}\lambda_s^2\Pi_l(g_s^2,h_{s,1}^2,\dots,h_{s,m}^2)+\sum_{s=1}^{\infty}\lambda_s^2(a_s^2-\Pi_l(g_s^2,h_{s,1}^2,\dots,h_{s,m}^2))\\&=:M_2+E_2.\end{aligned}$$

As before, obvious that

$$\|E_2\|_{H^1}\le\sum_{s=1}^{\infty}|\lambda_s^2|\|a_s^2-\Pi_l(g_s^2,h_{s,1}^2,\dots,h_{s,m}^2)\|_{H^1}\le\varepsilon\sum_{s=1}^{\infty}|\lambda_s^2|\le(C\varepsilon)^2\|f\|_{H^1}.$$

This gives us that

$$\begin{aligned}f&=\sum_{s=1}^{\infty}\lambda_s^1a_s^1\\&=\sum_{s=1}^{\infty}\lambda_s^1\Pi_l(g_s^1,h_{s,1}^1,\dots,h_{s,m}^1)+\sum_{s=1}^{\infty}\lambda_s^1(a_s^1-\Pi_l(g_s^1,h_{s,1}^1,\dots,h_{s,m}^1))\\&=M_1+E_1=M_1+M_2+E_2\end{aligned}$$

$$= \sum_{k=1}^{2}\sum_{s=1}^{\infty}\lambda_s^k \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k) + E_2.$$

Repeating this construction for each $1 \le k \le K$ produces functions $g_s^k \in L^{q'(\cdot)}(\mathbb{R}^n)$, $h_{s,1}^k \in L^{p_1(\cdot)}(\mathbb{R}^n), \dots, h_{s,m}^k \in L^{p_m(\cdot)}(\mathbb{R}^n)$ with

$$\|g_s^k\|_{L^{q'(\cdot)}}\|h_{s,1}^k\|_{L^{p_1(\cdot)}}\cdots\|h_{s,m}^k\|_{L^{p_m(\cdot)}} \le C(\varepsilon,\alpha), \qquad \forall s,$$

a sequence $\{\lambda_s^k\} \in \ell^1$ with $\|\lambda_s^k\|_{\ell^1} \le C^k\varepsilon^{k-1}\|f\|_{H^1}(\mathbb{R}^n)$, and a function $E_K \in H^1(\mathbb{R}^n)$ with

$$\|E_K\|_{H^1} \le (C\varepsilon)^K\|f\|_{H^1}$$

so that

$$f = \sum_{k=1}^{K}\sum_{s=1}^{\infty}\lambda_s^k \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k) + E_K.$$

Letting $K \to \infty$ gives the desired decomposition of

$$f = \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}\lambda_s^k \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k).$$

We conclude that

$$\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k| \le \sum_{k=1}^{\infty}\varepsilon^{-1}(C\varepsilon)^k\|f\|_{H^1} = \frac{C}{1-\varepsilon C}\|f\|_{H^1}.$$

Thus, we have completed the proof of Theorem 1.1. □

Finally, we dispense with the proof of Theorem 1.2.

Proof of Theorem 1.2. The upper bound in this theorem is contained in [7]. For the lower bound, suppose that $f \in H^1(\mathbb{R}^n)$, using the weak factorization in Theorem 1.1 and the boundedness of $[b, I_\alpha]_l$, we obtain

$$\begin{aligned}\langle b, f\rangle_{L^2} &= \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}\lambda_s^k\langle b, \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k)\rangle_{L^2}\\ &= \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}\lambda_s^k\langle g_s^k, [b, I_\alpha]_l(h_{s,1}^k, \dots, h_{s,m}^k)\rangle_{L^2}.\end{aligned}$$

Hence, we have that

$$\begin{aligned}\left|\langle b, f\rangle_{L^2}\right| &\le \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k|\|g_s^k\|_{L^{q'(\cdot)}}\|[b, I_\alpha]_l(h_{s,1}^k, \dots, h_{s,m}^k)\|_{L^{q(\cdot)}}\\ &\le \|[b, I_\alpha]_l\|_{L^{p_1(\cdot)}\times\cdots\times L^{p_m(\cdot)}\to L^{q(\cdot)}}\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k|\|g_s^k\|_{L^{q'(\cdot)}}\prod_{j=1}^{m}\|h_{s,j}^k\|_{L^{p_j(\cdot)}}\\ &\le C\|[b, I_\alpha]_l\|_{L^{p_1(\cdot)}\times\cdots\times L^{p_m(\cdot)}\to L^{q(\cdot)}}\|f\|_{H^1}.\end{aligned}$$

From the duality theorem between $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$, it follows that $b \in \mathrm{BMO}(\mathbb{R}^n)$. □

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