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AUTOMORPHISMS OF K3 SURFACES WITH PICARD NUMBER TWO

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Abstract. It is known that the automorphism group of a K3 surface with Picard number two is either an infinite cyclic group or an infinite dihedral group when it is infinite. In this paper, we study the generators of such automorphism groups. We use the eigenvector corresponding to the spectral radius of an automorphism of infinite order to determine the generators.

1. Introduction

The aim of this paper is to give some conditions on the generators of the automorphism group of a K3 surface of Picard number 2 (Theorem 1.1 and Theorem 1.2). For a K3 surface X with rank two Picard lattice, Galluzzi, Lombardo and Peters [4] applied the classical theory of binary quadratic forms to prove that the automorphism group $\mathrm{Aut}(X)$ is trivial or $\mathbb{Z}_2$ if it is finite. Moreover, if it is infinite, the automorphism group is an infinite cyclic group or an infinite dihedral group. In this paper, we find conditions for the generators of the automorphism group by using the eigenvector corresponding to the spectral radius of an automorphism of infinite order.

Let g be an automorphism of a compact complex surface X. It is known that the topological entropy $h(g)$ is determined by the spectral radius ρ of g^* acting on $H^*(X)$, that is, $h(g) = \log \rho(g^*|H^2(X))$. If $h(g) > 0$, then a minimal model for X is either a K3 surface, an Enriques surface, a complex torus or a rational surface [3]. In the sense of dynamics of automorphisms, it is a natural question to find a minimal possible entropy. For example, for a K3 surface, one constructs an automorphism synthetically by the lattice theory and Torelli theorem to find the minimal entropy ([9]). However, in this paper, we go the

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other way around. That is, using topological entropies, we determine automorphisms of X. More precisely, by finding eigenvectors of an automorphism, we can find the generators of the automorphism group of a K3 surface.

The main observation of this idea is the fact that the solutions of Pell equation associated with a non-square number form an infinite group generated by *Pell multiples* of finite solutions (cf. Section 2.1). For some $k \in \mathbb{Z} \setminus \{0, -1\}$, if we consider a non-empty set A_k (cf. (10) in Section 3) of divisors each of which has the self-intersection number $2k$, then by $g \in \mathrm{Aut}(X)$, $g^*(D) \in A_k$ for any $D \in A_k$, where $g^* := g^*|S_X$ and S_X is the Picard lattice of X. In particular, for $(u, v) \in A_k$, if we consider a sequence of $(u_n, v_n) := g^{n*}(u, v)$ and if the ratio u_n/v_n converges to U/V as n increases, then (U, V) will be the eigenvector corresponding to the spectral radius of $g^*|S_X$. We use this eigenvector to find g. Furthermore, using this eigenvector and g, we can also determine anti-symplectic involutions when $\mathrm{Aut}(X) \cong \mathbb{Z}_2 * \mathbb{Z}_2$. An automorphism g of X is said to be *symplectic* if $g^*\omega_X = \omega_X$ and *anti-symplectic* if $g^*\omega_X = -\omega_X$, where ω_X is a nowhere vanishing holomorphic 2-form of X.

Let X be a K3 surface with Picard lattice S_X whose self-intersection matrix is

$$Q_{S_X} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \tag{1}$$

for some basis with $d := -\mathrm{disc}(S_X) = b^2 - 4ac > 0$. Let g be an automorphism of X with $g^*|S_X$ given by the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{2}$$

Theorem 1.1. *For a* K3 *surface X with intersection matrix of Picard lattice S_X as above, an automorphism g of infinite order acting on S_X as in* (2) *satisfies*

$$\gamma = -\frac{a}{c}\beta,\ \delta = \alpha - \frac{b}{c}\beta \ \text{ and } \ \alpha^2 - \frac{b}{c}\alpha\beta + \frac{a}{c}\beta^2 = 1. \tag{3}$$

Moreover, $g^*|S_X$ is a power of an isometry h of S_X defined by the matrix

$$h = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\frac{a}{c}\beta_1 & \alpha_1 - \frac{b}{c}\beta_1 \end{pmatrix}, \tag{4}$$

where $(2\alpha_1 - b\frac{\beta_1}{c}, \frac{\beta_1}{c})$ is the minimal positive solution of Pell equation $x^2 - dy^2 = 4$.

Theorem 1.2. *Let X be a* K3 *surface with Picard lattice S_X whose intersection matrix is* (1)*. If $\mathrm{Aut}(X) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, then an involution ι acting on S_X by the matrix in* (2) *satisfies*

$$\delta = -\alpha,\ \gamma = -\frac{b}{c}\alpha + \frac{a}{c}\beta \ \text{ and } \ \alpha^2 - \frac{b}{c}\alpha\beta + \frac{a}{c}\beta^2 = 1. \tag{5}$$

Remark 1.3. By Lemma 2.6 in Section 2.2, we can easily see whether an isometry acts on S_X as $-\mathrm{id}$ or not. If it acts on S_X as $-\mathrm{id}$, then it extends to an isometry of $H^2(X,\mathbb{Z})$. Moreover, if it preserves the ample cone, then by Torelli theorem, it extends to an anti-symplectic involution.

The structure of this paper is the following: In Section 2 we recall some results about Pell equations and lattices. In Section 3 we prove Theorem 1.1 and Theorem 1.2. In Section 4 we apply our results to several examples to find the generators of the automorphism group of a K3 surface of Picard number 2.

2. Preliminaries

2.1. Pell equations

For a positive integer d, an equation of the form

$$u^2 - dv^2 = 1 \tag{6}$$

is called a *Pell equation.* We are interested in solutions (u,v), where u and v are integers. Solutions with $u>0$ and $v>0$ will be called *positive solutions.* It is known in [8] that for every non-square positive integer d, the equation (6) has a nontrivial solution with $v \neq 0$. Moreover, the solutions of Pell equation can be generated from the smallest positive solution (u_1,v_1) of (6).

Theorem 2.1 ([1], Sec. 6.6. Theorem 7)**.** *If d is a square, the only solutions of* (6) *are $u=\pm 1$ and $v=0$.*

If d is not a square, let (u_1,v_1) be the smallest positive solution of (6) *and write $\alpha = u_1 + v_1\sqrt{d}$, then all solutions of* (6) *are*

$$\{(\pm u_n, v_n) \mid u_n, v_n \in \mathbb{Z} \text{ such that } u_n + v_n\sqrt{d} = (\alpha)^n,\ n \in \mathbb{Z}\}.$$

Remark 2.2. All solutions of (6) are units of $\mathbb{Z}[\sqrt{d}]$.

More generally, for $m \in \mathbb{Z}\setminus\{0\}$, the equation

$$u^2 - dv^2 = m \tag{7}$$

is called a *generalized Pell equation.* Note that if (a,b) is a solution of (7), then for any solution (u_n,v_n) of (6), (u'_n,v'_n) defined by $u'_n + v'_n\sqrt{d} = (u_n + v_n\sqrt{d})(a+b\sqrt{d})$ is also a solution of (7). (u'_n,v'_n) is called a *Pell multiple* of $a+b\sqrt{d}$.

In particular, for $m=4$, we have the following result.

Theorem 2.3 ([2], Theorem 4.4.1)**.** *Let d be a non-square positive integer. If (u_1,v_1) is the smallest positive solution of $u^2-dv^2=4$, then all solutions are generated by powers of $\alpha = \frac{u_1+v_1\sqrt{d}}{2}$ in the sense that writing $\alpha^n = \frac{u_n+v_n\sqrt{d}}{2}$, $(\pm u_n, v_n)$ is a new solution and all solutions can be obtained in that way.*

2.2. Lattices

A *lattice* is a pair (L, b) of a free finite rank $\mathbb{Z}$-module L together with a bilinear form $b : L \times L \to \mathbb{Z}$. A lattice is *even* if $b(x, x) \in 2\mathbb{Z}$ for any $x \in L$, *odd* otherwise. The *discriminant* $\mathrm{disc}(L)$ is the determinant of the matrix of the bilinear form. A lattice is called *non-degenerate* if the discriminant is non-zero and *unimodular* if the discriminant is ± 1. If the lattice L is non-degenerate, the pair (s_+, s_-), where $s_\pm$ denotes the multiplicity of the eigenvalue ± 1 for the quadratic form associated to $L \otimes \mathbb{R}$, is called a *signature* of L. An *isometry* of a lattice is an isomorphism preserving the bilinear form. The *orthogonal group* $O(L)$ consists of all isometries of L.

For a lattice (L, b), the dual lattice L^* is defined by

$$L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q} \mid b(x, y) \in \mathbb{Z} \text{ for any } y \in L\}.$$

We have a natural inclusion $L \hookrightarrow L^*$ and the *discriminant group* of L is $A(L) = L^*/L$. The bilinear form on L induces a symmetric bilinear form $b^* : L^* \times L^* \to \mathbb{Q}$. Moreover, b^* induces a symmetric bilinear form $b_L : A(L) \times A(L) \to \mathbb{Q}/\mathbb{Z}$ and thus a quadratic form $q_L : A(L) \to \mathbb{Q}/\mathbb{Z}$.

Whenever L is even, q_L takes values in $\mathbb{Q}/2\mathbb{Z}$. By $O(A(L))$ we denote the group of automorphisms of $A(L)$ preserving q_L. The inclusion of L into L^* yields a homomorphism $\Phi : O(L) \to O(A(L))$. For a non-degenerate lattice L of signature $(1, k)$ with $k > 0$, we have the decomposition

$$\{x \in L \otimes \mathbb{R} \mid x^2 > 0\} = C_L \cup (-C_L) \tag{8}$$

into two disjoint cones. We define

$$O^+(L) := \{g \in O(L) \mid g(C_L) = C_L\}. \tag{9}$$

Note that $O^+(L)$ is a subgroup of $O(L)$ of index 2.

We state some results about lattices which will be used in later sections.

Theorem 2.4 ([12], Theorem 1.14.4)**.** *For any even lattice L of signature $(1, \rho)$ with $\rho \leq 9$, there exists a projective* K3 *surface X such that $S_X \cong L$.*

An embedding $S \hookrightarrow L$ of lattices is called *primitive* if L/S is free.

Proposition 2.5 ([11], Proposition 1.6.1)**.** *A primitive embedding of an even lattice S into an even unimodular lattice L, in which the orthogonal complement of S is isomorphic to K, is determined by an isomorphism $\gamma : A(S) \xrightarrow{\sim} A(K)$ for which $q_K \circ \gamma = -q_S$.*

Lemma 2.6 ([6], Lemma 1)**.** *Let L be a non-degenerate even lattice of rank n. For $g \in O(L)$ and $\epsilon \in \{\pm 1\}$, g acts on $A(L)$ as $\epsilon \cdot \mathrm{id}$ if and only if $(g - \epsilon \cdot I_n) \cdot Q_L^{-1}$ is an integer matrix, where Q_L is the intersection matrix of L.*

In [4], Galluzzi, Lombardo, and Peters classified the automorphism groups of K3 surfaces of Picard number 2.

Definition ([4], Section 3.2)**.** A lattice L is *ambiguous* if L admits an isometry P with $\det P = -1$.

Theorem 2.7 ([4], Corollary 1)**.** *For X a* K3 *surface with Picard number* 2 *the group* Aut(X) *is finite precisely when the Picard lattice S_X contains divisors L with $L^2 = 0$ or with $L^2 = -2$. If S_X does not contain such divisors and if moreover S_X is not ambiguous, then* Aut(X) *is infinite cyclic, but if S_X is ambiguous, then* Aut(X) *is either infinite cyclic or the infinite dihedral group.*

It is known that a symplectic involution on a projective K3 surface occurs only if Picard number is greater than 8.

Proposition 2.8 ([5], Section 2.1)**.** *A projective* K3 *surface with a symplectic involution has Picard number at least* 9*.*

3. Proof

Let L be an even lattice of signature $(1,1)$ with intersection matrix given by (1) with $d := -\mathrm{disc}(L) = b^2 - 4ac > 0$. By Theorem 2.4, there is a K3 surface X whose Picard lattice $S_X \cong L$. It is known that d is a square number if and only if there is a $D \in S_X$ with $D^2 = 0$. Suppose that d is not a square number. Moreover, we assume that S_X has no divisor D with self-intersection -2. Hence we have that Aut(X) is infinite (cf. Theorem 2.7).

We consider

$$A_k = \{D = (x, y) \in S_X \mid D^2 = 2ax^2 + 2bxy + 2cy^2 = 2k\} \tag{10}$$

for some $k \in \mathbb{Z}$ such that $A_k \neq \emptyset$. Then A_k consists of $(A_k)_\pm = \{(x, y) \mid x = \frac{-by \pm z}{2a}$ such that $z^2 - dy^2 = 4ak\}$. For $(x_0, y_0) \in (A_k)_\pm$ with $x_0 = \frac{-by_0 \pm z_0}{2a}$ and $z_0^2 - dy_0^2 = 4ak$, we have that $(x_n, y_n) \in (A_k)_\pm$, where $x_n = \frac{-by_n \pm z_n}{2a}$ with $z_n^2 - dy_n^2 = 4ak$ and (z_n, y_n) is a Pell multiple of (z_0, y_0).

Lemma 3.1. *For every $(x_n, y_n) \in (A_k)_\pm$ (resp.), $\frac{x_n}{y_n}$ converges to $\frac{-b \pm \sqrt{d}}{2a}$ (resp.) as n increases.*

Proof. For (x_n, y_n) with $x_n = \frac{-by_n + z_n}{2a}$, (z_n, y_n) is obtained by $z_n + y_n\sqrt{d} = (u_n + v_n\sqrt{d})(z_0 + y_0\sqrt{d})$ with $u_n^2 - dv_n^2 = 1$. Hence $\frac{z_n}{y_n} = \frac{u_n z_0 + dv_n y_0}{v_n z_0 + u_n y_0} = \frac{z_0 \frac{u_n}{v_n} + dy_0}{z_0 + \frac{u_n}{v_n} y_0}$ converges to $\sqrt{d}$ since $\frac{u_n}{v_n}$ converges to $\sqrt{d}$ as n increases. Now for $(x_n, y_n) \in (A_k)_+$, $\frac{x_n}{y_n} = \frac{-b + z_n/y_n}{2a}$ converges to $\frac{-b+\sqrt{d}}{2a}$ as n increases.

Similarly this also holds for $(x_n, y_n) \in (A_k)_-$. □

3.1. Proof of Theorem 1.1

Let g be an automorphism of infinite order of X whose action on S_X is given by (2). Then $A_k \neq \emptyset$ for some $k \neq 0, -2$ (cf. Theorem 2.7). For $(x_0, y_0) \in A_k$, let $(x_n, y_n) = g^{n*}(x_0, y_0)$. Then we have that $(x_n, y_n) \in A_k$. By Lemma 3.1, $\frac{x_n}{y_n}$ converges to $\frac{x}{y} = \frac{-b+\sqrt{d}}{2a}$ as n increases, hence the ratio $\frac{x}{y}$ indicates the direction of an eigenvector of g^*. Hence $g*$ preserves the ratio, i.e., for

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta y \\ \gamma x + \delta y \end{pmatrix}, \tag{11}$$

we have that

$$\frac{\alpha x+\beta y}{\gamma x+\delta y}=\frac{x}{y}=\frac{-b+\sqrt{d}}{2a} \quad \text{or} \quad \frac{\alpha\frac{-b+\sqrt{d}}{2a}+\beta}{\gamma\frac{-b+\sqrt{d}}{2a}+\delta}=\frac{-b+\sqrt{d}}{2a}. \tag{12}$$

Then we have that

$$\alpha(-b+\sqrt{d})+2a\beta=\frac{\gamma(2b^2-4ac-2b\sqrt{d})}{2a}+\delta(-b+\sqrt{d}). \tag{13}$$

Now since this is an element of $\mathbb{Q}[\sqrt{d}]$, we have that

$$\alpha+\frac{b}{a}\gamma-\delta=0 \quad \text{and} \quad a\beta+c\gamma=0. \tag{14}$$

This gives the first two conditions in (3). Since we assume that d is not a square number, $c\neq 0$. Moreover, since g^* is an isometry of S_X, we have that $g^{*tr}Q_{S_X}g^*=Q_{S_X}$, where Q_{S_X} is the intersection matrix of S_X as in (1). This and conditions in (14) give the last identity in (3). This proves the first part of the theorem.

Next, we will see that the generator of infinite order is a power of such a minimal isometry in the sense that the minimal isometry is associated with the minimal positive solution of some Pell equation. Suppose that

$$h'=\begin{pmatrix}\alpha' & \beta' \\ -\frac{a}{c}\beta' & \alpha'-\frac{b}{c}\beta'\end{pmatrix}, \tag{15}$$

where $\alpha'=\frac{b\frac{\beta'}{c}+z}{2}$ with $(z,\frac{\beta'}{c})$ being another solution of

$$u^2-dv^2=4. \tag{16}$$

We need to show that h' is a power of h in (4). For this, let $(z_1,\frac{\beta_1}{c})$ be the minimal positive solution of (16). Then by Theorem 2.3, any solution $(z_k,\frac{\beta_k}{c})$ is given by a power of $\frac{z_1+\frac{\beta_1}{c}\sqrt{d}}{2}$, that is, $\frac{z_k+\frac{\beta_k}{c}\sqrt{d}}{2}=(\frac{z_1+\frac{\beta_1}{c}\sqrt{d}}{2})^k$ with $k\in\mathbb{Z}$. Hence $z=z_l$ and $\beta'=\beta_l$ for some $l\in\mathbb{Z}$. The following claim shows that $h'=h^l$.

Claim. For all $k\in\mathbb{Z}$, let $\alpha_k=\frac{b\frac{\beta_k}{c}+z_k}{2}$. Then

$$h^k=\begin{pmatrix}\alpha_k & \beta_k \\ -\frac{a}{c}\beta_k & \alpha_k-\frac{b}{c}\beta_k\end{pmatrix}.$$

Proof. For the induction argument, suppose that

$$h^{k-1}=\begin{pmatrix}\alpha_{k-1} & \beta_{k-1} \\ -\frac{a}{c}\beta_{k-1} & \alpha_{k-1}-\frac{b}{c}\beta_{k-1}\end{pmatrix}.$$

Then h^k is given by the following matrix

$$\begin{pmatrix}\alpha_1\alpha_{k-1}-\frac{a}{c}\beta_1\beta_{k-1} & \alpha_1\beta_{k-1}+\alpha_{k-1}\beta_1-\frac{b}{c}\beta_1\beta_{k-1} \\ -\frac{a}{c}\beta_{k-1}(\alpha_1-\frac{b}{c}\beta_1)-\frac{a}{c}\alpha_{k-1}\beta_1 & \alpha_{k-1}(\alpha_1-\frac{b}{c}\beta_1)-\beta_{k-1}(\frac{b}{c}\alpha_1-\frac{b^2-ac}{c^2}\beta_1)\end{pmatrix}.$$

From $\frac{z_k+\frac{\beta_k}{c}\sqrt{d}}{2} = (\frac{z_1+\frac{\beta_1}{c}\sqrt{d}}{2})^k$, we have that $z_k = \frac{z_1 z_{k-1}+d\frac{\beta_1\beta_{k-1}}{c^2}}{2}$ and $\beta_k = \frac{z_1\beta_{k-1}+z_{k-1}\beta_1}{2}$. Hence $\beta_k = \alpha_1\beta_{k-1} + \alpha_{k-1}\beta_1 - \frac{b}{c}\beta_1\beta_{k-1}$ and $\alpha_k = \alpha_1\alpha_{k-1} - \frac{a}{c}\beta_1\beta_{k-1}$. □

In particular, $h' = h^l$ and this completes the proof of Theorem 1.1.

Remark 3.2. Note that for $\alpha_k = \frac{\frac{b}{c}\beta_k - z_k}{2}$, the corresponding isometry reflects the positive cone and the negative cone.

3.2. Proof of Theorem 1.2

In this section, we find some conditions on the involution if we have $\mathrm{Aut}(X) \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

Proof of Theorem 1.2. Suppose that σ and τ are the generators of $\mathbb{Z}_2 * \mathbb{Z}_2$. By Proposition 2.8, these are anti-symplectic involutions. Hence if we let $g = \sigma\circ\tau$, then g is symplectic of infinite order. Let ι be an involution of X. Then $\iota = g^n \circ \sigma$ or $\tau \circ g^n$ for some n. Let $\{v, w\}$ be the eigenvectors of $g^{n*}|S_X \otimes \mathbb{R}$ corresponding to eigenvalues ρ and $1/\rho$ such that $g^{n*}v = \rho v$ and $g^{n*}w = \frac{1}{\rho}w$, where ρ is the spectral radius of $g^{n*}|S_X$.

We see that v and w lie on each of the two extremal rays of the ample cone of X since g^{n*} preserves the ample cone. Otherwise, $g^{k*} = \mathrm{id}$ on S_X for some k. Since g is symplectic, $g^{k*} = \mathrm{id}$ on $H^2(X, \mathbb{Z})$. Then by Torelli theorem, g^k is an identity which is a contradiction. Moreover, the two involutions σ and τ interchange these two eigenvectors v and w. Indeed, the matrices corresponding to the actions of σ and τ on S_X have the same determinant -1. Otherwise, they trivially act on S_X which implies that $g = \sigma\circ\tau$ also acts trivially on S_X. Then g is an identity by Torelli theorem which is also a contradiction. Hence $\sigma(v) = r_0 w$ and $\sigma(w) = r_0^{-1}v$ for some $r_0 \in \mathbb{R}\setminus 0$.

Now for either $\iota = g^n\circ\sigma$ or $\tau\circ g^n$, $\iota(v) = rw$ and $\iota(w) = r^{-1}v$ for some $r \in \mathbb{R}\setminus 0$. If we let

$$\iota^*|S_X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{17}$$

then as in (12) this implies that for $v = (x, y)$ with $\frac{x}{y} = \frac{-b+\sqrt{d}}{2a}$, the ratio $\frac{x}{y}$ is interchanged with its conjugate via ι. Hence, we have that

$$\frac{\alpha\frac{-b\pm\sqrt{d}}{2a} + \beta}{\gamma\frac{-b\pm\sqrt{d}}{2a} + \delta} = \frac{-b \mp \sqrt{d}}{2a}. \tag{18}$$

This implies that

$$\frac{-b \pm \sqrt{d}}{2a}\alpha + \beta = \frac{c}{a}\gamma + \frac{-b \mp \sqrt{d}}{2a}\delta. \tag{19}$$

Thus we have that

$$\delta = -\alpha, \quad \gamma = -\frac{b}{c}\alpha + \frac{a}{c}\beta. \tag{20}$$

Moreover, since $\det(\iota) = -1$, we also have that

$$\alpha^2 - \frac{b}{c}\alpha\beta + \frac{a}{c}\beta^2 = 1. \tag{21}$$

□

By the similarity of automorphisms in Theorem 1.1 and Theorem 1.2, we have the following lemma. Let $E_{12} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ be the matrix interchanging two columns.

Lemma 3.3. *Let X be a* K3 *surface with the intersection matrix* (1) *of Picard lattice S_X. If $\mathrm{Aut}(X) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ and $a = c$ in* (1)*, then for any involution ι, $\iota^*|S_X = h^m E_{12}$ for some m, where h is in* (4)*.*

Proof. Suppose $a = c$. Let ι be an involution given by (2) with conditions (5). Then by interchanging the two columns of (2), we will have a matrix with the conditions in (3). The proof of Theorem 1.1 shows that any such matrix with the conditions in (3) is a power of h, hence we will have the lemma. □

4. Applications

In this section, we apply our results to several examples.

4.1. Example 1

In [10], Mori showed that there is a non-singular quartic surface X in $\mathbb{P}^3$ with a non-singular curve C of degree d and genus g if and only if (1) $g = d^2/8 + 1$, or (2) $g < d^2/8$ and $(d, g) \neq (5, 3)$. More generally, we refer to [7].

Let X be a quartic hypersurface in $\mathbb{P}^3$ whose Picard lattice S_X has the intersection matrix

$$\begin{pmatrix} 4 & d \\ d & 2g-2 \end{pmatrix} \tag{22}$$

generated by $\{H = \mathcal{O}_X(1), C\}$. If $g = d^2/8 + 1$, then the discriminant of (22) is zero. In this case, by Theorem 2.7, the automorphism group is finite. Hence we assume that $g < d^2/8$.

We consider a K3 surface X whose Picard lattice S_X has the following intersection matrix

$$Q_{S_X} = \begin{pmatrix} 4 & 2n \\ 2n & 4 \end{pmatrix} \tag{23}$$

with $d := -\mathrm{disc}(S_X) = 4(n^2 - 4) > 0$. Since X has no divisors of self-intersection number 0 or -2, $\mathrm{Aut}(X)$ is either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$ by Theorem 2.7.

First, we determine the generator of automorphisms of infinite order in $\mathrm{Aut}(X)$. By the relations (3), an automorphism g of infinite order with $g^*|S_X$ in (2) satisfies $\gamma = -\beta$, $\delta = \alpha - n\beta$ and $\alpha^2 - n\alpha\beta + \beta^2 = 1$. Now by Theorem 1.1, $g^*|S_X$ is a power of h, where

$$h = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix}. \tag{24}$$

Indeed, we have the following.

Proposition 4.1. *For a* K3 *surface X whose intersection matrix of Picard lattice S_X is given in* (23) *with $d = -\mathrm{disc}(S_X) = 4(n^2-4) > 0$, $\mathrm{Aut}(X) \cong \mathbb{Z}$. The generator is as follows*:

(1) *if n is even, $g^* = h^4$ is the generator of $\mathbb{Z}$ and symplectic.*
(2) *if n is odd ($\neq 3$), $g^* = h^6$ is the generator of $\mathbb{Z}$ and symplectic.*
(3) *if $n = 3$, $g^* = h^3$ is the generator of $\mathbb{Z}$ and anti-symplectic.*

Proof. As Claim in Section 3.1, let α_k, β_k be the first row of h^k. Then we have $\alpha_{k+1} = n\alpha_k - \beta_k$ and $\beta_{k+1} = \alpha_k$. For example, $(\alpha_1, \beta_1) = (n, 1)$, $(\alpha_0, \beta_0) = (1, 0)$ and $(\alpha_{-1}, \beta_{-1}) = (0, -1)$, etc. It is easily shown that $(h^4 - I_2)Q_{S_X}^{-1}$ is an integer matrix for even n and $(h^6 - I_2)Q_{S_X}^{-1}$ is an integer matrix for odd n. Moreover, these powers are the minimal in order to be an integer matrix. In other words, for even number n, $(h^k \pm I_2)Q_{S_X}^{-1}$ is not an integer matrix for any $k \leq 3$. Similarly, for $n \neq 3$ odd, $(h^k \pm I_2)Q_{S_X}^{-1}$ is not an integer matrix for any $k \leq 5$.

Now by Proposition 2.5, h^4 (n even) or h^6 ($n \neq 3$ odd) can be extended to an isometry of $H^2(X, \mathbb{Z})$ and by Torelli theorem it defines an automorphism of X. Hence h^4 for n even (h^6 for $n \neq 3$ odd) is the generator of automorphisms of infinite order of $\mathrm{Aut}(X)$. Moreover, by Lemma 2.6, both h^4 and h^6 are symplectic. By the same argument, h^3 is the generator of automorphisms of infinite order of $\mathrm{Aut}(X)$ for $n = 3$ and is anti-symplectic.

Now by assuming $\mathrm{Aut}(X) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, we will derive a contradiction. If we assume that $\mathrm{Aut}(X) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, then, by Theorem 1.2,

$$\tau^*|S_X = \begin{pmatrix} p & q \\ q - np & -p \end{pmatrix}, \tag{25}$$

where $p^2 - npq + q^2 = 1$. Moreover, by Lemma 3.3, both $\tau^*|S_X$ and $\sigma^*|S_X$ are the matrices obtained from h^k by interchanging columns for some k. For example, $\tau^*|S_X := \tau_k = h^k E_{12}$, i.e.,

$$\tau^*|S_X = \begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k - n\beta_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta_k & \alpha_k \\ \alpha_k - n\beta_k & -\beta_k \end{pmatrix}, \tag{26}$$

where (α_k, β_k) is the first row of h^k.

Since τ is anti-symplectic, by Lemma 2.6, $(\tau^*|S_X + \mathrm{id})Q_{S_X}^{-1}$ is an integer matrix. Note that if

(27)
$$\begin{aligned}(\tau^*|S_X + \mathrm{id})Q_{S_X}^{-1} &= \frac{1}{2(n^2-4)} \begin{pmatrix} n\alpha_k - 2\beta_k - 2 & -2\alpha_k + n\beta_k + n \\ -2\alpha_k + n\beta_k + n & n\alpha_k - 2\beta_k - 2 - (n^2-4)\beta_k \end{pmatrix} \\ &=: \frac{1}{2(n^2-4)} (a_{ij})_{i,j=1,2}\end{aligned}$$

is an integer matrix, then β_k is even because $2(n^2 - 4)$ divides both a_{11} and a_{22}, hence their sum.

Now if n is odd, then since $a_{12} = -2\alpha_k + n(\beta_k + 1)$ is even, β_k is odd, a contradiction.

If n is even, β_k is even only if k is even and α_k is even only if k is odd. Hence, $\tau^*|S_X := \tau_{2l}^*|S_X = h^{2l}E_{12}$ for some l. Now by multiplying $g = h^4$ or $g^{-1} = h^{-4}$, we may assume that h^2E_{12} or E_{12} is an isometry of an anti-symplectic automorphism on S_X since g is symplectic. However, by Lemma 2.6, this is not possible. $\square$

4.2. Example 2

We consider a K3 surface X whose Picard lattice S_X has the following intersection matrix

$$\begin{pmatrix} 2 & n \\ n & 2 \end{pmatrix}. \tag{28}$$

Note that $d := -\mathrm{disc}(S_X) = n^2 - 4 > 0$ and for $n \neq 3$, $\mathrm{Aut}(X)$ is either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$ by Theorem 2.7. In [4, Example 4], Galluzzi, Lombardo and Peters proved that $\mathrm{Aut}(X) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ by finding generators. We can also find the same generators by using our results.

By Theorem 1.1, an automorphism g of infinite order in (2) satisfies $\gamma = -\beta, \delta = \alpha - n\beta$ and $\alpha^2 - n\alpha\beta + \beta^2 = 1$. Moreover, by Theorem 1.1, $g^*|S_X$ is a power of h, where

$$h = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix}. \tag{29}$$

As the example above, we have that h^2 is a symplectic automorphism.

Moreover, by Theorem 1.2, we may have an anti-symplectic involution σ which satisfies

$$\sigma^*|S_X = \begin{pmatrix} p & q \\ q - np & -p \end{pmatrix}, \tag{30}$$

where $p^2 - npq + q^2 = 1$. By Lemma 3.3, $h^2 = \sigma^*|S_X \circ \tau^*|S_X$ and $\sigma^*|S_X = hE_{12}, \tau^*|S_X = h^{-1}E_{12}$, hence

$$\sigma^*|S_X = \begin{pmatrix} 1 & n \\ 0 & -1 \end{pmatrix} \text{ and } \tau^*|S_X = \begin{pmatrix} -1 & 0 \\ n & 1 \end{pmatrix}. \tag{31}$$

Moreover, by Lemma 2.6 and Torelli theorem, σ and τ are anti-symplectic involutions. Hence $\mathrm{Aut}(X) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ with generators σ and τ.

Remark 4.2. When $n = 4$, X is the complete intersection of bidegree $(1, 1)$ and $(2, 2)$ hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$. When $n = 5$, X is the complete intersection of bidegree $(1, 2)$ and $(2, 1)$ hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$.

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