TRIGONOMETRIC GENERATED RATE OF CONVERGENCE
OF SMOOTH PICARD SINGULAR INTEGRAL OPERATORS

GEORGE A. ANASTASSIOU

Abstract. In this article we continue the study of smooth Picard singular integral operators that started in [2], see there chapters 10-14. This time the foundation of our research is a trigonometric Taylor’s formula. We establish the convergence of our operators to the unit operator with rates via Jackson type inequalities engaging the first modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not positive. Our results are pointwise and uniform.

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1. Introduction

We are motivated by [1], [2] chapters 10-14, and [3], [4]. We use a trigonometric new Taylor formula from [3], see also [4]. Here we consider some very general operators, the smooth Picard singular integral operators over the real line and we study further their convergence properties quantitatively. We establish related inequalities involving the first modulus of continuity with respect to uniform norm and the estimates are pointwise and uniform. We provide a detailed proof.

2. Results

By [3], [4], for \( f \in C^2(\mathbb{R}) \) and \( a, x \in \mathbb{R} \), we have by trigonometric Taylor formula

\[
\begin{align*}
    f(x) - f(a) &= f'(a) \sin (x - a) + 2f''(a) \sin^2 \left( \frac{x - a}{2} \right) + \\
    &\quad \int_a^x \left[ (f''(t) + f(t)) - (f''(a) + f(a)) \right] \sin (x - t) \, dt.
\end{align*}
\]

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For \( r \in \mathbb{N} \) and \( n \in \mathbb{Z}^+ \), we set
\[
\alpha_j := \begin{cases} 
(-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, \ldots, r, \\
1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 0,
\end{cases}
\]
that is
\[
\sum_{j=0}^{r} \alpha_j = 1.
\]

\( C_U(\mathbb{R}) \) denotes the space of uniformly continuous functions on \( \mathbb{R} \), and \( C_B(\mathbb{R}) \) denotes the space of bounded continuous functions on \( \mathbb{R} \).

Here we consider both \( f, f'' \in C_U(\mathbb{R}) \cup C_B(\mathbb{R}) \).

For \( x \in \mathbb{R} \), \( \xi > 0 \) we consider the Lebesgue integrals, so called smooth Picard operators
\[
P_{r,\xi}(f, x) = \frac{1}{2\xi} \int_{-\infty}^{\infty} \left( \sum_{j=0}^{r} \alpha_j f(x + jt) \right) e^{-|t|/\xi} dt,
\]
see [1]; \( P_{r,\xi} \) are not in general positive operators, see [2].

We notice by
\[
\frac{1}{2\xi} \int_{-\infty}^{\infty} e^{-|t|/\xi} dt = 1,
\]
that
\[
P_{r,\xi}(c, x) = c, \text{ where } c \text{ is a constant},
\]
and
\[
P_{r,\xi}(f, x) - f(x) = \frac{1}{2\xi} \left( \sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x + jt) - f(x)) e^{-|t|/\xi} dt \right).
\]

Denote by
\[
\omega_1(f, \delta) := \sup_{x, y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x - y|}, \quad \delta > 0,
\]
the first modulus of continuity of \( f \).

We set
\[
\Delta(x) := P_{r,\xi}(f, x) - f(x) - f''(x) \left( \sum_{j=0}^{r} \frac{\alpha_j j^2}{j^2\xi^2 + 1} \right) \xi^2; \quad \xi > 0, \ x \in \mathbb{R}.
\]

We present our uniform approximation result.

**Theorem 2.1.** It holds
\[
|\Delta(x)| \leq \|\Delta(x)\|_{\infty} \leq \xi^2 \omega_1(f'' + f, \xi) \left( \sum_{j=0}^{r} |\alpha_j| j^2 (j + 1) \right) =: A; \quad \xi > 0, \ x \in \mathbb{R}.
\]
And \( \| \Delta (x) \|_{\infty} \to 0 \), as \( \xi \to 0 \). If \( f''(x) = 0 \), then \( |P_{r,\xi} (f, x) - f(x)| \leq A \).

Proof. By (1) we get that
\[
\begin{align*}
& f(x + jt) - f(x) = f'(x) \sin (jt) + 2 f''(x) \sin^2 \left( \frac{jt}{2} \right) + \\
& \int_{x}^{x+jt} [(f''(s) + f(s)) - (f''(x) + f(x))] \sin (x + jt - s)\ ds,
\end{align*}
\]
(11)
or better
\[
\begin{align*}
& f(x + jt) - f(x) = f'(x) \sin (jt) + 2 f''(x) \sin^2 \left( \frac{jt}{2} \right) + \\
& \int_{0}^{jt} [(f''(x + z) + f(x + z)) - (f''(x) + f(x))] \sin (jt - z)\ dz.
\end{align*}
\]
(12)
Furthermore, it holds
\[
\sum_{j=0}^{r} \alpha_{j} [f(x + jt) - f(x)] =
\]
\[
f'(x) \sum_{j=0}^{r} \alpha_{j} \sin (jt) + 2 f''(x) \sum_{j=0}^{r} \alpha_{j} \sin^2 \left( \frac{jt}{2} \right) + \\
\sum_{j=0}^{r} \alpha_{j} \int_{0}^{jt} [(f''(x + z) + f(x + z)) - (f''(x) + f(x))] \sin (jt - z)\ dz,
\]
(13)
or better
\[
\sum_{j=0}^{r} \alpha_{j} [f(x + jt) - f(x)] =
\]
\[
f'(x) \sum_{j=0}^{r} \alpha_{j} \sin (jt) + 2 f''(x) \sum_{j=0}^{r} \alpha_{j} \sin^2 \left( \frac{jt}{2} \right) + \\
\sum_{j=0}^{r} \alpha_{j} \int_{0}^{t} [(f''(x + jw) + f(x + jw)) - (f''(x) + f(x))] \sin j(t - w)\ dw.
\]
(14)
Call
\[
R := R(t)
\]
\[
:= \sum_{j=0}^{r} \alpha_{j} \int_{0}^{t} [(f''(x + jw) + f(x + jw)) - (f''(x) + f(x))] \sin j(t - w)\ dw,
\]
\quad \forall \ t \in \mathbb{R}.
\]
(15)
Then, for \( t \geq 0 \),
\[
|R| \leq
\]

\[\]
\[
\sum_{j=0}^{r} |\alpha_j| j \int_0^t |(f''(x + jw) + f(x + jw)) - (f''(x) + f(x))||\sin j(t - w)| \, dw \leq \\
\sum_{j=0}^{r} |\alpha_j| j \int_0^t \omega_1 (f'' + f, jw) j (t - w) \, dw = \\
(\xi > 0)
\]

(16)

\[
\sum_{j=0}^{r} |\alpha_j| j^2 \omega_1 (f'' + f, \xi) \int_0^t \left(1 + \frac{jw}{\xi}\right) (t - w) \, dw = \\
\omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \int_0^t (t - w) \, dw + \frac{j}{\xi} \int_0^t w^{2-1} (t - w)^{2-1} \, dw \right] = \\
\omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{(t - w)^2}{2} \bigg|_t^0 + \frac{j}{\xi} \frac{1}{6} t^3 \right] = \\
\omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{t^2}{2} + \frac{j t^3}{6\xi} \right] .
\]

Hence \((t \geq 0)\)

\[
|R| \leq \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{t^2}{2} + \frac{j t^3}{6\xi} \right].
\]

(17)

Let now \(t < 0\), then

\[
|R| \leq \\
\sum_{j=0}^{r} |\alpha_j| j \left| \int_0^t |(f''(x + jw) + f(x + jw)) - (f''(x) + f(x))| \sin j(t - w) \, dw \right| \leq \\
\sum_{j=0}^{r} |\alpha_j| j \int_0^t |(f''(x + jw) + f(x + jw)) - (f''(x) + f(x))| |\sin j(t - w)| \, dw \leq \\
\sum_{j=0}^{r} |\alpha_j| j \int_0^t \omega_1 (f'' + f, -jw) j (w - t) \, dw = \\
\sum_{j=0}^{r} |\alpha_j| j \int_0^t \omega_1 \left(f'' + f, -jw\frac{\xi}{\xi}\right) j (w - t) \, dw = \\
\omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \int_0^t \left(1 - \frac{j}{\xi} w\right) (w - t) \, dw =
\]

(18)
\[ \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \int_{t}^{0} (w - t) \, dw + \frac{j}{\xi} \int_{t}^{0} (0 - w)^{2-1} (w - t)^{2-1} \, dw \right] = \]

\[ \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{(w - t)^2}{2} \bigg|_{t}^{0} + \frac{j}{\xi} \frac{1}{6} (-t)^3 \right] = \]

\[ \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{t^2}{2} + j \frac{(-t)^3}{6\xi} \right]. \]

We found that \((t < 0)\)

\[ |R| \leq \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{t^2}{2} + j \frac{(-t)^3}{6\xi} \right]. \quad (19) \]

Consequently, for \(t \in \mathbb{R}\), we obtain

\[ |R(t)| \leq \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{t^2}{2} + j \frac{(-t)^3}{6\xi} \right], \quad \xi > 0. \quad (20) \]

So, we have

\[ \sum_{j=0}^{r} \alpha_j [f(x + jt) - f(x)] - f'(x) \sum_{j=0}^{r} \alpha_j \sin(jt) - 2f''(x) \sum_{j=0}^{r} \alpha_j \sin^2 \left( \frac{jt}{2} \right) = R(t). \]

Therefore, it holds

\[ \Delta_1 (x) := P_{r, \xi} (f, x) - f(x) - f'(x) \sum_{j=0}^{r} \alpha_j \frac{1}{2\xi} \left( \int_{-\infty}^{\infty} \sin(jt) e^{-\frac{|t|}{\tau}} \, dt \right) \]

\[ -2f''(x) \sum_{j=0}^{r} \alpha_j \frac{1}{2\xi} \left( \int_{-\infty}^{\infty} \sin^2 \left( \frac{jt}{2} \right) e^{-\frac{|t|}{\tau}} \, dt \right) = \]

\[ \frac{1}{2\xi} \int_{-\infty}^{\infty} R(t) e^{-\frac{|t|}{\tau}} \, dt. \quad (22) \]

Hence we get

\[ |\Delta_1 (x)| \leq \frac{1}{2\xi} \int_{-\infty}^{\infty} |R(t)| e^{-\frac{|t|}{\tau}} \, dt \leq \]

\[ \frac{1}{2\xi} \int_{-\infty}^{\infty} \left[ \omega_1 (f'' + f, \xi) \sum_{j=0}^{r} |\alpha_j| j^2 \left[ \frac{t^2}{2} + j \frac{(-t)^3}{6\xi} \right] \right] e^{-\frac{|t|}{\tau}} \, dt \]

(we use

\[ \int_{-\infty}^{\infty} t^k e^{-\frac{|t|}{\tau}} \, dt = \begin{cases} 0, & k \text{ odd}, \\ 2k!\xi^{k+1}, & k \text{ even} \end{cases} \quad (24) \]
Notice \(-\infty \leq 0 \leq \infty\))

Furthermore, we have that

We observe that

Next we simplify left hand side (22).

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Notice that \(\Delta_1(x) \to 0\), as \(\xi \to 0\).

Next we simplify left hand side (22).

We observe that

Notice \(-\infty \leq t \leq 0 \Rightarrow \infty \geq -t \geq 0\). So that

Therefore, it is

Furthermore, we have that

\[
\int_{-\infty}^{\infty} \sin^2 \left( \frac{jt}{2} \right) e^{-\frac{\omega}{\tau} t} dt = 2 \int_0^{\infty} \sin^2 \left( \frac{jt}{2} \right) e^{-\frac{\omega}{\tau} t} dt, \quad j = 0, 1, \ldots, r. \tag{30}
\]
The last follows by
\[ \int_{-\infty}^{0} \sin^2 \left( \frac{j t}{2} \right) e^{-|t|} \, dt = - \int_{-\infty}^{0} \left( -\sin \left( \frac{j (-t)}{2} \right) \right)^2 e^{-|(-t)|} \, d(-t) \quad (z=-t) \]  
(31)

Next, we calculate
\[ \int_{0}^{\infty} \sin^2 \left( \frac{j z}{2} \right) e^{-|z|} \, dz = \int_{0}^{\infty} \sin^2 \left( \frac{j z}{2} \right) e^{-\frac{z}{2}} \, dz. \]

(call \( \frac{z}{\xi} =: x \) and \( \frac{j \xi}{2} =: a_1 \))

\[ \xi \int_{0}^{\infty} \sin^2 (a_1 x) e^{-x} \, dx = \]  
(32)

(by Wolfram Alpha Computational Intelligence)
\[ \xi \left( \frac{2a_1^2}{4a_1^4 + 1} \right) = \frac{j^2 \xi^3}{2(j^2 \xi^2 + 1)}. \]

Thus
\[ \int_{-\infty}^{\infty} \sin^2 \left( \frac{j t}{2} \right) e^{-|t|} \, dt = \frac{j^2 \xi^3}{2(j^2 \xi^2 + 1)}, \quad j = 0, 1, ..., r. \]  
(33)

Consequently, it holds
\[ \frac{1}{2\xi} \int_{-\infty}^{\infty} \sin^2 \left( \frac{j t}{2} \right) e^{-\frac{|t|}{\xi}} \, dt = \frac{j^2 \xi^2}{2(j^2 \xi^2 + 1)} \to 0, \text{ as } \xi \to 0, \quad j = 0, 1, ..., r. \]  
(34)

Finally we obtain
\[ -2f''(x) \sum_{j=0}^{r} \alpha_j \frac{1}{2\xi} \left( \int_{-\infty}^{\infty} \sin^2 \left( \frac{j t}{2} \right) e^{-\frac{|t|}{\xi}} \, dt \right) = -f''(x) \left( \sum_{j=0}^{r} \alpha_j \frac{j^2}{(j^2 \xi^2 + 1)} \right) \xi^2. \]  
(35)

Clearly now it is \( \Delta_1(x) = \Delta(x) \).

The proof of the theorem is finally completed. \( \square \)

We finish with

**Corollary 2.2.** It follows (\( \xi > 0, x \in \mathbb{R} \))
\[ \|P_{r,\xi} (f, x) - f(x)\|_{\infty} \leq \omega_1 (f'' + f, \xi) \left( \sum_{j=0}^{r} |\alpha_j| \frac{j^2}{1+j^2 \xi^2} \right) \xi^2 + \]  
(36)
\[ \|f''(x)\|_{\infty} \left( \sum_{j=0}^{r} |\alpha_j| \frac{j^2}{1+j^2 \xi^2} \right) \xi^2 \to 0, \text{ as } \xi \to 0. \]

**Proof.** Easy, by (9) and (10). \( \square \)
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REFERENCES


George A. Anastassiou received his B.Sc. degree in Mathematics from Athens University, Greece in 1975. He received his Diploma in Operations Research from Southampton University, UK in 1976. He also received his MA in Mathematics from University of Rochester, USA in 1981. He was awarded his Ph.D. in Mathematics from University of Rochester, USA in 1984. During 1984-86 he served as a visiting assistant professor at the University of Rhode Island, USA. Since 1986 till now 2020, he is a faculty member at the University of Memphis, USA. He is currently a full Professor of Mathematics since 1994. His research area is "Computational Analysis" in the very broad sense. He has published over 550 research articles in international mathematical journals and over 43 monographs, proceedings and textbooks in well-known publishing houses. Several awards have been awarded to George Anastassiou. In 2007 he received the Honorary Doctoral Degree from University of Oradea, Romania. He is associate editor in over 70 international mathematical journals and editor in-chief in 3 journals, most notably in the well-known "Journal of Computational Analysis and Applications".

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.
é-mail: ganastss@memphis.edu