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NUMERICAL INVESTIGATION OF ZEROS OF THE FULLY q-POLY-EULER NUMBERS AND POLYNOMIALS OF THE SECOND TYPE

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ABSTRACT. In this paper, we construct a fully modified q-poly-Euler numbers and polynomials of the second type and give some properties. Finally, we investigate the zeros of the fully modified q-poly-Euler numbers and polynomials of the second type by using computer.

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1. Introduction

In this paper, we investigate the zeros of the fully modified q-poly-Euler numbers and polynomials of the second type. Throughout this paper, the symbol, $\mathbb{N}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{R}$ and $\mathbb{C}$ denotes the set of natural numbers, the set of integers, the set of nonnegative integers, the set of real numbers and the set of complex numbers, respectively.

The q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q},$$

where $x, q \in \mathbb{R}$ with $q \neq 1$. We note that $\lim_{q\to 1}[x]_q = x$. From the definition of q-number, many mathematicians studied the this field such as q-differential equations, q-series, q-trigonometric function, and so on, see [1-2, 10-16]. Of course, mathematicians constructed and researched about Gaussian binomial coefficients.

Definition 1.1. The Gaussian binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q.$$

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We note $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$.

Definition 1.2. Two forms of q-exponential functions can be expressed as

$$E_q(t) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{t^n}{[n]_q!} \qquad e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}.$$

We remember that the classical Stirling numbers of the second kind $S_2(n,m)$ are defined by the relations (see [14])

$$(x)_n = \sum_{m=0}^{n} S_2(n,m)x^m.$$

Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}.$$

The familiar tangent polynomials $T_n(x)$ are defined by the generating function (see [8, 9]):

$$\left(\frac{2}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!}, \quad (|2t| < \pi). \tag{1.1}$$

When $x = 0$, $T_n(0) = T_n$ are called the tangent numbers.

For $k \in \mathbb{Z}, 0 < q < 1$, the q-poly-tangent polynomials $T_{n,q}^{(k)}(x)$, the q-poly-Bernoulli polynomials $B_{n,q}^{(k)}(x)$, the q-poly-Euler polynomials $E_{n,q}^{(k)}(x)$ are defined by means of the following generating functions:

$$\begin{aligned} \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{e^{2t}+1}e^{xt} &= \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x)\frac{t^n}{n!}, \\ \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e^{t}-1}e^{xt} &= \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x)\frac{t^n}{n!}, \\ \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{e^{t}+1}e^{xt} &= \sum_{n=0}^{\infty} E_{n,q}^{(k)}(x)\frac{t^n}{n!}, \end{aligned} \tag{1.2}$$

where

$$\mathrm{Li}_{k,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k}$$

is the kth q-polylogarithm function(see [5, 6]).

In [4], we construct modified poly-tangent numbers and polynomials.

Definition 1.3. For any integer k, the modified poly-tangent polynomials $T_n^{(k)}(x)$ are defined by means of the generating function

$$\sum_{n=0}^{\infty} T_n^{(k)}(x)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}. \tag{1.3}$$

The numbers $T_n^{(k)}(0) := T_n^{(k)}$ are called the modified poly-tagent numbers. If $k = 1$, then $T_n^{(1)}(x) = T_n(x), T_n^{(1)} = T_n$. It is more natural than (1.2) because it becomes a tangent polynomial when $k = 1$ in Definition 1.3.

Many kinds of of generalizations of these polynomials and numbers have been presented in the literature (see [1-14]). In the following section, we give some relationships, both between these polynomials and tangent polynomials and between these polynomials and other polynomials. Finally, we investigate the zeros of the fully modified q-poly-tangent polynomials of the second type by using computer.

2. The fully modified q-poly-tangent polynomials of the second type

In this section, we define the fully modified q-poly-Bernoulli and tangent numbers and polynomials of the second type. We also derive several identities with each other and investigate some properties that are concerned with q-Stirling numbers. In [4], we construct the fully modified q-poly-tangent polynomials $\widetilde{T}_{n,q}^{(k)}(x)$ of the first type.

Definition 2.1. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$, we define fully modified q-poly-tangent polynomials $\widetilde{T}_{n,q}^{(k)}(x)$ of the first type by

$$\frac{[2]_q Li_{k,q}(1 - e_q(-t))}{t(e_q(2t) + 1)} e_q(xt) = \sum_{n=0}^{\infty} \widetilde{T}_{n,q}^{(k)}(x) \frac{t^n}{[n]_q!}. \tag{2.1}$$

When $x = 0$, $\widetilde{T}_{n,q}^{(k)} = \widetilde{T}_{n,q}^{(k)}(0)$ are called the fully modified q-poly-tangent numbers of the first type. Now we construct a new type of the fully modified q-poly-tangent polynomials of the first type.

Definition 2.2. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, and $0 < q < 1$, we define the fully modified q-poly-tangent polynomials $\mathbf{T}_{n,q}^{(k)}(x)$ of the second type by

$$\frac{[2]_q Li_{k,q}(1 - E_q(-t))}{t(E_q(2t) + 1)} E_q(xt) = \sum_{n=0}^{\infty} \mathbf{T}_{n,q}^{(k)}(x) \frac{t^n}{[n]_q!}. \tag{2.2}$$

When $x = 0$, $\mathbf{T}_{n,q}^{(k)} = \mathbf{T}_{n,q}^{(k)}(0)$ are called fully modified q-poly-tangent numbers of the second type.

The modified q-Stirling numbers of the second kind, $S_q^*(n, m)$ are defined by the following generating function

$$\sum_{n=m}^{\infty} S_q^*(n, m) \frac{t^n}{[n]_q!} = \frac{(E_q(t) - 1)^m}{[m]_q!}. \tag{2.3}$$

For $0 \leq n, m \leq 5$, a few values of the modified q-Stirling numbers of second kind are given as below.

We also define the fully modified q-poly-Bernoulli polynomials $\mathbf{B}_{n,q}^{(k)}(x)$ of the second type. Using the generating functions of the polynomials, we derive some

TABLE 1. The modified q-Stirling numbers of second kind $S_q^*(n,m)$

n \ m	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	$q^{\binom{2}{2}}$	1	0	0	0
3	0	$q^{\binom{3}{2}}$	$\frac{2q[3]_q!}{[2]_q![2]_q!}$	1	0	0
4	0	$q^{\binom{4}{2}}$	$\frac{q^2[4]_q!}{([2]_q!)^3}+\frac{2q^3[4]_q!}{[2]_q![3]_q!}$	$\frac{3q[4]_q!}{[2]_q![3]_q!}$	1	0
5	0	$q^{\binom{5}{2}}$	$\frac{2q^4[5]_q!}{[2]_q![3]_q!}+\frac{2q^6[5]_q!}{[2]_q![4]_q!}$	$\frac{3q^3[5]_q!}{([3]_q!)^2}+\frac{3q^2[5]_q!}{([2]_q)^2[3]_q!}$	$\frac{4q[5]_q!}{[2]_q![4]_q!}$	1

identities that are related with the fully modified q-poly-tangent polynomials of the second type and the q-analogue of ordinary tangent polynomials of the second type.

Definition 2.3. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, and $0 < q < 1$, we define fully modified q-poly-Bernoulli polynomials $\mathbf{B}_{n,q}^{(k)}(x)$ of the second type by

$$\frac{Li_{k,q}(1-E_q(-t))}{E_q(t)-1}E_q(xt) = \sum_{n=0}^{\infty}\mathbf{B}_{n,q}^{(k)}(x)\frac{t^n}{[n]_q!}. \tag{2.4}$$

Corollary 2.4. *If $k=1$ in polylogarithm function, we get the q-Bernoulli polynomials $\mathbf{B}_{n,q}(x)$ and q-tangent polynomials $\mathbf{T}_{n,q}(x)$ of the second type,*

$$\frac{t}{E_q(t)-1}E_q(xt) = \sum_{n=0}^{\infty}\mathbf{B}_{n,q}(x)\frac{t^n}{[n]_q!},$$

$$\frac{[2]_q}{E_q(2t)+1}E_q(xt) = \sum_{n=0}^{\infty}\mathbf{T}_{n,q}(x)\frac{t^n}{[n]_q!},$$

respectively.

Corollary 2.5. *If $q \to 1$ in Definition 2.2 and 2.3, we obtain the poly-Bernoulli polynomials $B_n^{(k)}(x)$ and poly-tangent polynomials $T_n^{(k)}(x)$*

$$\frac{Li_k(1-e^{-t})}{e^t-1}e^{xt} = \sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!},$$

$$\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt} = \sum_{n=0}^{\infty}T_n^{(k)}(x)\frac{t^n}{n!},$$

respectively.

Using (2.2), it is clear that next theorem is obtained.

Theorem 2.6. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$. Then we have*

$$\mathbf{B}_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{\binom{n-l}{2}} \mathbf{B}_{l,q}^{(k)}\ x^{n-l},$$
$$\mathbf{T}_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{\binom{n-l}{2}} \mathbf{T}_{l,q}^{(k)}\ x^{n-l}.$$

Using the q-exponential functions, we introduce the following fully modified polynomials of the second type with two variables.

Definition 2.7. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, and $0 < q < 1$. We define the fully modified q-poly-tangent polynomials $\mathbf{T}_{n,q}^{(k)}(x, y)$ of the second type with two variables as below

$$\frac{[2]_q Li_{k,q}(1 - E_q(-t))}{t(E_q(2t) - 1)} E_q(xt)e_q(yt) = \sum_{n=0}^{\infty} \mathbf{T}_{n,q}^{(k)}(x, y) \frac{t^n}{[n]_q!}. \tag{2.5}$$

Theorem 2.8. *Let n be a nonnegative integer, $k \in \mathbb{Z}$ and $0 < q < 1$. Then we get*

$$\mathbf{T}_{n,q}^{(k)}(x, y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{l,q}^{(k)}(x) y^{n-l}. \tag{2.6}$$

Proof. Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbf{T}_{n,q}^{(k)}(x, y) \frac{t^n}{n!} &= \frac{[2]_q Li_{k,q}(1 - E_q(-t))}{t(E_q(2t) + 1)} E_q(xt)e_q(yt) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{l,q}^{(k)}(x) y^{n-l} \right) \frac{t^n}{[n]_q!}. \end{aligned}$$

Hence, we get

$$\mathbf{T}_{n,q}^{(k)}(x, y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{l,q}^{(k)}(x) y^{n-l}.$$

□

Corollary 2.9. *Let n be a nonnegative integer, $k \in \mathbb{Z}$, and $0 < q < 1$. If we take $y = -x$, we obtain*

$$\mathbf{T}_{n,q}^{(k)}(x, -x) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{l,q}^{(k)}(x)\ (-x)^{n-l}.$$

The fully modified q-poly-tangent polynomials of the second type with two variables are expressed by the following recurrence formula.

Theorem 2.10. *For $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $0 < q < 1$, then we derive*

$$\mathbf{T}_{n,q}^{(k)}(x,y) - \mathbf{T}_{n,q}^{(k)}(x) = \sum_{l=0}^{n-1} \begin{bmatrix} n \\ l \end{bmatrix}_q y^{n-l} \mathbf{T}_{l,q}^{(k)}(x).$$

Proof. Let $k \in \mathbb{Z}$ and $0 < q < 1$. Using (2.2), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathbf{T}_{n,q}^{(k)}(x,y)\frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} \mathbf{T}_{n,q}^{(k)}(x)\frac{t^n}{[n]_q!} \\
&\qquad = \frac{[2]_q Li_{k,q}(1-E_q(-t))}{t(E_q(2t)+1)} E_q(xt)(e_q(yt)-1) \\
&\qquad = \sum_{n=0}^{\infty} \mathbf{T}_{l,q}^{(k)}(x)\frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} y^{n+1}\frac{t^{n+1}}{[n+1]_q!} \\
&\qquad = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1} \begin{bmatrix} n \\ l \end{bmatrix}_q y^{n-l}\mathbf{T}_{l,q}^{(k)}(x)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we have the above result. In particular, if $y = 1$, we have

$$\mathbf{T}_{n,q}^{(k)}(x,1) - \mathbf{T}_{n,q}^{(k)}(x) = \sum_{l=0}^{n-1} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{l,q}^{(k)}(x).$$

□

By the Gaussian binomial coefficients and the q-polylogarithm function, we derive next theorem that is related with q-tangent polynomials of the second type.

Theorem 2.11. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$, we have*

$$\mathbf{T}_{n,q}^{(k)}(x) = \sum_{a=0}^{n}\sum_{l=0}^{a} \begin{bmatrix} n \\ a \end{bmatrix}_q \begin{bmatrix} a \\ l \end{bmatrix}_q \frac{q^{\binom{n-a+1}{2}}}{[n-a+1]_q} \mathbf{B}_{l,q}^{(k)}\ \mathbf{T}_{a-l,q}(x).$$

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$. From the definition of q-polylogarithm function, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbf{T}_{n,q}^{(k)}(x)\frac{t^n}{[n]_q!} &= \frac{Li_{k,q}(1-E_q(-t))}{E_q(t)-1}\frac{[2]_q(E_q(t)-1)}{t(E_q(2t)+1)}E_q(xt) \\
&= \sum_{n=0}^{\infty} \mathbf{B}_{n,q}^{(k)}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} \mathbf{T}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{q^{\binom{n+1}{2}}}{[n+1]_q}\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{l=0}^{a} \begin{bmatrix} n \\ a \end{bmatrix}_q \begin{bmatrix} a \\ l \end{bmatrix}_q \frac{q^{\binom{n+1}{2}}}{[n-a+1]_q}\mathbf{B}_{l,q}^{(k)}\ \mathbf{T}_{a-l,q}(x)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Hence, it is equivalent to write by the fully modified q-poly-Bernoulli polynomials of the second type and the q-tangent polynomials of the second type.

□

We also can see the relationship that include the the fully modified q-tangent polynomials of the second type and the modified q-Stirling numbers of second kind.

Theorem 2.12. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, and $0 < q < 1$. Then we obtain*

$$\mathbf{T}^{(k)}_{n,q}(x) = \sum_{a=0}^{n}\sum_{l=1}^{a+1}\begin{bmatrix} n \\ a \end{bmatrix}_q \frac{(-1)^{l+a+1}[l-1]_q!}{[l]_q^{k-1}}\frac{S^*_q(a+1,l)}{[a+1]_q}\mathbf{T}_{n-a,q}(x).$$

Proof. Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{T}^{(k)}_{n,q}(x)\frac{t^n}{n!} &= \frac{[2]_q Li_{k,q}(1-E_q(-t))}{t(E_q(2t)+1)}E_q(xt) \\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{l+n+l}[l]_q!}{[l]_q^k[n+1]_q}S^*_q(n+1,l)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathbf{T}_{n,q}(x)\frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{l=1}^{a+1}\begin{bmatrix} n \\ a \end{bmatrix}_q \frac{(-1)^{l+a+1}[l-1]_q!}{[l]_q^{k-1}}\frac{S^*_q(a+1,l)}{[a+1]_q}\mathbf{T}_{n-a,q}(x)\frac{t^n}{[n]_q!}.\end{aligned}$$

□

The fully modified q-poly tangent polynomials can be indicated by the formula that is related to the modified q-Stirling numbers of second kind, the fully modified q-Bernoulli numbers of the second type of order l and the fully modified q-poly tangent numbers of the second type.

Theorem 2.13. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$, we get*

$$\mathbf{T}^{(k)}_{n,q}(x) = \sum_{a=0}^{n}\sum_{i=0}^{a}\frac{\begin{bmatrix} n+l \\ a \end{bmatrix}_q \begin{bmatrix} a \\ i \end{bmatrix}_q}{\begin{bmatrix} n+l \\ l \end{bmatrix}_q}S^*_q(n-a+l,l)\mathbf{B}^{<l>}_{i,q}(x)\mathbf{T}^{(k)}_{a-i,q}.$$

where $\mathbf{B}^{<l>}_{i,q}(x)$ is q-Bernoulli polynomials of second type of order l.

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. Then we obtain

$$\begin{aligned}&\frac{[2]_q Li_{k,q}(1-E_q(-t))}{t(E_q(2t)+1)}E_q(xt) \\ &\quad = \sum_{n=0}^{\infty}\mathbf{T}^{(k)}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}S^*_q(n+l,l)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathbf{B}^{<l>}_{n,q}(x)\frac{t^n}{[n]_q!} \\ &\quad = \sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{i=0}^{a}\frac{\begin{bmatrix} n+l \\ a \end{bmatrix}_q \begin{bmatrix} a \\ i \end{bmatrix}_q}{\begin{bmatrix} n+l \\ l \end{bmatrix}_q}S^*_q(n-a+l,l)\ \mathbf{B}^{<l>}_{i,q}(x)\mathbf{T}^{(k)}_{n-a,q}\frac{t^n}{[n]_q!}.\end{aligned}$$

Comparing the coefficients on the both sides, we get

$$\mathbf{T}_{n,q}^{(k)}(x)=\sum_{a=0}^{n}\sum_{i=0}^{a}\frac{\begin{bmatrix}n+l\\a\end{bmatrix}_q\begin{bmatrix}a\\i\end{bmatrix}_q}{\begin{bmatrix}n+l\\l\end{bmatrix}_q}S_q^*(n-a+l,l)\mathbf{B}_{i,q}^{<l>}(x)\mathbf{T}_{n-a,q}^{(k)}.$$

□

Theorem 2.14. *For $n\in\mathbb{N},k\in\mathbb{Z}$ and $0<q<1$, the following identity holds*

$$\begin{aligned}&\mathbf{T}_{n,q}^{(k)}(x,2)-\mathbf{T}_{n,q}^{(k)}(x)\\&=\sum_{r=0}^{n-1}\sum_{a=0}^{r}\sum_{l=0}^{a+1}\begin{bmatrix}n\\r+1\end{bmatrix}_q\begin{bmatrix}r+1\\a\end{bmatrix}_q\frac{(-1)^{l+a+1}[l-1]_q!2^{r-a+1}}{[l]_q^{k-1}[a+1]_q}\\&\qquad\times S_q^*(a+1,l)\mathbf{T}_{n-r+1,q}(x).\end{aligned}$$

Proof. Let $n\in\mathbb{Z}_+,k\in\mathbb{Z}$ and $0<q<1$. Then we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathbf{T}_{n,q}^{(k)}(x,2)\frac{t^n}{n!}-\sum_{n=0}^{\infty}\mathbf{T}_{n,q}^{(k)}(x)\frac{t^n}{n!}\\&=\frac{[2]_qLi_{k,q}(1-E(-t))}{t(E_q(2t)+1)}E_q(xt)\left(e_q(2t)-1\right)\\&=\sum_{n=0}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{n+l+1}[l-1]_q!}{[l]_q^{k-1}[n+1]_q}S_q^*(n+1,l)\frac{t^n}{[n]_q!}\\&\qquad\times\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}2^{n+1}\frac{t^{n+1}}{[n+1]_q!}\;\mathbf{T}_{n,q}(x)\frac{t^n}{[n]_q!}\\&=\sum_{n=1}^{\infty}\sum_{r=0}^{n-1}\sum_{a=0}^{r}\sum_{l=0}^{a+1}\begin{bmatrix}n\\r+1\end{bmatrix}_q\begin{bmatrix}r+1\\a\end{bmatrix}_q\frac{(-1)^{l+a+1}[l-1]_q!}{[l]_q^{k-1}[a+1]_q}\\&\qquad\times 2^{r-a+1}S_q^*(a+1,l)\mathbf{T}_{n-r-1,q}(x)\frac{t^n}{[n]_q!}.\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{[n]_q!}$, for $n\in\mathbb{N}$, we get the result as below

$$\begin{aligned}&\mathbf{T}_{n,q}^{(k)}(x,2)-\mathbf{T}_{n,q}^{(k)}(x)\\&=\sum_{r=0}^{n-1}\sum_{a=0}^{r}\sum_{l=0}^{a+1}\begin{bmatrix}n\\r+1\end{bmatrix}_q\begin{bmatrix}r+1\\a\end{bmatrix}_q\frac{(-1)^{l+a+1}[l-1]_q!}{[l]_q^{k-1}[a+1]_q}\\&\qquad\times 2^{r-a+1}S_q^*(a+1,l)\mathbf{T}_{n-r+1,q}(x).\end{aligned}$$

□

3. Zeros of the fully q-poly-tangent polynomials of the second type

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the fully modified q-poly-tangent polynomials of the second type $\mathbf{T}_{n,q}^{(k)}(x)$. The fully modified q-poly-tangent polynomials of the second type $\mathbf{T}_{n,q}^{(k)}(x)$ can be determined explicitly. A few of them are

$$\mathbf{T}_{0,q}^{(k)}(x) = \frac{[2]_q}{2},$$

$$\mathbf{T}_{1,q}^{(k)}(x) = -\frac{1}{2} - \frac{q}{2} - \frac{q}{2(1-q^2)} + \frac{q^3}{2(1-q^2)} + \frac{[2]_q^{1-k}}{2(1-q^2)} - \frac{q^2[2]_q^{1-k}}{2(1-q^2)} + \frac{x}{2} + \frac{qx}{2},$$

$$\begin{aligned}\mathbf{T}_{2,q}^{(k)}(x) = {} & -q - q^2 + \frac{q^3}{2(1-q^3)} - \frac{q^5}{2(1-q^3)} + \frac{qx^2}{2} + \frac{q^2x^2}{2} + \frac{q\begin{bmatrix}2\\1\end{bmatrix}_q}{2(1-q^2)} - \frac{q^3\begin{bmatrix}2\\1\end{bmatrix}_q}{2(1-q^2)} \\ & - \frac{[2]_q^{1-k}\begin{bmatrix}2\\1\end{bmatrix}_q}{2(1-q^2)} + \frac{q^2[2]_q^{1-k}\begin{bmatrix}2\\1\end{bmatrix}_q}{2(1-q^2)} - \frac{qx\begin{bmatrix}2\\1\end{bmatrix}_q}{2(1-q^2)} + \frac{q^3x\begin{bmatrix}2\\1\end{bmatrix}_q}{2(1-q^2)} + \frac{[2]_q^{1-k}x\begin{bmatrix}2\\1\end{bmatrix}_q}{2(1-q^2)} \\ & - \frac{q^2[2]_q^{1-k}x\begin{bmatrix}2\\1\end{bmatrix}_q}{2(1-q^2)} + \frac{1}{2}[2]_q! + \frac{1}{2}q[2]_q! + \frac{[3]_q^{1-k}[2]_q!}{2(1-q^3)} - \frac{q^2[3]_q^{1-k}[2]_q!}{2(1-q^3)} \\ & - \frac{1}{2}x[2]_q! - \frac{1}{2}qx[2]_q! - \frac{q[2]_q^{1-k}[3]_q!}{(1-q^3)([2]_q!)^2} + \frac{q^3[2]_q^{1-k}[3]_q!}{(1-q^3)([2]_q!)^2}.\end{aligned}$$

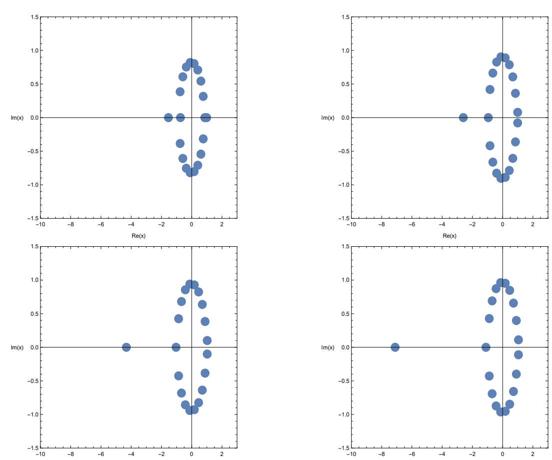

Figure 1. Zeros of $\mathbf{T}_{n,q}^{(k)}(x)$

We investigate the zeros of the fully modified q-poly-tangent polynomials of the second type $\mathbf{T}_{n,q}^{(k)}(x)$ by using a computer. We plot the zeros of the fully modified q-poly-tangent polynomials of the second type $\mathbf{T}_{n,q}^{(k)}(x)$ for $n = 20$ and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 20, q = 9/10$, and $k = -2$. In Figure 1(top-right), we choose $n = 20, q = 9/10$, and $k = 1$. In Figure 1(bottom-left), we choose $n = 20, q = 9/10$, and $k = 1$. In Figure 1(bottom-right), we choose $n = 20, q = 9/10$, and $k = 2$.

Stacks of zeros of $\mathbf{T}_{n,q}^{(k)}(x)$ for $1 \leq n \leq 20$ from a 3-D structure are presented (Figure 2). In Figure 2(left), we choose $n = 20, q = 9/10$, and $k = -2$. In Figure

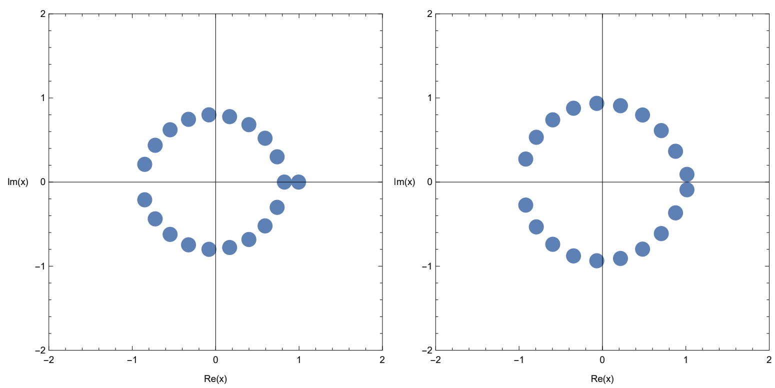

FIGURE 2. Stacks of zeros of $\mathbf{T}_{n,q}^{(k)}(x)$ for $1 \leq n \leq 20$

2(right), we choose $n = 20, q = 9/10,$, and $k = 2$.

Our numerical results for approximate solutions of real zeros of $\mathbf{T}_{n,q}^{(k)}(x)$ are displayed (Tables 2, 3, 4).

Table 2. Numbers of real and complex zeros of $\mathbf{T}_{n,q}^{(k)}(x)$

degree n	$k = -2, q = 9/10$		$k = 2, q = 9/10$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	4	0	4	0
5	5	0	5	0
6	4	2	4	2
7	5	2	3	4
8	6	2	4	4
9	7	2	5	4
10	6	4	6	4

The plot of real zeros of $\mathbf{T}_{n,,q}^{(k)}(x)$ for $1 \leq n \leq 20$ structure are presented (Figure 3). In Figure 3(left), we choose $n = 20, q = 9/10$, and $k = -2$. In Figure

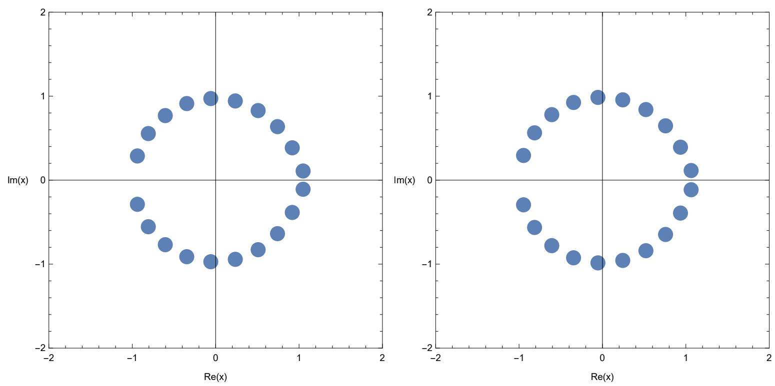

FIGURE 3. Real zeros of $\mathbf{T}_{n,q}^{(k)}(x)$ for $1 \leq n \leq 20$

3(right), we choose $n = 20, q = 9/10,$, and $k = 2$.

We observe a remarkable regular structure of the real roots of the fully modified q-poly-tangent polynomials of the second type $\mathbf{T}_{n,q}^{(k)}(x)$. We also hope to verify a remarkable regular structure of the real roots of the fully modified q-poly-tangent polynomials of the second type $\mathbf{T}_{n,q}^{(k)}(x)$ (Table 1). Next, we calculated an approximate solution satisfying fully q-poly-tangent polynomials of the second type $\mathbf{T}_{n,q}^{(k)}(x) = 0$ for $x \in \mathbb{R}$. The results are given in Table 3 and Table 4.

Table 3. Approximate solutions of $\mathbf{T}_{n,q}^{(-2)}(x) = 0, q = 9/10$

degree n	x
1	-2.13632
2	$-3.99049, \quad -0.519511$
3	$-5.8933, \quad -1.54803, \quad 0.29391$
4	$-7.94479, \quad -2.52723, \quad -0.503691, \quad 0.897817$
5	$-10.1943, \quad -3.46791, \quad -1.45091, \quad 0.483989, \quad 1.29515$
6	$-12.6777, \quad -4.38276, \quad -2.45606, \quad -0.383495$
7	$-15.4276, \quad -5.27475, \quad -3.52244, \quad -1.34303, \quad 0.713521$

Table 4. Approximate solutions of $\mathbf{T}_{n,q}^{(2)}(x) = 0, q = 9/10$

degree n	x
1	1.19668
2	0.244866, 2.28145
3	−0.513357, 1.30301, 3.21404
4	−1.13991, 0.357342, 2.36829, 4.05951
5	−1.61625, −0.60465, 1.40764, 3.45328, 4.82913
6	0.462563, 2.47063, 4.67171, 5.45367
7	−0.487024, 1.51284, 3.56407

By numerical computations, we will make a series of the following conjectures:

Conjecture 3.1. Prove that $\mathbf{T}_{n,q}^{(k)}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions. However, $\mathbf{T}_{n,q}^{(k)}(x)$ has not $Re(x) = a$ reflection symmetry for $a \in \mathbb{R}$.

Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n (see Figures 1, 2, 3). We are able to decide if $\mathbf{T}_{n,q}^{(k)}(x) = 0$ has n distinct solutions (see Tables 1, 2, 3).

Conjecture 3.2. Prove that $\mathbf{T}_{n,q}^{(k)}(x) = 0$ has n distinct solutions.

As a result of the numerical experiment, the following results can be inferred (see Table 1).

Conjecture 3.3. For $n \geq 2$, prove that

$$S_q^*(n, n-1) = \frac{(n-1)q[n]_q}{[2]_q}, \quad S_q^*(n, 1) = q^{\binom{n}{2}}.$$

Since n is the degree of the polynomial $\mathbf{T}_{n,q}^{(k)}(x)$, the number of real zeros $R_{\mathbf{T}_{n,q}^{(k)}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{\mathbf{T}_{n,q}^{(k)}(x)} = n - C_{\mathbf{T}_{n,q}^{(k)}(x)}$, where $C_{\mathbf{T}_{n,q}^{(k)}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{\mathbf{T}_{n,q}^{(k)}(x)}$ and $C_{\mathbf{T}_{n,q}^{(k)}(x)}$. The authors have no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the fully modified q-poly-tangent polynomials of the second type $\mathbf{T}_{n,q}^{(k)}(x)$ which appear in mathematics and physics.

Conflicts of interest : The author declares no conflict of interest.

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