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A NEW CRITERION FOR SUBDIVISION ITERATION DETERMINATION OF GENERALIZED STRICTLY DIAGONALLY DOMINANT MATRICES†

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Abstract. Generalized strictly diagonally dominant matrices have a wide range of applications in matrix theory and practical applications, so it is of great theoretical and practical value to study their numerical determination methods. In this paper, we study the numerical determination of generalized strictly diagonally dominant matrices by using the properties of generalized strictly diagonally dominant matrices. We obtain a new criterion for subdivision iteration determination of the generalized strictly diagonally dominant matrices by subdividing the set of non-prevailing row indices and constructing new iteration factors for the set of predominant row indices, new elements of the positive diagonal factors are derived. Advantages are illustrated by numerical examples.

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1. Introduction

The generalized strictly diagonally dominant matrices are widely used in many fields such as eigenvalue estimation, economic mathematics, power system theory and cybernetics in application. Whether a matrix is a generalized strictly diagonally dominant matrix has become a hot issue for many scholars, because many problems are attributed to the determination of generalized strictly diagonally dominant matrix. In [1], Fan Y S et al gave a set of criteria for subdivision iteration of generalized strictly diagonally dominant matrix by subdividing the non-dominant row index set of matrices and constructing progressive positive diagonal factors. In this paper, we based on the research of reference [1], a new

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criterion of subdivision iteration for generalized strictly diagonally dominant matrix is given by constructing a new positive diagonal matrix.

Let $C^{n\times n}$ be all $n\times n$-order complex matrices, $A=(a_{ij})\in C^{n\times n}$,

$$N=\{1,2,\cdots,n\},\Lambda_i=\Lambda_i(A)=\sum_{j\neq i}|a_{ij}|\ (i,j\in N).$$

Definition 1.1 ([3]). A matrix $A=(a_{ij})\in C^{n\times n}$ is called a strictly diagonally dominant matrix, if $|a_{ii}|>\Lambda_i$, for any $i\in N$, and is denoted by $A\in D$. A matrix A is called a generalized strictly diagonally dominant matrix(ie. non-singular H-matrix), if there exists a positive diagonal matrix, such that $AX\in D$, and is denoted by $A\in D^*$.

Definition 1.2 ([3]). A irreducible matrix $A=(a_{ij})\in C^{n\times n}$ is called a irreducible diagonally dominant matrix, if for any $i\in N$, $|a_{ii}|\geq\Lambda_i$, and at least one strictly inequality holds.

Definition 1.3 ([3]). A matrix $A=(a_{ij})\in C^{n\times n}$ is called a diagonally dominant matrix with nonzero element chains, if $|a_{ii}|\geq\Lambda_i$, for any $i\in N$, and at least one strict inequality holds. There always exists nonzero element sequence $a_{ij_1}a_{j_1j_2}\cdots a_{j_pj}$, such $|a_{jj}|>\Lambda_j$, for the i of $|a_{ii}|=\Lambda_i$.

Lemma 1.4 ([4]). *A irreducible diagonally dominant matrix $A=(a_{ij})\in C^{n\times n}$, A is called a non-singular H-matrix, and at least one strictly diagonally dominant row exists.*

Lemma 1.5 ([4]). *A matrix $A=(a_{ij})\in C^{n\times n}$ is called a non-singular H-matrix, where A is a diagonally dominant matrix with nonzero element chains.*

In this paper, we always set $|a_{ii}|\neq 0$, $\Lambda_i\neq 0$, and $\sum\limits_{t\in\varnothing}\bullet=0$. Let $A=(a_{ij})\in C^{n\times n}$,

$$N_1=\{i\in N:0<|a_{ii}|<\Lambda_i\}\,,\ N_2=\{i\in N:0<|a_{ii}|=\Lambda_i\}\,,$$
$$N_3=\{i\in N:|a_{ii}|>\Lambda_i\}\,,\ \mathbf{Z}=\{0,1,2,\cdots\},\ \mathbf{Z}^+=\{1,2,\cdots\}.$$

Obviously, $N=N_1\cup N_2\cup N_3$, and $A\in D$, if $N_1\cup N_2$ is empty. $A\notin D$, if N_3 is empty. Therefore, we always assumes that $N_1\cup N_2$ is not empty, so as N_3.

Let $A=(a_{ij})\in C^{n\times n}$. Divide N_1 into $N_1^{(1)}\cup N_1^{(2)}\cup\cdots\cup N_1^{(m)}$(m is an arbitrary positive integer), where

$$N_1^{(1)}=\left\{i\in N_1:0<|a_{ii}|<\frac{1}{m}\Lambda_i\right\},$$
$$N_1^{(k)}=\left\{i\in N_1:\frac{k-1}{m}\Lambda_i\leq|a_{ii}|<\frac{k}{m}\Lambda_i\right\},\ k=2,3,\cdots,m,$$

and $N_1^{(k)}$ may be empty. We set:

$$\overline{x}_{1i}^{(k)}=\frac{k}{m}-\frac{|a_{ii}|}{\Lambda_i}\ (i\in N_1^{(k)},\ k=1,2,\cdots,m),\ \overline{x}_{2i}=\frac{1}{m}\ (i\in N_2),$$

$$\overline{r}_0 = 1,\ \overline{r}_1 = \max_{i\in N_3}\left(\frac{\Lambda_i}{|a_{ii}|}\right),$$

$$\overline{r}_{l+1} = \max_{i\in N_3}\left\{\frac{\sum\limits_{k=1}^{m}(\sum\limits_{t\in N_1^{(k)}}|a_{it}|\overline{x}_{1t}^{(k)}) + \sum\limits_{t\in N_2}|a_{it}|\overline{x}_{2t} + \overline{r}_l\sum\limits_{t\in N_3,t\neq i}|a_{it}|}{|a_{ii}|}\right\}\ (l\in \mathbf{Z}^+),$$

$$\overline{h}_{l+1,i} = \frac{\sum\limits_{k=1}^{m}(\sum\limits_{t\in N_1^{(k)}}|a_{it}|\overline{x}_{1t}^{(k)}) + \sum\limits_{t\in N_2}|a_{it}|\overline{x}_{2t} + \overline{r}_l\sum\limits_{t\in N_3,t\neq i}|a_{it}|}{|a_{ii}|}\ (i\in N_3,\ l\in \mathbf{Z}).$$

In 2012, Fan Y S et al gave the following results :

Theorem 1.6 ([1])**.** *Let $A=(a_{ij})\in C^{n\times n}$, if there exist $l\in \boldsymbol{Z}$, make*

$$|a_{ii}|\overline{x}_{1i}^{(k)} > \sum_{k=1}^{m}(\sum_{t\in N_1^{(k)},t\neq i}|a_{it}|\overline{x}_{1t}^{(k)}) + \sum_{t\in N_2}|a_{it}|\overline{x}_{2t} + \sum_{t\in N_3}|a_{it}|\overline{h}_{l+1,t}$$

$$(i\in N_1^{(k)},\ k=1,2,\cdots,m),$$

$$|a_{ii}|\overline{x}_{2i} > \sum_{k=1}^{m}(\sum_{t\in N_1^{(k)}}|a_{it}|\overline{x}_{1t}^{(k)}) + \sum_{t\in N_2,t\neq i}|a_{it}|\overline{x}_{2t} + \sum_{t\in N_3}|a_{it}|\overline{h}_{l+1,t}\ (i\in N_2),$$

then $A\in D^*$.

On this basis, this paper obtained a new positive diagonal matrix by constructing a new iterative factor, and gave a new criterion for determining the subdivision iteration of generalized strictly diagonally dominant matrices, which finally extended the main results of reference in [1].

For convenience, we employ the following notations.

$$x_{1i}^{(k)} = \frac{k}{m} - \frac{|a_{ii}|}{\Lambda_i}\ (i\in N_1^{(k)},\ k=1,2,\cdots,m),\ x_{3i} = \frac{\Lambda_i}{|a_{ii}|}\ (i\in N_3),$$

$$x_{2i} = \frac{\sum\limits_{k=1}^{m}(\sum\limits_{t\in N_1^{(k)}}|a_{it}|x_{1t}^{(k)}) + \sum\limits_{t\in N_2,t\neq i}|a_{it}| + \sum\limits_{t\in N_3}|a_{it}|x_{3t}}{|a_{ii}|}\ (i\in N_2),$$

$r_0 = 1$,

$$r_{l+1} = \max_{i\in N_3}\left\{\frac{\sum\limits_{k=1}^{m}(\sum\limits_{t\in N_1^{(k)}}|a_{it}|x_{1t}^{(k)}) + \sum\limits_{t\in N_2}|a_{it}|x_{2t} + r_l\sum\limits_{t\in N_3,t\neq i}|a_{it}|x_{3t}}{|a_{ii}|x_{3i}}\right\}\ (l\in \mathbf{Z}),$$

$$h_{l+1,i} = \frac{\sum\limits_{k=1}^{m}(\sum\limits_{t\in N_1^{(k)}} |a_{it}|x_{1t}^{(k)}) + \sum\limits_{t\in N_2} |a_{it}|x_{2t} + r_l \sum\limits_{t\in N_3, t\neq i} |a_{it}|x_{3t}}{|a_{ii}|x_{3i}} \ (i \in N_3,\ l \in \mathbf{Z}),$$

$$f_{l+1,i} = \frac{\sum\limits_{k=1}^{m}(\sum\limits_{t\in N_1^{(k)}} |a_{it}|x_{1t}^{(k)}) + \sum\limits_{t\in N_2} |a_{it}|x_{2t} + \sum\limits_{t\in N_3, t\neq i} |a_{it}|x_{3t}h_{l+1,t}}{|a_{ii}|} \ (i \in N_3,\ l \in \mathbf{Z}),$$

$$\delta_{l+1} = \max_{i\in N_3}\left\{ \frac{\sum\limits_{k=1}^{m}(\sum\limits_{t\in N_1^{(k)}} |a_{it}|x_{1t}^{(k)}) + \sum\limits_{t\in N_2} |a_{it}|x_{2t}}{|a_{ii}|f_{l+1,i} - \sum\limits_{t\in N_3, t\neq i} |a_{it}|f_{l,t}} \right\} \ (l \in \mathbf{Z}^+).$$

2. Main results

Theorem 2.1. *Let $A = (a_{ij}) \in C^{n\times n}$, if there exists $l_0 \in \boldsymbol{Z}^+$, make*

$$|a_{ii}|x_{1i}^{(k)} > \sum_{k=1}^{m}(\sum_{t\in N_1^{(k)}, t\neq i} |a_{it}|x_{1t}^{(k)}) + \sum_{t\in N_2} |a_{it}|x_{2t} + \delta_{l_0+1} \sum_{t\in N_3} |a_{it}|f_{l_0+1,t}$$

$$(i \in N_1^{(k)},\ k = 1, 2, \cdots, m),$$

then$A \in D^*$, where for any $i \in N_2$, existing $t \in N_3$ to make $|a_{it}| \neq 0$.

Proof. $0 < x_{1i}^{(k)} < 1$ is established, for any $i \in N_1^{(k)}$, $k = 1, 2, \cdots, m$; and $0 < x_{3i} < 1$, for any $i \in N_3$. According to the theorem conditions and definition of x_{2i}, we have $0 < x_{2i} < 1$, for any $i \in N_2$; $r_0 = 1$, $x_{3i} = \frac{\Lambda_i}{|a_{ii}|}$, for any $i \in N_3$, we get

$$\Lambda_i = |a_{ii}|x_{3i},$$

$$\sum_{k=1}^{m}(\sum_{t\in N_1^{(k)}} |a_{it}|x_{1t}^{(k)}) + \sum_{t\in N_2} |a_{it}|x_{2t} + r_0 \sum_{t\in N_3, t\neq i} |a_{it}|x_{3t} < \Lambda_i = |a_{ii}|x_{3i},$$

then

$$\frac{\sum\limits_{k=1}^{m}(\sum\limits_{t\in N_1^{(k)}} |a_{it}|x_{1t}^{(k)}) + \sum\limits_{t\in N_2} |a_{it}|x_{2t} + r_0 \sum\limits_{t\in N_3, t\neq i} |a_{it}|x_{3t}}{|a_{ii}|x_{3i}} < 1.$$

From definitions of $h_{1,i}$, r_1, we also get

$$h_{1,i} \leq r_1 < r_0 = 1 \ (i \in N_3). \tag{1}$$

By (1) and r_2, then

$$\sum_{k=1}^{m}(\sum_{t\in N_1^{(k)}} |a_{it}|x_{1t}^{(k)}) + \sum_{t\in N_2} |a_{it}|x_{2t} + r_1 \sum_{t\in N_3, t\neq i} |a_{it}|x_{3t} < \Lambda_i = |a_{ii}|x_{3i}, \tag{2}$$

$$r_1 \sum_{t\in N_3, t\neq i} |a_{it}| x_{3t} \leq \sum_{t\in N_3, t\neq i} |a_{it}| x_{3t}. \tag{3}$$

Based on (2), (3), $h_{2,i}$, and r_2, then

$$h_{2,i} \leq r_2 \leq r_1 < 1 \ (i \in N_3).$$

We assume $h_{s+1,i} \leq r_{s+1} \leq r_s < 1$, when $l = s$, then

$$\sum_{k=1}^{m} (\sum_{t\in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \sum_{t\in N_2} |a_{it}| x_{2t} + r_s \sum_{t\in N_3, t\neq i} |a_{it}| x_{3t} < \Lambda_i = |a_{ii}| x_{3i}, \tag{4}$$

$$\sum_{k=1}^{m} (\sum_{t\in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \sum_{t\in N_2} |a_{it}| x_{2t} + r_{s+1} \sum_{t\in N_3, t\neq i} |a_{it}| x_{3t} < \Lambda_i = |a_{ii}| x_{3i}, \tag{5}$$

$$r_{s+1} \sum_{t\in N_3, t\neq i} |a_{it}| x_{3t} \leq r_s \sum_{t\in N_3, t\neq i} |a_{it}| x_{3t}. \tag{6}$$

We obtain $h_{s+2,i} \leq r_{s+2} \leq r_{s+1} < 1$ stem from the above findings, for any $i \in N_3$. Therefore, it can be known from Mathematical Induction,

$$h_{l+1,i} \leq r_{l+1} \leq r_l \leq \cdots \leq r_1 < r_0 = 1 \ (i \in N_3, \ l \in \mathbf{Z}^+). \tag{7}$$

By r_{l+1}, $h_{l+1,i}$, $f_{l+1,i}$, (7), we have

$$h_{l+1,i} x_{3i} \leq r_{l+1} x_{3i} < 1 \ (i \in N_3, \ l \in \mathbf{Z}), \tag{8}$$

$$f_{l+1,i} \leq h_{l+1,i} x_{3i} \leq r_{l+1} x_{3i} \leq r_l x_{3i} \leq \cdots \leq r_1 x_{3i} < 1 \ (i \in N_3, \ l \in \mathbf{Z}^+),$$

$$h_{l+1,i} \leq h_{l,i} \leq \cdots \leq h_{1,i} < 1 \ (i \in N_3, \ l \in \mathbf{Z}^+), \tag{9}$$

then

$$f_{l+1,i} \leq f_{l,i} \leq \cdots \leq f_{1,i} < x_{3i} < 1 \ (i \in N_3, \ l \in \mathbf{Z}^+). \tag{10}$$

As $f_{l+1,i}$ defined and (8), we also get

$$0 \leq x_{3i}(r_l - h_{l,i}) < 1 \ (i \in N_3, \ l \in \mathbf{Z}^+),$$

and for any $i \in N_3$,

$$\begin{aligned} |a_{ii}| f_{l+1,i} - \sum_{t\in N_3, t\neq i} |a_{it}| f_{l,t} &= \sum_{k=1}^{m} (\sum_{t\in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \sum_{t\in N_2} |a_{it}| x_{2t} \\ &\quad + \sum_{t\in N_3, t\neq i} |a_{it}| \frac{\sum_{t\in N_3, t\neq i} |a_{it}| x_{3t}(r_l - h_{l,t})}{|a_{tt}|} \\ &\geq \sum_{k=1}^{m} (\sum_{t\in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \sum_{t\in N_2} |a_{it}| x_{2t}. \end{aligned} \tag{11}$$

By (11), x_{3i}, and δ_{l+1}, we can obtain

$$0 < \delta_{l+1} \leq 1 \ (i \in N_3, \ l \in \mathbf{Z}^+). \tag{12}$$

$$\delta_{l+1} f_{l+1,i} < x_{3i} < 1 \ (i \in N_3, \ l \in \mathbf{Z}^+). \tag{13}$$

Since for the theorem conditions and x_{2i} , (13), for any $i \in N_2$, then

$$|a_{ii}| x_{2i} > \sum_{k=1}^{m} (\sum_{t \in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \sum_{t \in N_2, t \neq i} |a_{it}| x_{2t} + \delta_{l+1} \sum_{t \in N_3} |a_{it}| f_{l+1,t}. \tag{14}$$

According to the above formula and theorem conditions, we can find $l_0 \in \mathbf{Z}^+$, and a positive number ε that can be sufficiently small, so that the following results is true,

$$|a_{ii}| x_{1i}^{(k)} - [\sum_{k=1}^{m} (\sum_{t \in N_1^{(k)}, t \neq i} |a_{it}| x_{1t}^{(k)}) + \sum_{t \in N_2} |a_{it}| x_{2t} + \delta_{l_0+1} \sum_{t \in N_3} |a_{it}| f_{l_0+1,t}] > \varepsilon \sum_{t \in N_3} |a_{it}|, \tag{15}$$

for any $i \in N_1^{(k)} (k = 1, 2, \cdots, m)$, and

$$|a_{ii}| x_{2i} - [\sum_{k=1}^{m} (\sum_{t \in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \sum_{t \in N_2, t \neq i} |a_{it}| x_{2t} + \delta_{l_0+1} \sum_{t \in N_3} |a_{it}| f_{l_0+1,t}] > \varepsilon \sum_{t \in N_3} |a_{it}|. \tag{16}$$

for any $i \in N_2$. We construct a positive diagonal matrix $X = \mathrm{diag}(x_1, x_2, \cdots, x_n)$, where

$$x_i = \begin{cases} x_{1i}^{(k)}, & i \in N_1^{(k)}, \ k = 1, 2, \cdots, m; \\ x_{2i}, & i \in N_2; \\ \delta_{l_0+1} f_{l_0+1,i} + \varepsilon, & i \in N_3. \end{cases}$$

Let $B = AX = (b_{ij})$. By (15), for any $i \in N_1^{(k)} (k = 1, 2, \cdots, m)$, then

$$\begin{aligned} |b_{ii}| - \Lambda_i(B) &= |a_{ii}| x_{1i}^{(k)} - [\sum_{k=1}^{m} (\sum_{t \in N_1^{(k)}, t \neq i} |a_{it}| x_{1t}^{(k)}) + \sum_{t \in N_2} |a_{it}| x_{2t} \\ &\quad + \sum_{t \in N_3} |a_{it}| (\delta_{l_0+1} f_{l_0+1,t} + \varepsilon)] > 0. \end{aligned}$$

From (16) , for $i \in N_2$, we have

$$\begin{aligned} |b_{ii}| - \Lambda_i(B) &= |a_{ii}| x_{2i} - [\sum_{k=1}^{m} (\sum_{t \in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \sum_{t \in N_2, t \neq i} |a_{it}| x_{2t} \\ &\quad + \sum_{t \in N_3} |a_{it}| (\delta_{l_0+1} f_{l_0+1,t} + \varepsilon)] > 0. \end{aligned}$$

For any $i \in N_3$, $|a_{ii}| > \Lambda_i(A) > \sum\limits_{t \in N_3, t \neq i} |a_{it}|$, $\varepsilon > 0$, then

$$\varepsilon(|a_{ii}| - \sum_{t \in N_3, t \neq i} |a_{it}|) > 0. \tag{17}$$

We can obtain the following results by (10), (18), and (12), and δ_{l+1},

$$|a_{ii}|\delta_{l_0+1} f_{l_0+1,i} - [\sum_{k=1}^{m}(\sum_{t \in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \sum_{t \in N_2} |a_{it}| x_{2t} + \delta_{l_0+1} \sum_{t \in N_3, t \neq i} |a_{it}| f_{l_0,t}] \geq 0, \tag{18}$$

$$|a_{ii}|\delta_{l_0+1} f_{l_0+1,i} - [\sum_{k=1}^{m}(\sum_{t \in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \sum_{t \in N_2} |a_{it}| x_{2t} + \delta_{l_0+1} \sum_{t \in N_3, t \neq i} |a_{it}| f_{l_0+1,t}] \geq 0. \tag{19}$$

For any $i \in N_3$,

$$\begin{aligned} |b_{ii}| - \Lambda_i(B) &= |a_{ii}|(\delta_{l_0+1} f_{l_0+1,i} + \varepsilon) - [\sum_{k=1}^{m}(\sum_{t \in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) \\ &\quad + \sum_{t \in N_2} |a_{it}| x_{2t} + \sum_{t \in N_3, t \neq i} |a_{it}|(\delta_{l_0+1} f_{l_0+1,t} + \varepsilon)] \\ &= |a_{ii}|\delta_{l_0+1} f_{l_0+1,i} - [\sum_{k=1}^{m}(\sum_{t \in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \sum_{t \in N_2} |a_{it}| x_{2t} \\ &\quad + \delta_{l_0+1} \sum_{t \in N_3, t \neq i} |a_{it}| f_{l_0+1,t}] + \varepsilon(|a_{ii}| - \sum_{t \in N_3, t \neq i} |a_{it}|) > 0. \end{aligned}$$

from (17) and (19). In summary, $|b_{ii}| > \Lambda_i(B)$, for any $i \in N$, ie. $B \in D$, then $A \in D^*$. □

Remark 2.1. Assuming $N_2 = \varnothing$, and $\sum\limits_{t \in N_3, t \neq i} |a_{it}| \neq 0$ for any $i \in N_3$, then $x_{3i} \leq \overline{r}_1$, where $r_0 = 1$, $l = 1$, and $\overline{r}_1 = \max\limits_{i \in N_3} \left(\frac{\Lambda_i}{|a_{ii}|}\right)$. Furthermore, by $0 < r_1 < 1$, and (7), (12), $h_{2,i}$, $f_{2,i}$, δ_2, and $\overline{h}_{2,i}$, we can get

$$\frac{\sum\limits_{k=1}^{m}(\sum\limits_{t \in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + r_1 \sum\limits_{t \in N_3, t \neq i} |a_{it}| x_{3t}}{|a_{ii}|} < \frac{\sum\limits_{k=1}^{m}(\sum\limits_{t \in N_1^{(k)}} |a_{it}| x_{1t}^{(k)}) + \overline{r}_1 \sum\limits_{t \in N_3, t \neq i} |a_{it}|}{|a_{ii}|},$$

$$f_{2,i} \leq h_{2,i} x_{3i} < \overline{h}_{2,i} \ (i \in N_3),$$

$$\delta_2 f_{2,i} \leq h_{2,i} x_{3i} < \overline{h}_{2,i} \ (i \in N_3).$$

To sum up, theorem 2.1 in this paper generalizes the main results of reference [1], when $N_2 \neq \varnothing$, $l = 1$.

Meanwhile, the main results of reference [6, 7] are also generalized. In theorem 2.1 we always have $0 < x_{2i} < 1$ for any positive integer m, when $N_2 \neq \varnothing$. And it is illustrated by numerical examples.

The criterion of Theorem 2.1 can be determined by computer using the following algorithm:

INPUT: A matrix $A = (a_{it}) \in C^{n\times n}$, and positive integer m, L.

OUTPUT: $A \in D^*$, $X =\mathrm{diag}(x_1, x_2, \cdots, x_n)$.

Step1. Compute Λ_i, N_1, N_2, N_3, and $x_{1i}^{(k)}$, x_{3i}, x_{2i}.

Step2. Let $r_0 = 1$, $l = 1$, compute $h_{1,i}$, r_1, $f_{1,i}$.

Step3. Compute$h_{l+1,i}$, r_{l+1}, $f_{l+1,i}$, δ_{l+1}.

Step4. If

$$|a_{ii}|x_{1i}^{(k)} > \sum_{k=1}^{m}\Big(\sum_{t\in N_1^{(k)},t\neq i}|a_{it}|x_{1t}^{(k)}\Big) + \sum_{t\in N_2}|a_{it}|x_{2t} + \delta_{l+1}\sum_{t\in N_3}|a_{it}|f_{l+1,t},$$

for any $i \in N_1^{(k)}(k = 1, 2, \cdots, m)$, and

$$\sum_{t\in N_3}|a_{it}| \neq 0,$$

for any $i \in N_2$, then $A \in D^*$. And output $X =\mathrm{diag}(x_1, x_2, \cdots, x_n)$, STOP, where

$$x_i = \begin{cases} x_{1i}^{(k)}, & i \in N_1^{(k)}, k = 1, 2, \cdots, m; \\ x_{2i}, & i \in N_2; \\ \delta_{l+1}f_{l+1,i}, & i \in N_3. \end{cases}$$

Otherwise, go to Step5.

Step5. Set $l = l + 1$, when $l < L$, and go to Step3. Otherwise, output "failure", STOP.

Similarly, we can generalize the criterion in the case of irreducibility and non-zero element chain, from lemma 1.4 and lemma 1.5.

Theorem 2.2. *Let $A = (a_{ij}) \in C^{n\times n}$ be a irreducible matrix, if there exists $l_0 \in \boldsymbol{Z}^+$, such that*

$$|a_{ii}|x_{1i}^{(k)} \geq \sum_{k=1}^{m}\Big(\sum_{t\in N_1^{(k)},t\neq i}|a_{it}|x_{1t}^{(k)}\Big) + \sum_{t\in N_2}|a_{it}|x_{2t} + \delta_{l_0+1}\sum_{t\in N_3}|a_{it}|f_{l_0+1,t}$$

$$(i \in N_1^{(k)},\ k = 1, 2, \cdots, m),$$

And a strict inequality holds for at least one $i \in N_1^{(k)}$, then $A \in D^*$.

Theorem 2.3. *Let $A = (a_{ij}) \in C^{n\times n}$, if there exists $l_0 \in \boldsymbol{Z}^+$, then*

$$|a_{ii}|x_{1i}^{(k)} \geq \sum_{k=1}^{m}\Big(\sum_{t\in N_1^{(k)},t\neq i}|a_{it}|x_{1t}^{(k)}\Big) + \sum_{t\in N_2}|a_{it}|x_{2t} + \delta_{l_0+1}\sum_{t\in N_3}|a_{it}|f_{l_0+1,t}$$

$$(i \in N_1^{(k)},\ k = 1, 2, \cdots, m),$$

and for i where the above equation holds, there always have nonzero element chain $a_{ij_1}a_{j_1j_2}\cdots a_{j_gj}$, such that

$$|a_{jj}|x_{1j}^{(k)} > \sum_{k=1}^{m}(\sum_{t\in N_1^{(k)}, t\neq j}|a_{jt}|x_{1t}^{(k)}) + \sum_{t\in N_2}|a_{jt}|x_{2t} + \delta_{l_0+1}\sum_{t\in N_3}|a_{jt}|f_{l_0+1,t}$$

$$(j \in N_1^{(k)},\ k = 1, 2, \cdots, m),$$

then $A \in D^*$.

3. Numerical example

Example 3.1. Consider matrix

$$A = \begin{pmatrix} 1.8 & 0 & 0.9 & 0 & 0 & 3.1 \\ 1 & 6 & 6 & 1 & 1 & 75 \\ 1 & 0 & 3 & 1 & 1 & 0 \\ 1 & 1 & 1 & 19 & 1 & 2 \\ 0 & 0 & 1 & 1 & 20 & 1 \\ 1 & 1 & 0 & 1 & 0 & 50 \end{pmatrix}.$$

Set $m = 1$, we get $N_1 = N_1^{(1)} = \{1, 2\}$, $N_2 = \{3\}$, $N_3 = \{4, 5, 6\}$. And $\overline{x}_{11}^{(1)} = 0.5500$, $\overline{x}_{12}^{(1)} = 0.9286$, $\overline{x}_{23} = 1$. From $\overline{r}_0 = 1$, $\overline{r}_1 = \max\limits_{i\in N_3}\{\frac{6}{19}, \frac{3}{20}, \frac{3}{50}\} = \frac{6}{19}$,

$$\overline{r}_{l+1} = \max_{i\in N_3}\left\{\frac{\sum\limits_{k=1}^{m}(\sum\limits_{t\in N_1^{(k)}}|a_{it}|\overline{x}_{1t}^{(k)}) + \sum\limits_{t\in N_2}|a_{it}|\overline{x}_{2t} + \overline{r}_l\sum\limits_{t\in N_3, t\neq i}|a_{it}|}{|a_{ii}|}\right\} \quad (l \in \mathbf{Z}^+),$$

and

$$\overline{h}_{l+1,i} = \frac{\sum\limits_{k=1}^{m}(\sum\limits_{t\in N_1^{(k)}}|a_{it}|\overline{x}_{1t}^{(k)}) + \sum\limits_{t\in N_2}|a_{it}|\overline{x}_{2t} + \overline{r}_l\sum\limits_{t\in N_3, t\neq i}|a_{it}|}{|a_{ii}|} \quad (i \in N_3, l \in \mathbf{Z}),$$

it significantly that $0 < \overline{h}_{l+1,i} < 1$ $(i \in N_3)$, and when $i = 2$, for any $l \in Z$, we have

$$\begin{aligned} |a_{22}|\overline{x}_{12}^{(1)} = 5.5714 \quad < \quad & |a_{21}|\overline{x}_{11}^{(1)} + |a_{23}|\overline{x}_{23} + |a_{24}|\overline{h}_{l+1,4} + |a_{25}|\overline{h}_{l+1,5} + |a_{26}|\overline{h}_{l+1,6} \\ & = 6.55 + |a_{24}|\overline{h}_{l+1,4} + |a_{25}|\overline{h}_{l+1,5} + |a_{26}|\overline{h}_{l+1,6}. \end{aligned}$$

Comparatively, it is impossible to determine that matrix A is a generalized strictly diagonally dominant matrix by using the criteria of theorem 1 in reference [1], [6], [7], where $m = 1$ for any $l \in Z$. However, when we set $m = 1$, $l_0 = 1$, we can obtain $x_{11}^{(1)} = 0.5500$, $x_{12}^{(1)} = 0.9286$, $x_{34} = 0.3158$, $x_{35} = 0.1500$, $x_{36} =$

0.0600, $x_{23} = 0.3386$, $f_{1,4} = 0.1013$, $f_{1,5} = 0.0242$, $f_{1,6} = 0.0318$, $f_{2,4} = 0.1006$, $f_{2,5} = 0.0238$, $f_{2,6} = 0.0317$, $\delta_2 = 0.9981$, and

$$|a_{11}|x_{11}^{(1)} = 0.9900 > |a_{12}|x_{12}^{(1)}+|a_{13}|x_{23}+\delta_2(|a_{14}|f_{2,4}+|a_{15}|f_{2,5}+|a_{16}|f_{2,6}) = 0.4027,$$

$$|a_{22}|x_{12}^{(1)} = 5.5714 > |a_{21}|x_{11}^{(1)}+|a_{23}|x_{23}+\delta_2(|a_{24}|f_{2,4}+|a_{25}|f_{2,5}+|a_{26}|f_{2,6}) = 5.0753.$$

Obviously, the matrix A satisfies the condition of theorem 2.1 in this paper, then $A \in D^*$.

In fact, we construct a positive diagonal matrix

$$X = \mathrm{diag}(0.5500, 0.9286, 0.3386, 0.1004, 0.0238, 0.0316),$$

then

$$AX = \begin{pmatrix} 0.9900 & 0 & 0.3047 & 0 & 0 & 0.0979 \\ 0.5500 & 5.5714 & 2.0316 & 0.1004 & 0.0238 & 2.3695 \\ 0.5500 & 0 & 1.0158 & 0.1004 & 0.0238 & 0 \\ 0.5500 & 0.9286 & 0.3386 & 1.9084 & 0.0238 & 0.0632 \\ 0 & 0 & 0.3386 & 0.1004 & 0.4752 & 0.0316 \\ 0.5500 & 0.9286 & 0 & 0.1004 & 0 & 1.5797 \end{pmatrix}.$$

It is easily to prove that $AX \in D^*$, namely $A \in D^*$.

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Data availability : Not applicable

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