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INTERVAL VALUED VECTOR VARIATIONAL INEQUALITIES AND VECTOR OPTIMIZATION PROBLEMS VIA CONVEXIFICATORS

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Abstract. In this study, we take into account interval-valued vector optimization problems $(IVOP)$ and obtain their relationships to interval vector variational inequalities $(IVVI)$ of Stampacchia and Minty kind in aspects of convexificators, as well as the $(IVOP)$ LU-efficient solution under the LU-convexity assumption. Additionally, we examine the weak version of the $(IVVI)$ of the Stampacchia and Minty kind and determine the relationships between them and the weakly LU-efficient solution of the $(IVOP)$. The results of this study improve and generalizes certain earlier results from the literature.

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1. Introduction

Giannessi [6] first proposed the idea of vector variational inequalities (VVI) in 1980, which have been broader applicability in optimization, adaptive control, finance, and stability problems, see, for example, [4, 13] and the references cited therein. In optimization theory, nonsmooth phenomenon frequently occur, which has prompted the development of several subdifferential and generalized directional derivative notions. A generalization of plenty of well subdifferentials, particularly Mordukhovich [23], Michel-Penot [20], and Clarke [3] subdifferentials is the idea of a convexificator. It has been demonstrated that the idea of convexificators is a helpful tool in the field of nonsmooth optimization. The concept of a convexificator was proposed by Demyanov [5] in the year 1994. Convexificators were recently employed by Golestani and Nobakhtian [7], Long and Huang [17] and Luu [19] to create the ideal circumstances for nonsmooth

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optimization problems. We refer to [12, 15, 16, 18, 27], and its sources for further details on convexificators.

One of the deterministic optimization models that can be used to deal with uncertain data is a problem known as an interval-valued optimization problem. There are three basic ways to describe constrained optimization with uncertainty, which are referred to as the stochastic programming approach, the fuzzy programming approach, and the interval-valued programming approach. To find solutions to these problems, numerous techniques have been established. Stochastic and fuzzy optimization problems, on the other hand, are notoriously difficult to resolve. In the method of optimization known as the interval-valued optimization problem, the coefficients of both the objective and constraint functions are represented by closed intervals. As a result, the solution to the stochastic or fuzzy optimization problem will be more difficult to achieve than the solution to the $(IVOP)$. This is the primary reason why the $(IVOP)$ has recently attracted increased interest in the optimization community, see for example [8, 10, 24, 25, 26, 28] and the references contained therein for more information. For both smooth and nonsmooth vector-valued objective functions, numerous results proving optimality criteria in terms of (VVI) have been developed, see [2]. Concerning optimal solutions with interval values, Zhang et al. [29] studied LU-convexity as an extension of convexity to determine the optimality criteria for real-valued maps. Jenname [9] examine the case of $(IVOP)$ and demonstrate how they relate to interval (VVI) of Stampacchia and Minty kind. Motivated and inspired by ongoing research work, we adapt the concept of the LU-convex function and generalize it to an interval-valued vector function. Afterward, we will use these concepts as a tool to find the relationship between $(IVOP)$ and (VVI) of Stampacchia and Minty types.

The work done in this paper is divided into five sections. Sections 1 and 2 deal with the introduction and preliminaries required for a basic understanding of the topic. Section 3 deals with the basics of intervals and their features. Section 4 deduces relationships between $(IVOP)$, $(IVVI)$ of Stampacchia and Minty kind in terms of convexificators and LU-efficient solution of $(IVOP)$ under the LU-convexity condition. Finally, in section 5 we conclude our paper.

2. Main results

In this paper, we take $\mathbb{R}^n$ as n-dimensional Euclidean space, $\mathbb{R}^n_+$ and $int\mathbb{R}^n_+$ as its nonnegative and positive orthant, respectively. $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ signify the extended real line and $\langle ., . \rangle$ denotes the Euclidean inner product. Further, we assume that $0 \neq D \subseteq \mathbb{R}^n$ contains the Euclidean norm $\|.\|$.

The convention for equality and inequalities is as follows:
If $\upsilon, \omega \in \mathbb{R}^n$, then

$\upsilon \geqq \omega \;\Leftrightarrow\; \upsilon_j \geq \omega_j,\; j = 1, 2, 3, ..., n;$
$\upsilon > \omega \;\Leftrightarrow\; \upsilon_j > \omega_j,\; j = 1, 2, 3, ..., n;$
$\upsilon \geq \omega \;\Leftrightarrow\; \upsilon_j \geq \omega_j,\; j = 1, 2, 3, ..., n,$ but $\upsilon \neq \omega;$
$\upsilon \leqq \omega \;\Leftrightarrow\; \upsilon_j \leq \omega_j,\; j = 1, 2, 3, ..., n;$
$\upsilon < \omega \;\Leftrightarrow\; \upsilon_j < \omega_j,\; j = 1, 2, 3, ..., n;$
$\upsilon \leq \omega \;\Leftrightarrow\; \upsilon_j \leq \omega_j,\; j = 1, 2, 3, ..., n,$ but $\upsilon \neq \omega.$
First of all, we recall some definitions from [11] as follows:

Definition 2.1. Suppose $\Gamma : D \to \overline{\mathbb{R}}$ be an extended real valued function, $\upsilon \in D$ and $\Gamma(\upsilon)$ be finite. Then, the *lower and upper Dini derivatives* of Γ at $\upsilon \in D$ in the direction $\omega \in \mathbb{R}^n$, are denoted and defined as follows:

$$\Gamma^-(\upsilon, \omega) = \liminf_{\lambda \to 0} \frac{\Gamma(\upsilon + \lambda\omega) - \Gamma(\upsilon)}{\lambda}.$$
$$\Gamma^+(\upsilon, \omega) = \limsup_{\lambda \to 0} \frac{\Gamma(\upsilon + \lambda\omega) - \Gamma(\upsilon)}{\lambda}.$$

Definition 2.2. Suppose $\Gamma : D \to \overline{\mathbb{R}}$ be an extended real valued function, $\upsilon \in D$ and $\Gamma(\upsilon)$ be finite. Then Γ is called:

(i) *an upper convexificator* $\partial^*\Gamma(\upsilon) \subseteq \mathbb{R}^n$ at $\upsilon \in D$, if and only if $\partial^*\Gamma(\upsilon)$ is closed and for every $\omega \in \mathbb{R}^n$, we have
$$\Gamma^-(\upsilon, \omega) \leq \sup_{\zeta \in \partial^*\Gamma(\upsilon)} \langle \zeta, \omega \rangle,$$
(ii) *a lower convexificator* $\partial_*\Gamma(\upsilon) \subseteq \mathbb{R}^n$ at $\upsilon \in D$, if and only if $\partial_*\Gamma(\upsilon)$ is closed and for every $\omega \in \mathbb{R}^n$, we have
$$\Gamma^+(\upsilon, \omega) \geq \inf_{\zeta \in \partial_*\Gamma(\upsilon)} \langle \zeta, \omega \rangle,$$
(iii) *a convexificator* $\partial^*_*\Gamma(\upsilon) \subseteq \mathbb{R}^n$ at $\upsilon \in D$, if and only if $\partial^*_*\Gamma(\upsilon)$ is both upper and lower convexificator of Γ at υ.
That is, for every $\omega \in \mathbb{R}^n$, we have
$$\Gamma^-(\upsilon, \omega) \leq \sup_{\zeta \in \partial^*_*\Gamma(\upsilon)} \langle \zeta, \omega \rangle,\; \Gamma^+(\upsilon, \omega) \geq \inf_{\zeta \in \partial^*_*\Gamma(\upsilon)} \langle \zeta, \omega \rangle.$$

Theorem 2.3. [11] *Suppose* $a, b \in D$ *and* $\Gamma : D \to \overline{\mathbb{R}}$ *be finite and continuous on* (a, b). *Suppose* $\partial^*_*\Gamma(\omega)$ *is a bounded convexificator for all* $\omega \in [a, b]$. *Then exists* $c \in (a, b)$ *such that*
$$\Gamma(b) - \Gamma(a) = \langle \zeta, b - a \rangle,\; \textit{for}\; \zeta \in co\partial^*_*\Gamma(c).$$

The notion of convexity for locally Lipschitz vector-valued functions using convexificators is defined as follows:

Definition 2.4. [14] Suppose $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p) : D \to \mathbb{R}^p$ be a vector-valued function such that $\Gamma_k : D \to \mathbb{R}$ is locally Lipschitz at $\omega \in D$ and admits a bounded convexificator $\partial^*_*\Gamma(\omega)$ at ω for all $k \in \ell = \{1, 2, ..., p\}$. Then Γ is called
:

(i) ∂_*^*-convex at $\omega \in D$ if

$$\Gamma(\upsilon) - \Gamma(\omega) \geqq \langle \zeta, \upsilon - \omega \rangle_p, \ \forall \ \upsilon \in D, \ \zeta \in \partial_*^*\Gamma(\omega),$$

(ii) strictly ∂_*^*-convex at $\omega \in D$ if

$$\Gamma(\upsilon) - \Gamma(\omega) > \langle \zeta, \upsilon - \omega \rangle_p, \ \forall \ \upsilon \in D, \ \zeta \in \partial_*^*\Gamma(\omega).$$

Theorem 2.5. [14] *Suppose $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p) : D \to \mathbb{R}^p$ be a vector-valued function such that $\Gamma_k : D \to \Re$ are locally Lipschitz functions on D and admit bounded convexificators $\partial_*^*\Gamma_k(\upsilon)$, for any $\upsilon \in D$, $\forall \ k \in \ell$. Then Γ is ∂_*^*-convex (strictly) on D if and only if $\partial_*^*\Gamma$ is monotone (strictly) on D.*

3. Interval-valued vector functions

First, we review several fundamental operations that can be performed at real intervals. For further information on interval analysis, we refer to [21, 22]. Let's denote the set of all closed intervals in $\mathbb{R}$ by $\Re$. Suppose $\mathbf{P} = [p^L, p^U]$, $\mathbf{Q} = [q^L, q^U] \in \Re$, then the sum and the product are defined by

$$\mathbf{P} + \mathbf{Q} = \{p + q : p \in \mathbf{P}, \ q \in \mathbf{Q}\} = [p^L + q^L, p^U + q^U],$$

$$\mathbf{P} \times \mathbf{Q} = \{pq : p \in \mathbf{P}, \ q \in \mathbf{Q}\} = [min \ Z, max \ Z],$$

where $Z = \{p^U q^U, p^U q^L, p^L q^U, p^L q^L\}$. It is important to note that any real number p can be interpreted as the closed interval $\mathbf{P}_p = [p, p]$, which means that the sum of $p + \mathbf{Q}$ is $\mathbf{P}_p + \mathbf{Q}$.

Based on the previous procedures, we can describe the product by multiplying an interval by a real number α as

$$\alpha\mathbf{P} = \{\alpha p : p \in \mathbf{P}\} = \begin{cases} [\alpha p^L, \alpha p^U], & \text{if } \alpha \geq 0, \\ \left[\alpha p^U, \alpha p^L\right], & \text{if } \alpha < 0. \end{cases}$$

Note that $-\mathbf{P} = \{-p : p \in \mathbf{P}\} = [-p^U, -p^L]$. Thus the difference between the two sets will be defined as

$$\mathbf{P} - \mathbf{Q} = \mathbf{P} + (-\mathbf{Q}) = [p^L - q^U, p^U - q^L].$$

For intervals, an order relation is defined as

(1) $\mathbf{P} \preceq_{LU} \mathbf{Q} \iff p^L \leq q^L$ and $p^U \leq q^U$,
(2) $\mathbf{P} \prec_{LU} \mathbf{Q} \iff \mathbf{P} \preceq_{LU} \mathbf{Q}$ and $\mathbf{P} \neq \mathbf{Q}$, that is one of following holds:
  (a) $p^U < q^U$ and $p^L < q^L$, or
  (b) $p^U < q^U$ and $p^L \leq q^L$, or
  (c) $p^U \leq q^U$ and $p^L < q^L$.

Remark 3.1. Suppose $\mathbf{P} = [p^L, p^U]$, $\mathbf{Q} = [q^L, q^U] \in \Re$, then $\mathbf{P}$ and $\mathbf{Q}$ are comparable if $\mathbf{P} \preceq_{LU} \mathbf{Q}$ or $\mathbf{P} \succeq_{LU} \mathbf{Q}$.

If any of the following is true, then $\mathbf{P}$ and $\mathbf{Q}$ cannot be compared to one another:

$$p^U > q^U \text{ and } p^L < q^L; \ p^U \geq q^U \text{ and } p^L < q^L; \ p^U > q^U \text{ and } p^L \leq q^L;$$

$$p^U < q^U \text{ and } p^L > q^L; \ p^U \leq q^U \text{ and } p^L > q^L; \ p^U < q^U \text{ and } p^L \geq q^L;$$

Suppose $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_n)$ be an interval-valued vector, where every component $\mathbf{P}_k = [c_k^L, c_k^U],\ k = 1, 2, ..., n$ is a closed interval. We take into consideration two interval-valued vectors denoted by $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_n)$ and $\mathbf{Q} = (\mathbf{Q}_1, \mathbf{Q}_2, ..., \mathbf{Q}_n)$ in such a way that $\mathbf{P}_k$ and $\mathbf{Q}_k$ are comparable for every k values ranging from 1 to n, then

(a) $\mathbf{P} \preceq_{LU} \mathbf{Q}$ if $\mathbf{P}_k \preceq_{LU} \mathbf{Q}_k$ for every $k = 1, 2, ..., n$,

(b) $\mathbf{P} \prec_{LU} \mathbf{Q}$ if $\mathbf{P}_k \preceq_{LU} \mathbf{Q}_k$ for every $k = 1, 2, ..., n$, and $\mathbf{P}_i \prec_{LU} \mathbf{Q}_i$ for at least one i.

A function $\Gamma : D \to \Re$ is called an interval-valued function if $\Gamma(\omega) = [\Gamma^L(\omega), \Gamma^U(\omega)]$, where Γ^L and Γ^U are real-valued functions defined on D satisfying $\Gamma^L(\omega) \leq \Gamma^U(\omega)$, for every $\omega \in D$. If $\Gamma_1, \Gamma_2, ..., \Gamma_p : D \to \Re$ are p interval-valued functions, then we refer to the function $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p) : D \to \Re^p$ an interval-valued vector function.

Definition 3.1. [29] Suppose $\Gamma = [\Gamma^L, \Gamma^U] : D \to \Re$ be an interval-valued function, then Γ is called locally Lipschitz at $\omega_0 \in D$ w.r.t. the Hausdorff metric if there exists $M > 0$ and $\delta > 0$ such that

$$d_H(\Gamma(\omega), \Gamma(\upsilon)) \leq M\|\omega - \upsilon\|,$$

where $d_H(\Gamma(\omega), \Gamma(\upsilon))$ is the Hausdorff metric between $\Gamma(\omega)$ and $\Gamma(\upsilon)$, defined by

$$d_H(\Gamma(\omega), \Gamma(\upsilon)) = max\{|\Gamma(\omega)^L - \Gamma(\upsilon)^L|, |\Gamma(\omega)^U - \Gamma(\upsilon)^U|\}.$$

If every $\omega_0 \in D$ is Lipschitz, then f is locally Lipschitz on D.

Proposition 3.2. [29] *Suppose* $\Gamma = [\Gamma^L, \Gamma^U] : D \to \Re$ *be a locally Lipschitz on* D*, then* Γ^L *and* Γ^U *both are locally Lipschitz on* D.

Definition 3.3. A function $\Gamma : D \to \Re$ is called ∂_*^*-LU-convex on D if the real valued functions Γ^L and Γ^U are ∂_*^*-convex on D.

Suppose $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p) : D \to \Re^p$ be an interval-valued vector function. Every component function $\Gamma_k = [\Gamma_k^L, \Gamma_k^U],\ k \in \ell = \{1, 2, ..., p\}$ is a locally Lipschitz interval-valued function defined on D. The nonsmooth interval-valued vector optimization problem (in short, $(IVOP)$) is defined as:

$$\text{Min } \{\Gamma(\omega) = (\Gamma_1(\omega), \Gamma_2(\omega), ..., \Gamma_p(\omega))\} \text{ such that } \omega \in D.$$

Definition 3.4. [29] A vector $\upsilon \in D$ is

(i) an LU-efficient solution of the $(IVOP)$ if $\exists$ no $\omega \in D$ such that $\Gamma(\omega) \prec_{LU} \Gamma(\upsilon)$
or equivalently

$$\Gamma_k(\omega) \preceq_{LU} \Gamma_k(\upsilon),\ \forall\ k \in \ell,\ k \neq j.$$

$$\Gamma_j(\omega) \prec_{LU} \Gamma_j(\upsilon),\ \text{for some } j \in \ell.$$

(ii) a weakly LU-efficient solution of the $(IVOP)$ if $\exists$ no $\omega \in D$ such that

$$\Gamma_k(\omega) \prec_{LU} \Gamma_k(\upsilon),\ \forall\ k \in \ell.$$

4. Interval Valued Minty and Stampacchia Vector Variational Inequalities in terms of convexificators

An interval-valued (VVI) problem of Minty type in terms of convexificators (for short, $(IMVVIP)$) for a nonsmooth case, is to find $\omega \in D$ such that the following inequality cannot hold

$$\begin{cases} \left\langle \zeta^L, \upsilon - \omega \right\rangle_p = \left(\left\langle \zeta_1^L, \upsilon - \omega \right\rangle, \left\langle \zeta_2^L, \upsilon - \omega \right\rangle, ..., \left\langle \zeta_p^L, \upsilon - \omega \right\rangle \right) \leqq 0, \\ \left\langle \zeta^U, \upsilon - \omega \right\rangle_p = \left(\left\langle \zeta_1^U, \upsilon - \omega \right\rangle, \left\langle \zeta_2^U, \upsilon - \omega \right\rangle, ..., \left\langle \zeta_p^U, \upsilon - \omega \right\rangle \right) \leqq 0, \end{cases}$$

for all $\upsilon \in D$ and all $\zeta_k^L \in \partial_*^*\Gamma_k(\upsilon),\ \zeta_k^U \in \partial_*^*\Gamma_k(\upsilon),\ k \in \ell$.

An interval-valued (VVI) problem of Stampacchia type in terms of convexificators (for short, $(ISVVIP)$) for a nonsmooth case, is to find $\omega \in D$ such that the following inequality cannot hold

$$\begin{cases} \left\langle \xi^L, \upsilon - \omega \right\rangle_p = \left(\left\langle \xi_1^L, \upsilon - \omega \right\rangle, \left\langle \xi_2^L, \upsilon - \omega \right\rangle, ..., \left\langle \xi_p^L, \upsilon - \omega \right\rangle \right) \leqq 0, \\ \left\langle \xi^U, \upsilon - \omega \right\rangle_p = \left(\left\langle \xi_1^U, \upsilon - \omega \right\rangle, \left\langle \xi_2^U, \upsilon - \omega \right\rangle, ..., \left\langle \xi_p^U, \upsilon - \omega \right\rangle \right) \leqq 0, \end{cases}$$

for all $\upsilon \in D$ and all $\xi_k^L \in \partial_*^*\Gamma_k(\omega),\ \xi_k^U \in \partial_*^*\Gamma_k(\omega),\ k \in \ell$.

We propose essential conditions, which are both necessary and sufficient for an effective solution to the $(IVOP)$.

Theorem 4.1. *Suppose $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p) : D \to \Re^p$ be an interval-valued vector function such that $\Gamma_k : D \to \Re$ are locally Lipschitz functions on D and admit bounded convexificators $\partial_*^*\Gamma_k(\upsilon)$ for any $\upsilon \in D$, $\forall\ k \in \ell$. Also, suppose that Γ is ∂_*^*-LU-convex on D. Then $\omega \in D$ is an LU-efficient solution of $(IVOP)$ if and only if ω is a solution of $(IMVVIP)$.*

Proof. Suppose that ω is not a solution of $(IMVVIP)$, then there exists $\upsilon \in D$, $\zeta_k^L \in \partial_*^*\Gamma_k(\upsilon),\ \zeta_k^U \in \partial_*^*\Gamma_k(\upsilon),\ k \in \ell$ such that

$$\begin{cases} \left\langle \zeta^L, \upsilon - \omega \right\rangle_p = \left(\left\langle \zeta_1^L, \upsilon - \omega \right\rangle, \left\langle \zeta_2^L, \upsilon - \omega \right\rangle, ..., \left\langle \zeta_p^L, \upsilon - \omega \right\rangle \right) \leqq 0, \\ \left\langle \zeta^U, \upsilon - \omega \right\rangle_p = \left(\left\langle \zeta_1^U, \upsilon - \omega \right\rangle, \left\langle \zeta_2^U, \upsilon - \omega \right\rangle, ..., \left\langle \zeta_p^U, \upsilon - \omega \right\rangle \right) \leqq 0. \end{cases} \quad (1)$$

Since each Γ_k is ∂_*^*-LU-convex. Therefore Γ_k^L and Γ_k^U are ∂_*^*-convex, so we have

$$\begin{cases} \Gamma_k^L(\upsilon) - \Gamma_k^L(\omega) \geqq \left\langle \zeta_k^L, \upsilon - \omega \right\rangle_p, \\ \Gamma_k^U(\upsilon) - \Gamma_k^U(\omega) \geqq \left\langle \zeta_k^U, \upsilon - \omega \right\rangle_p, \end{cases} \quad (2)$$

for all $\omega \in D$ and $k \in \ell$. From (1) and (2), there exists υ such that

$$\Gamma(\upsilon) \prec_{LU} \Gamma(\omega),$$

which is a contradiction.

Conversely, suppose that $\omega \in D$ is a solution of $(IMVVIP)$ but not an LU-efficient solution of $(IVOP)$. Then there exists $\upsilon \in D$ such that

$$\Gamma(\upsilon) \prec_{LU} \Gamma(\omega). \tag{3}$$

Using convexity of D, take $\mu(t) = \omega + t(\upsilon - \omega) \in D$ for any $t \in [0, 1]$. As Γ is ∂_*^*-LU-convex on D, by Proposition 2.5, $\forall\ t \in [0, 1]$, we have

$$\begin{cases} \Gamma^L(\omega + t(\upsilon - \omega)) - \Gamma^L(\omega) \leqq t[\Gamma^L(\upsilon) - \Gamma^L(\omega)], \\ \Gamma^U(\omega + t(\upsilon - \omega)) - \Gamma^U(\omega) \leqq t[\Gamma^U(\upsilon) - \Gamma^U(\omega)], \end{cases}$$

or equivalently, for every $k \in \ell$ and $t \in [0, 1]$, we have

$$\begin{cases} \Gamma_k^L(\omega + t(\upsilon - \omega)) - \Gamma_k^L(\omega) \leq t[\Gamma_k^L(\upsilon) - \Gamma_k^L(\omega)], \\ \Gamma_k^U(\omega + t(\upsilon - \omega)) - \Gamma_k^U(\omega) \leq t[\Gamma_k^U(\upsilon) - \Gamma_k^U(\omega)]. \end{cases}$$

By Mean value Theorem 2.3 on convexificators, for any $k \in \ell$, there exists $\bar{t_k} \in (0, t)$ and $\bar{\zeta_k^L} \in co\partial_*^*\Gamma_k(\mu(\bar{t_k}))$, $\bar{\zeta_k^U} \in co\partial_*^*\Gamma_k(\mu(\bar{t_k}))$ such that

$$\begin{cases} \left\langle \bar{\zeta_k^L}, t(\upsilon - \omega) \right\rangle = \Gamma_k^L(\omega + t(\upsilon - \omega)) - \Gamma_k^L(\omega), \\ \left\langle \bar{\zeta_k^U}, t(\upsilon - \omega) \right\rangle = \Gamma_k^U(\omega + t(\upsilon - \omega)) - \Gamma_k^U(\omega), \end{cases}$$

which implies that for any $k \in \ell$ and for some $\bar{\zeta_k^L} \in co\partial_*^*\Gamma_k(\mu(\bar{t_k}))$, $\bar{\zeta_k^U} \in co\partial_*^*\Gamma_k(\mu(\bar{t_k}))$, we have

$$\begin{cases} \left\langle \bar{\zeta_k^L}, \upsilon - \omega \right\rangle \leq \Gamma_k^L(\upsilon) - \Gamma_k^L(\omega), \\ \left\langle \bar{\zeta_k^U}, \upsilon - \omega \right\rangle \leq \Gamma_k^U(\upsilon) - \Gamma_k^U(\omega). \end{cases} \tag{4}$$

Suppose $\bar{t_1} = \bar{t_2} = ... = \bar{t_p} = \bar{t}$. Multiplying both side of (4) by $\bar{t}$, for $k \in \ell$ and $\bar{\zeta_k^L} \in co\partial_*^*\Gamma_k(\mu(\bar{t}))$, $\bar{\zeta_k^U} \in co\partial_*^*\Gamma_k(\mu(\bar{t}))$, we have

$$\begin{cases} \left\langle \bar{\zeta_k^L}, \mu(\bar{t}) - \omega \right\rangle \leq \bar{t}(\Gamma_k^L(\upsilon) - \Gamma_k^L(\omega)), \\ \left\langle \bar{\zeta_k^U}, \mu(\bar{t}) - \omega \right\rangle \leq \bar{t}(\Gamma_k^U(\upsilon) - \Gamma_k^U(\omega)). \end{cases} \tag{5}$$

Combining (3) and (5), we see that υ is not a solution of $(IMVVIP)$, which is a contradiction.

Consider the case when $\bar{t_1}, \bar{t_2}, ..., \bar{t_p}$ are not all equal. Suppose $\bar{t_1} \neq \bar{t_2}$. Then from (4), we have

$$\begin{cases} \left\langle \bar{\zeta_1^L}, \upsilon - \omega \right\rangle \leq \Gamma_1^L(\upsilon) - \Gamma_1^L(\omega), \\ \left\langle \bar{\zeta_1^U}, \upsilon - \omega \right\rangle \leq \Gamma_1^U(\upsilon) - \Gamma_1^U(\omega), \end{cases}$$

for some $\bar{\zeta_1^L} \in co\partial_*^*\Gamma_1(\mu(\bar{t_1}))$, $\bar{\zeta_1^U} \in co\partial_*^*\Gamma_1(\mu(\bar{t_1}))$ and

$$\begin{cases} \left\langle \bar{\zeta_2^L}, \upsilon - \omega \right\rangle \leq \Gamma_2^L(\upsilon) - \Gamma_2^L(\omega), \\ \left\langle \bar{\zeta_2^U}, \upsilon - \omega \right\rangle \leq \Gamma_2^U(\upsilon) - \Gamma_2^U(\omega), \end{cases}$$

for some $\bar{\zeta_2^L} \in co\partial_*^*\Gamma_2(\mu(\bar{t_2}))$, $\bar{\zeta_2^U} \in co\partial_*^*\Gamma_2(\mu(\bar{t_2}))$.
Since Γ_1 and Γ_2 are ∂_*^*-LU-convex. Therefore Γ_1^L and Γ_2^U are ∂_*^*-convex, so by Theorem 2.5, we have

$$\begin{cases} \left\langle \bar{\zeta_1^L} - \bar{\zeta_{12}^L}, \mu(\bar{t_1}) - \mu(\bar{t_2}) \right\rangle \geq 0, \ \forall \ \bar{\zeta_{12}^L} \in co\partial_*^*\Gamma_1(\mu(\bar{t_2})), \\ \left\langle \bar{\zeta_1^U} - \bar{\zeta_{12}^U}, \mu(\bar{t_1}) - \mu(\bar{t_2}) \right\rangle \geq 0, \ \forall \ \bar{\zeta_{12}^U} \in co\partial_*^*\Gamma_1(\mu(\bar{t_2})), \end{cases}$$

and

$$\begin{cases} \left\langle \bar{\zeta_2^L} - \bar{\zeta_{21}^L}, \mu(\bar{t_2}) - \mu(\bar{t_1}) \right\rangle \geq 0, \ \forall \ \bar{\zeta_{21}^L} \in co\partial_*^*\Gamma_2(\mu(\bar{t_1})), \\ \left\langle \bar{\zeta_2^U} - \bar{\zeta_{21}^U}, \mu(\bar{t_2}) - \mu(\bar{t_1}) \right\rangle \geq 0, \ \forall \ \bar{\zeta_{21}^U} \in co\partial_*^*\Gamma_2(\mu(\bar{t_1})). \end{cases}$$

If $\bar{t_1} - \bar{t_2} > 0$, then

$$\begin{cases} \left\langle \bar{\zeta_{12}^L}, \upsilon - \omega \right\rangle \leq \Gamma_1^L(\upsilon) - \Gamma_1^L(\omega), \\ \left\langle \bar{\zeta_{12}^U}, \upsilon - \omega \right\rangle \leq \Gamma_1^U(\upsilon) - \Gamma_1^U(\omega). \end{cases}$$

If $\bar{t_2} - \bar{t_1} > 0$, then

$$\begin{cases} \left\langle \bar{\zeta_{21}^L}, \upsilon - \omega \right\rangle \leq \Gamma_2^L(\upsilon) - \Gamma_2^L(\omega), \\ \left\langle \bar{\zeta_{21}^U}, \upsilon - \omega \right\rangle \leq \Gamma_2^U(\upsilon) - \Gamma_2^U(\omega). \end{cases}$$

For $\bar{t_1} \neq \bar{t_2}$, set $\bar{t} = min\{\bar{t_1}, \bar{t_2}\}$, there exists $\bar{\zeta_k^L} \in co\partial_*^*\Gamma_k^L(\mu(\bar{t}))$, $\bar{\zeta_k^L} \in co\partial_*^*\Gamma_k^U(\mu(\bar{t}))$, for any $k = 1, 2$ such that

$$\begin{cases} \left\langle \bar{\zeta_k^L}, \upsilon - \omega \right\rangle \leq \Gamma_k^L(\upsilon) - \Gamma_k^L(\omega), \\ \left\langle \bar{\zeta_k^U}, \upsilon - \omega \right\rangle \leq \Gamma_k^U(\upsilon) - \Gamma_k^U(\omega). \end{cases}$$

Continuing this process, we can find $\hat{t} \in (0, t)$ such that $\hat{t} = min\{\bar{t_1}, \bar{t_2}, ..., \bar{t_p}\}$ and $\hat{\zeta_k^L} \in co\partial_*^*\Gamma_k^L(\mu(\hat{t}))$, $\hat{\zeta_k^U} \in co\partial_*^*\Gamma_k^U(\mu(\hat{t}))$, $\forall \ k \in \ell$ such that

$$\begin{cases} \left\langle \hat{\zeta_k^L}, \upsilon - \omega \right\rangle \leq \Gamma_k^L(\upsilon) - \Gamma_k^L(\omega), \\ \left\langle \hat{\zeta_k^U}, \upsilon - \omega \right\rangle \leq \Gamma_k^U(\upsilon) - \Gamma_k^U(\omega). \end{cases}$$

Multiplying the above inequalities by $\hat{t}$, we have

$$\begin{cases} \left\langle \hat{\zeta_k^L}, \mu(\hat{t}) - \omega \right\rangle \leq \hat{t}(\Gamma_k^L(\upsilon) - \Gamma_k^L(\omega)), \\ \left\langle \hat{\zeta_k^U}, \mu(\hat{t}) - \omega \right\rangle \leq \hat{t}(\Gamma_k^U(\upsilon) - \Gamma_k^U(\omega)). \end{cases}$$

From (3), for some $\mu(\hat{t}) \in D$, we have

$$\begin{cases} \left\langle \hat{\zeta^L}, \mu(\hat{t}) - \omega \right\rangle_p \leqq 0, \\ \left\langle \hat{\zeta^U}, \mu(\hat{t}) - \omega \right\rangle_p \leqq 0, \end{cases}$$

which is a contradiction. □

Theorem 4.2. *Suppose $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p) : D \to \Re^p$ be an interval-valued vector function such that $\Gamma_k : D \to \Re$ are locally Lipschitz functions on D and admit bounded convexificators $\partial^*_*\Gamma_k(\upsilon)$ for any $\upsilon \in D$, $\forall\ k \in \ell$. Also, suppose that Γ is ∂^*_*-LU-convex on D. If $\omega \in D$ is a solution of $(ISVVIP)$, then ω is an LU-efficient solution of $(IVOP)$.*

Proof. Suppose $\omega \in D$ is a solution of $(ISVVIP)$ but not an LU-efficient solution of $(IVOP)$. Then there exists $\upsilon \in D$ such that

$$\Gamma(\upsilon) \prec_{LU} \Gamma(\omega). \tag{6}$$

Since Γ is ∂^*_*-LU-convex. Therefore Γ^L and Γ^U are ∂^*_*-convex, for any $\xi^L \in \partial^*_*\Gamma(\omega)$ and $\xi^U \in \partial^*_*\Gamma(\omega)$, there exists $\omega \in D$ such that

$$\begin{cases} \left\langle \xi^L, \upsilon - \omega \right\rangle_p \leqq 0, \\ \left\langle \xi^U, \upsilon - \omega \right\rangle_p \leqq 0, \end{cases}$$

which is a contradiction to the fact $\omega \in D$ is a solution of $(ISVVIP)$. □

Theorem 4.3. *Suppose $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p) : D \to \Re^p$ be an interval-valued vector function such that $\Gamma_k : D \to \Re$ are locally Lipschitz functions on D and admit bounded convexificators $\partial^*_*\Gamma_k(\upsilon)$ for any $\upsilon \in D$, $\forall\ k \in \ell$. Also, suppose that Γ is ∂^*_*-LU-convex on D. If $\omega \in D$ is a solution of $(ISVVIP)$, then ω is a solution of $(IMVVIP)$.*

Proof. Suppose $\omega \in D$ is a solution of $(ISVVIP)$, then for any $\upsilon \in D$, $\xi^L \in \partial^*_*\Gamma(\omega)$, $\xi^U \in \partial^*_*\Gamma(\omega)$ the following cannot hold

$$\begin{cases} \left\langle \xi^L, \upsilon - \omega \right\rangle_p \leqq 0, \\ \left\langle \xi^U, \upsilon - \omega \right\rangle_p \leqq 0. \end{cases}$$

Since Γ is ∂^*_*-LU-convex. Therefore Γ^L and Γ^U are ∂^*_*-convex, so by Theorem 2.5, $\partial^*_*\Gamma^L$ and $\partial^*_*\Gamma^U$ are monotone over D, which implies that, for any $\upsilon \in D$ and $\zeta^L \in \partial^*_*\Gamma(\upsilon)$, $\zeta^U \in \partial^*_*\Gamma(\upsilon)$ the following cannot hold

$$\begin{cases} \left\langle \zeta^L, \upsilon - \omega \right\rangle_p \leqq 0, \\ \left\langle \zeta^U, \upsilon - \omega \right\rangle_p \leqq 0. \end{cases}$$

Hence $\omega \in D$ is a solution of $(IMVVIP)$. □

4.1. Interval Valued Weak Vector Variational Inequalities in terms of convexificators.

An interval-valued weak (VVI) problem of Minty type in terms of convexificators (for short, $(IWMVVIP)$) for a nonsmooth case, is to find $\omega \in D$ such that the following inequality cannot hold

$$\begin{cases} \left\langle \zeta^L, \upsilon-\omega \right\rangle_p = \left(\left\langle \zeta_1^L, \upsilon-\omega \right\rangle, \left\langle \zeta_2^L, \upsilon-\omega \right\rangle, ..., \left\langle \zeta_p^L, \upsilon-\omega \right\rangle \right) < 0, \\ \left\langle \zeta^U, \upsilon-\omega \right\rangle_p = \left(\left\langle \zeta_1^U, \upsilon-\omega \right\rangle, \left\langle \zeta_2^U, \upsilon-\omega \right\rangle, ..., \left\langle \zeta_p^U, \upsilon-\omega \right\rangle \right) < 0, \end{cases}$$

for all $\upsilon \in D$ and all $\zeta_k^L \in \partial_*^*\Gamma_k(\upsilon),\ \zeta_k^U \in \partial_*^*\Gamma_k(\upsilon),\ k \in \ell$.

An interval-valued weak (VVI) problem of Stampacchia type in terms of convexificators (for short, $(IWSVVIP)$) for a nonsmooth case, is to find $\omega \in D$ such that the following inequality cannot hold

$$\begin{cases} \left\langle \xi^L, \upsilon-\omega \right\rangle_p = \left(\left\langle \xi_1^L, \upsilon-\omega \right\rangle, \left\langle \xi_2^L, \upsilon-\omega \right\rangle, ..., \left\langle \xi_p^L, \upsilon-\omega \right\rangle \right) < 0, \\ \left\langle \xi^U, \upsilon-\omega \right\rangle_p = \left(\left\langle \xi_1^U, \upsilon-\omega \right\rangle, \left\langle \xi_2^U, \upsilon-\omega \right\rangle, ..., \left\langle \xi_p^U, \upsilon-\omega \right\rangle \right) < 0, \end{cases}$$

for all $\upsilon \in D$ and all $\xi_k^L \in \partial_*^*\Gamma_k(\omega),\ \xi_k^U \in \partial_*^*\Gamma_k(\omega),\ k \in \ell$.

Theorem 4.4. *Suppose* $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p) : D \to \Re^p$ *be an interval-valued vector function such that* $\Gamma_k : D \to \Re$ *are locally Lipschitz functions on* D *and admit bounded convexificators* $\partial_*^*\Gamma_k(\upsilon)$ *for any* $\upsilon \in D,\ \forall\ k \in \ell$*. Also, suppose that* Γ *is* ∂_*^**-LU-convex on* D*. Then* $\omega \in D$ *is a weakly LU-efficient solution of* $(IVOP)$ *if and only if* ω *is a solution of* $(IWSVVIP)$*.*

Proof. Suppose that ω is a weakly LU-efficient solution of $(IVOP)$. Then $\exists$ no $\upsilon \in D$ such that

$$\Gamma_k(\upsilon) \prec_{LU} \Gamma_k(\omega),\ \forall\ k \in \ell.$$

Thus $\exists$ no $\upsilon \in D$ such that

$$\begin{cases} (\Gamma_1^L(\upsilon) - \Gamma_1^L(\omega), \Gamma_2^L(\upsilon) - \Gamma_2^L(\omega), ..., \Gamma_p^L(\upsilon) - \Gamma_p^L(\omega)) < 0, \\ (\Gamma_1^U(\upsilon) - \Gamma_1^U(\omega), \Gamma_2^U(\upsilon) - \Gamma_2^U(\omega), ..., \Gamma_p^U(\upsilon) - \Gamma_p^U(\omega)) < 0. \end{cases}$$

Using convexity of D, $\omega + t(\upsilon-\omega) \in D$, for any $t \in [0,1]$, which implies that

$$\frac{\Gamma(\omega + t(\upsilon-\omega))}{t} < 0,\ \text{for } t \in [0,1].$$

Taking limit inf as $t \to 0$, we have

$$\begin{cases} (\Gamma_1^{L-}(\omega, \upsilon-\omega), \Gamma_2^{L-}(\omega, \upsilon-\omega), ..., \Gamma_p^{L-}(\omega, \upsilon-\omega)) < 0, \\ (\Gamma_1^{U-}(\omega, \upsilon-\omega), \Gamma_2^{U-}(\omega, \upsilon-\omega), ..., \Gamma_p^{U-}(\omega, \upsilon-\omega)) < 0. \end{cases}$$

Since Γ_k admit bounded convexificators $\partial_*^*\Gamma_k(\upsilon)$, for any $k \in \ell$, there exists no $\upsilon \in D$ such that

$$\begin{cases} \left\langle \xi^L, \upsilon-\omega \right\rangle_p < 0, \text{for all } \xi_k^L \in \partial_*^*\Gamma_k(\omega), \\ \left\langle \xi^U, \upsilon-\omega \right\rangle_p < 0, \text{for all } \xi_k^U \in \partial_*^*\Gamma_k(\omega). \end{cases}$$

Hence υ is a solution of $(IWSVVIP)$.
Conversely, suppose that υ is a solution of $(IWSVVIP)$ but not a weakly LU-efficient solution of $(IVOP)$. Then there exists $\omega \in D$ such that

$$\begin{cases} (\Gamma_1^L(\upsilon) - \Gamma_1^L(\omega), \Gamma_2^L(\upsilon) - \Gamma_2^L(\omega), ..., \Gamma_p^L(\upsilon) - \Gamma_p^L(\omega)) < 0, \\ (\Gamma_1^U(\upsilon) - \Gamma_1^U(\omega), \Gamma_2^U(\upsilon) - \Gamma_2^U(\omega), ..., \Gamma_p^U(\upsilon) - \Gamma_p^U(\omega)) < 0. \end{cases}$$

Using ∂_*^*-LU-convex of Γ at υ, there exists $\omega \in D$ such that

$$\begin{cases} \left\langle \xi^L, \upsilon - \omega \right\rangle_p < 0, \text{ for all } \xi_k^L \in \partial_*^* \Gamma_k(\omega), \\ \left\langle \xi^U, \upsilon - \omega \right\rangle_p < 0, \text{ for all } \xi_k^U \in \partial_*^* \Gamma_k(\omega), \end{cases}$$

which is a contradiction. □

Theorem 4.5. *Suppose $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p) : D \to \Re^p$ be an interval-valued vector function such that $\Gamma_k : D \to \Re$ are locally Lipschitz functions on D and admit bounded convexificators $\partial_*^* \Gamma_k(\upsilon)$ for any $\upsilon \in D$, $\forall\ k \in \ell$. Also, suppose that Γ is ∂_*^*-LU-convex on D. Then $\omega \in D$ is a solution of $(IWMVVIP)$ if and only if ω is a solution of $(IWSVVIP)$.*

Proof. Suppose that ω is a solution of $(IWMVVIP)$. Consider any sequence $\{t_l\}$ with $t_l \in (0, 1]$ such that $t_l \to 0$ as $l \to \infty$. As D is convex, we have $\omega_l := \omega + t_l(\upsilon - \omega) \in D$, for all $\upsilon \in D$. Since ω is a solution of $(IWMVVIP)$, for $\xi_p^L \in \partial_*^* \Gamma_p(\omega_l)$, $\xi_p^U \in \partial_*^* \Gamma_p(\omega_l)$ there exists no $\upsilon \in D$ such that

$$\begin{cases} \left\langle \xi_p^L, \omega_l - \upsilon \right\rangle_p < 0, \\ \left\langle \xi_p^U, \omega_l - \upsilon \right\rangle_p < 0. \end{cases}$$

Since each Γ_i is locally Lipschitz and admits bounded convexificators on D, there exists $k > 0$ such that $\|\xi_{l_i}\| \leq k$, which means that the sequence ξ_{l_i} converges to ξ_i for all $i \in \ell$. Also the convexificators $\partial_*^* \Gamma_i(\upsilon)$ are closed for all $i \in \ell$ and $\upsilon \in D$, it follows that $\omega_l \to \omega$ and $\xi_{l_i} \to \bar{\xi_i}$ as $l \to \infty$ with $\bar{\xi_i} \in \partial_*^* \Gamma_i(\omega)$ for all $i \in \ell$. Thus for $\xi^L \in \partial_*^* \Gamma(\omega)$ and $\xi^U \in \partial_*^* \Gamma(\omega)$ there exists no $\upsilon \in D$ such that

$$\begin{cases} \left\langle \xi^L, \upsilon - \omega \right\rangle_p < 0, \\ \left\langle \xi^U, \upsilon - \omega \right\rangle_p < 0. \end{cases}$$

Hence ω is a solution of $(IWSVVIP)$.
Conversely, suppose that ω is a solution of $(IWSVVIP)$. Then, for any $\upsilon \in D$ and $\xi^L \in \partial_*^* \Gamma(\omega)$, $\xi^U \in \partial_*^* \Gamma(\omega)$ the following cannot hold

$$\begin{cases} \left\langle \xi^L, \upsilon - \omega \right\rangle_p < 0, \\ \left\langle \xi^U, \upsilon - \omega \right\rangle_p < 0. \end{cases}$$

Since Γ is ∂_*^*-LU-convex on D. Therefore Γ^L and Γ^U are ∂_*^*-convex, so by Theorem 2.5 $\partial_*^* \Gamma^L$ and $\partial_*^* \Gamma^U$ are monotone over D, which implies that for any

$\upsilon \in D$ and $\xi^L \in \partial_*^*\Gamma(\upsilon)$, $\xi^U \in \partial_*^*\Gamma(\upsilon)$ the following cannot hold

$$\begin{cases} \left\langle \xi^L, \upsilon - \omega \right\rangle_p < 0, \\ \left\langle \xi^U, \upsilon - \omega \right\rangle_p < 0. \end{cases}$$

Hence ω is a solution of $(IWMVVIP)$. □

Theorem 4.6. *Suppose* $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p) : D \to \Re^p$ *be an interval-valued vector function such that* $\Gamma_k : D \to \Re$ *are locally Lipschitz functions at* $\upsilon \in D$ *and admit bounded convexificators* $\partial_*^*\Gamma(\upsilon)$, $\forall\ k \in \ell$. *Also, suppose that* Γ *is strictly* ∂_*^**-LU-convex on* D. *Then* $\omega \in D$ *is an LU-efficient solution of* $(IVOP)$ *if and only if* ω *is a weakly LU-efficient solution of* $(IVOP)$.

Proof. Every LU-efficient solution is a weakly LU-efficient solution of $(IVOP)$. Conversely, suppose that ω is a weakly LU-efficient solution of $(IVOP)$, but not an LU-efficient solution of $(IVOP)$. Then there exists $\upsilon \in D$ such that

$$\Gamma(\upsilon) \prec_{LU} \Gamma(\omega).$$

Since Γ is strictly ∂_*^*-LU-convex on D. Therefore Γ^L and Γ^U are ∂_*^* strictly convex, so for any $\xi^L \in \partial_*^*\Gamma(\upsilon)$ and $\xi^U \in \partial_*^*\Gamma(\upsilon)$, there exists $\omega \in D$

$$\begin{cases} \left\langle \xi^L, \upsilon - \omega \right\rangle_p < 0, \\ \left\langle \xi^U, \upsilon - \omega \right\rangle_p < 0, \end{cases}$$

which is not a solution of $(IWMVVIP)$. By Theorem 4.4, ω is not a weakly LU-efficient solution of $(IVOP)$, which is a contradiction. □

5. Conclusion

In this study, we looked at a class of nonsmooth $(IVOP)$ and Stampacchia and Minty type (VVI) in terms of convexificators, which are a weaker version of the notion of subdifferentials. We developed relationships between Stampacchia and Minty type (VVI) and LU-efficient $(IVOP)$ solutions using LU-convexity. Also, we study the weak version of the $(IVVI)$ of the Stampacchia and Minty kind and determine the relationships between them and the weakly LU-efficient solution of the $(IVOP)$.

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