AN ASYMPTOTIC EXPANSION FOR THE FIRST DERIVATIVE OF THE HURWITZ-TYPE EULER ZETA FUNCTION†

MIN-SOO KIM

Abstract. The Hurwitz-type Euler zeta function \( \zeta_E(z, q) \) is defined by the series
\[
\zeta_E(z, q) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+q)^z},
\]
for \( \text{Re}(z) > 0 \) and \( q \neq 0, -1, -2, \ldots \), and it can be analytic continued to the whole complex plane. An asymptotic expansion for \( \zeta_E'(-m, q) \) has been proved based on the calculation of Hermite’s integral representation for \( \zeta_E(z, q) \).

AMS Mathematics Subject Classification : 11M35, 33F05.
Key words and phrases : Asymptotic expansions, alternating Hurwitz zeta functions, Hermite’s integral representation.

1. Introduction

Elizalde [10] gave an asymptotic expansion for the first derivative
\[
\zeta'(-n, q) = \frac{\partial}{\partial z} \zeta(z, q) \bigg|_{z=-n}, \quad n = 0, 1, 2, \ldots
\]
of the Hurwitz zeta function
\[
\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}, \quad \text{Re}(z) > 1, \ q \neq 0, -1, -2, \ldots
\]
in inverse powers of \( q \). The procedure employed is similar to the standard method: Watson’s Lemma and Laplace’s method. The Hurwitz zeta function \( \zeta(z, q) \) admits an analytic continuation to the entire complex plane except for the simple pole at \( z = 1 \), and the Riemann zeta function \( \zeta(z) \) is a special case of \( \zeta(z, q) \):
\[
\zeta(z, 1) = \zeta(z)
\]

†This work was supported by the Kyungnam University Foundation Grant, 2022.
© 2023 KSCAM.
The first order derivative of $\zeta(z, q)$ for $z$ has also been linked to some integrals involving cyclotomic polynomials and iterated logarithms in [2], polygamma functions of negative order in [3], the multiple gamma functions in [4, 5, 7], and a log-gamma integral in [8] and [11]. In [15], Seri obtained an asymptotic formula for higher derivatives of the Hurwitz zeta function $\zeta(z, q)$ with respect to its first argument as $\zeta^{(m)}(z, q) = \partial^m \zeta(z, q)/\partial z^m$.

The Hurwitz-type Euler zeta function (or, equivalently, the alternating Hurwitz zeta function) is defined by the series (see [17, p. 37, (2.2)] and [9, p. 514, (3.1)])

$$\zeta_E(z, q) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + q)^z},$$

where $\Re(z) > 0$ and $q \neq 0, -1, -2, \ldots$ (cf. [14, p. 308, (3.4)]). It can be analytic continue as an entire function in the complex plane. The Dirichlet eta function (or, the alternating Riemann zeta function) $\eta(z)$ is a special case of $\zeta_E(z, q)$:

$$\zeta_E(z, 1) = \eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^z},$$

where $\Re(z) > 0$ (see [9, p. 514, (3.2)]). As in [10], we denote by

$$\zeta'_E(z, q) \equiv \frac{\partial}{\partial z} \zeta_E(z, q).$$

In this note, we shall prove an asymptotic expansion for $\zeta'(-m, q)$ based on the following calculation of Hermite’s integral representation for $\zeta_E(z, q)$.

**Proposition 1.1** ([17, p. 38, Proposition 1]). For all $z$ and $\Re(q) > 0$

$$\zeta_E(z, q) = \frac{1}{2} q^{-z} + 2 \int_{0}^{\infty} \left( q^2 + t^2 \right)^{-z/2} \sin \left[ z \tan^{-1} \left( \frac{t}{q} \right) \right] e^{\pi t} dt \frac{e^{\pi t}}{e^{2\pi t} - 1}.$$

**Remark 1.1.** This expression exhibits the non-singularity structure of $\zeta_E(z, q)$ (and $J(z, q)$ in Williams and Zhang’s [17] notation) explicitly, since the integral inside can be analytic continued to all complex numbers $z \in \mathbb{C}$ due to it uniformly convergents for $|z| < R$ with $R > 0$.

**Proposition 1.2.** For $n = 0, 1, 2, \ldots$, we have an asymptotic expansion

$$\zeta_E(z, q) = \frac{1}{2} q^{-z} - \frac{1}{2} \sum_{k=0}^{n} \frac{E_{2k+1}(0) \Gamma(2k + z + 1)}{(2k+1)!\Gamma(z)} q^{-2k-z-1} + O \left( \frac{1}{q^{2n+z+3}} \right)$$

for $|q|$ tending to $\infty$, where the gamma function $\Gamma(z)$ is defined by the following Mellin integral

$$\Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} dt.$$
and $E_n(x)$ are the Euler polynomials defined by the generating function
\[
\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\] (6)

**Remark 1.2.** This formula was studied by Hu and Kim in [12] from a different method. In particular, Hu and Kim [12] has derived the asymptotic expansions for higher order derivatives \( \frac{\partial}{\partial z}^m \zeta_E(z, q) \), where \( |q| \to \infty \) and \( z \in \mathbb{C} \).

**Example 1.3.** For \( n = 0 \) and \( n = 1 \), Proposition 1.2 yields the asymptotic:
\[
\zeta_E(z, q) = \frac{1}{2} q^{z-1} + \frac{1}{4} z q^{z-1} - \frac{1}{12} z (z+1)(z+2) q^{z-3} + O\left(\frac{1}{q^{z+3}}\right).
\]
From this, we immediately get \( \zeta_E(0, q) = \frac{1}{2} \).

Now we using the similar method in [10, pp. 348–349, (6)–(17)]. By Proposition 1.1, we arrive to the following expression for the first derivative
\[
\zeta'_E(z, q) = -\frac{1}{2} q^{-z} \log q + I_{-z}(q),
\] (7)
where
\[
I_{-z}(q) = 2 \int_0^\infty (q^2 + t^2)^{-\frac{z}{2}} \cos \left( z \tan^{-1} \left( \frac{t}{q} \right) \right) \tan^{-1} \left( \frac{t}{q} \right) \frac{e^{\pi t} dt}{e^{2\pi t} - 1} - \int_0^\infty (q^2 + t^2)^{-\frac{z}{2}} \sin \left( z \tan^{-1} \left( \frac{t}{q} \right) \right) \log(q^2 + t^2) \frac{e^{\pi t} dt}{e^{2\pi t} - 1}
\] (8)
(cf. [10, (8)]).

**Theorem 1.4.** For \( m = 0, 1, 2, \ldots \), we have the following asymptotic expansion
\[
\zeta'_E(-m, q) \sim -\frac{1}{2} q^m \log q + \frac{1}{4} (1 + m \log q) q^{m-1} - \sum_{k=1}^{\infty} a_{2k}(m) q^{-(2k-m+1)},
\]
which is valid for large \( |q| \) and \( \arg q \leq \pi - \delta < \pi \) with any fixed \( 0 < \delta \leq \pi \), where the coefficients \( a_{2k}(m) \) are given by
\[
a_{2k}(m) = \begin{cases} 
\frac{1}{2} E_{2k+1}(0) \binom{m}{2k+1} \log q + \sum_{h=0}^{2k} \binom{m}{h} \frac{(-1)^h}{2^{k-h+1}}, & 2k \leq m - 1, \\
\frac{1}{2} E_{2k+1}(0) \sum_{h=0}^{m} \binom{m}{h} \frac{(-1)^h}{2^{k-h+1}}, & 2k \geq m.
\end{cases}
\] (9)

**Example 1.5.** The formula (9) gives a closed form evaluation of the coefficients \( a_{2k}(m) \) in terms of the numbers \( E_{2k+1}(0) \). For \( k = 1, 2, 3, \ldots \), the first few values
are
\[ a_{2k}(0) = \frac{1}{2} E_{2k+1}(0) \left( \frac{1}{2k+1} \right), \]
\[ a_{2k}(1) = \frac{1}{2} E_{2k+1}(0) \left( \frac{1}{2k+1} - \frac{1}{2k} \right), \]
\[ a_{2k}(2) = \frac{1}{2} E_{2k+1}(0) \left( \frac{1}{2k+1} - \frac{1}{2k} + \frac{1}{2k-1} \right). \]
Therefore we obtain
\[ \zeta'_{E}(0, q) \sim -\frac{1}{2} \log q + \frac{1}{4} q^{-1} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{E_{2k+1}(0)}{2k + 1} q^{-(2k+1)}, \tag{10} \]
\[ \zeta'_{E}(-1, q) \sim \frac{1}{4} + \frac{1}{4} \log q - \frac{1}{2} q \log q + \frac{1}{2} \sum_{k=1}^{\infty} \frac{E_{2k+1}(0)}{(2k+1)2k} q^{-2k}, \tag{11} \]
and
\[ \zeta'_{E}(-2, q) \sim \left( \frac{1}{2} \log q + \frac{1}{4} \right) q - \frac{1}{2} q^{2} \log q - \sum_{k=1}^{\infty} \frac{E_{2k+1}(0)}{(2k+1)2k(2k-1)2k-1} q^{-(2k-1)}. \tag{12} \]
In a similar way, we can derive the expansion for the coefficients \(a_{2k}(3)\):
\[ a_{2k}(3) = \begin{cases} \frac{1}{2} E_{2k+1}(0) \left( \frac{3}{2k+1} \right) \log q + \sum_{h=0}^{2k} \binom{3}{h} \frac{(-1)^{h}}{2k-h+1}, & 2k \leq 3, \\
\frac{3}{2} E_{2k+1}(0) \sum_{h=0}^{2k} \binom{3}{h} \frac{(-1)^{h}}{2k-h+1}, & 2k \geq 3. \end{cases} \tag{13} \]
Setting \(k = 1\), we have
\[ a_{2}(3) = \frac{1}{8} \log q + \frac{11}{48}, \tag{14} \]
since \(E_{3}(0) = \frac{1}{4}\). And for \(k = 2, 3, 4, \ldots\), we get
\[ a_{2k}(3) = -\frac{3E_{2k+1}(0)}{(2k+1)2k(2k-1)(2k-2)}. \tag{15} \]
Then substituting these two expressions into Theorem 1.4 with \(m = 3\), we get that
\[ \zeta'_{E}(-3, q) \sim -\left( \frac{11}{48} + \frac{1}{8} \log q \right) q + \frac{1}{4} \left( 1 + 3 \log q \right) q^{2} - \frac{1}{2} q^{3} \log q \]
\[ + \sum_{k=2}^{\infty} \frac{3E_{2k+1}(0)}{(2k+1)2k(2k-1)(2k-2)} q^{-(2k-2)}. \tag{16} \]
Similarly, we also have
\[ \zeta'_{E}(-4, q) \sim -\left( \frac{13}{24} \log q + \frac{1}{4} \left( 1 + 4 \log q \right) q^{3} - \frac{1}{2} q^{4} \log q \right. \]
\[ - \sum_{k=2}^{\infty} \frac{12E_{2k+1}(0)}{(2k+1)2k(2k-1)(2k-2)(2k-3)} q^{-(2k-3)}. \tag{17} \]
In what follows, we will use the usual convention that an empty sum is taken to be zero. Applying different methods with [17, p. 41, (3.8)], by Proposition 1.1 with $z = -m$ for $m = 0, 1, 2, \ldots$, we get the following proposition.

**Proposition 1.6.** For $m = 0, 1, 2, \ldots$, we have

$$\zeta_E(-m, q) = \frac{1}{2} q^m + \frac{1}{2} \sum_{k=0}^{|m-1|} \binom{m}{2k+1} q^{m-2k-1} E_{2k+1}(0),$$

where $E_k(x)$ denotes the $k$-th Euler polynomials and $\lfloor \cdot \rfloor$ denotes the floor function. This implies $\zeta_E(-m, q) = \frac{1}{2} E_m(q)$. In particular, when $m = 0$, we obtain $\zeta_E(0, q) = \frac{1}{2}$.

### 2. Main results

First, we go to the proof Proposition 1.2.

**Proof of Proposition 1.2.** We rewrite Proposition 1.1 in an equivalent form as (see [17, p. 40, (3.1)])

$$\zeta_E(z, q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{e^{(1-q)t} t^{z-1}}{1 + e^t} dt. \quad (18)$$

And by noticing that

$$\frac{1}{1 + e^t} = \frac{1}{2} \sum_{k=0}^N \frac{E_k(0)}{k!} t^k + O(t^{N+1}), \quad (19)$$

we get the following integral representation

$$\zeta_E(z, q + 1) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-qt} t^{z-1} \left( \frac{1}{1 + e^t} - \frac{1}{2} \sum_{k=0}^N \frac{E_k(0)}{k!} t^k \right) dt$$

$$+ \frac{1}{2\Gamma(z)} \sum_{k=0}^N \frac{E_k(0)}{k!} \int_0^\infty e^{-qt} t^{k+z-1} dt. \quad (20)$$

Since the term in the bracket equals to $O(t^{N+1})$ as $t \to 0$, the above integral yields a function which is analytic for $\text{Re}(z) > -N - 1$. Evaluating the second integral and making use of the functional equation for $\zeta_E(z, q)$

$$\zeta_E(z, q + 1) + \zeta_E(z, q) = \frac{1}{q^z}, \quad \text{Re}(q) > 0,$$

we obtain an asymptotic expansion at infinity

$$\zeta_E(z, q) = \frac{1}{2} q^{-z} - \frac{1}{2} \sum_{k=1}^N \frac{E_k(0) \Gamma(k + z)}{k! \Gamma(z)} q^{-k-z} + O\left( \frac{1}{q^{N+z+1}} \right), \quad (21)$$
where $\text{Re}(q) > 0$. By noticing that $E_0(0) = 1$ and $E_{2k}(0) = 0$ for $k = 1, 2, 3, \ldots$, (21) can also be written in an equivalent form
\[
\zeta_E(z, q) = \frac{1}{2} q^{-z} - \frac{1}{2} \sum_{k=0}^{n} E_{2k+1}(0) \frac{\Gamma(2k + z + 1)}{(2k + 1)! \Gamma(z)} q^{-2k-z-1} + O \left( \frac{1}{q^{2n+z+3}} \right),
\]
for $n = 0, 1, 2, \ldots$. This completes the proof. \qed

Proof of Theorem 1.4. The proof of Theorem 1.4 is based on the following two lemmas.

Lemma 2.1 ([10, p. 348, (9)]). We have
\[
\tan^{-1} \left( \frac{t}{q} \right) = \sum_{h=0}^{\infty} \frac{(-1)^h}{2h + 1} \left( \frac{t}{q} \right)^{2h+1},
\]
\[
\log(q^2 + t^2) = 2 \log q + \sum_{h=1}^{\infty} \frac{(-1)^{h-1}}{h} \left( \frac{t}{q} \right)^{2h}.
\]

Lemma 2.2. For all $m = 0, 1, 2, \ldots$, we have
\[
I_m(q) = \frac{1}{4} (1 + m \log q) q^{m-1} - \sum_{k=1}^{\infty} a_{2k}(m) q^{-(2k-m+1)},
\]
where $a_{2k}(m)$ is defined in (9).

Proof. From Euler’s formula, a nonzero complex number $z = q + it$ can be broken down into
\[
z^m = (q + it)^m = \left( \sqrt{q^2 + t^2} e^{i \tan^{-1} \left( \frac{t}{q} \right)} \right)^m
= (q^2 + t^2) \frac{m}{\pi} \left( \cos \left( m \tan^{-1} \left( \frac{t}{q} \right) \right) + i \sin \left( m \tan^{-1} \left( \frac{t}{q} \right) \right) \right),
\]
where $m \geq 0$. Thus, we may write
\[
\text{Re} \left( \left( \tan^{-1} \left( \frac{t}{q} \right) - \frac{i}{2} \log(q^2 + t^2) \right) (q + it)^m \right)
= (q^2 + t^2) \frac{m}{\pi} \cos \left( m \tan^{-1} \left( \frac{t}{q} \right) \right) \tan^{-1} \left( \frac{t}{q} \right)
+ \frac{1}{2} (q^2 + t^2) \frac{m}{\pi} \sin \left( m \tan^{-1} \left( \frac{t}{q} \right) \right) \log(q^2 + t^2).
\]
Recall Riemann’s integral (see [6, p. 251, Theorem 12.2])
\[
\Gamma(z) \zeta(z, q) = \int_{0}^{\infty} t^{z-1} e^{(1-q)t} \frac{dt}{e^t - 1}, \quad \text{Re}(z) > 1,
\]
An Asymptotic expansion for the first Derivative of the Hurwitz-type Euler zeta function 1415

which enables \( \zeta(z, q) \) to be analytically continued to the whole complex plane except for a simple pole at \( z = 1 \) with residue 1. If setting \( z = \ell + 1, q = \frac{1}{2} \) and letting \( t \to 2\pi t \) in (25), then we have

\[
\int_0^\infty \frac{t^\ell e^{\pi t}}{e^{2\pi t} - 1} dt = \int_0^\infty \left( \frac{\ell}{2\pi} \right)^{\ell+1} \frac{t^\ell e^{\pi t}}{e^{2\pi t} - 1} dt = \left( \frac{1}{2\pi} \right)^{\ell+1} \int_0^\infty \frac{t^\ell e^{\pi t}}{e^{2\pi t} - 1} dt = \left( \frac{1}{2\pi} \right)^{\ell+1} \Gamma(\ell + 1) \zeta(\ell + 1, \frac{1}{2}).
\]

In what follows, we shall prove asymptotic expansions for the integrals on the right hand side of (8). Firstly we immediately see from (8) with \( z = -m \) and (24) that

\[
I_m(q) = 2\text{Re} \int_0^\infty \left( \tan^{-1} \left( \frac{t}{q} \right) - \frac{i}{2} \log(q^2 + t^2) \right) (q + it)^m \frac{e^{\pi t} dt}{e^{2\pi t} - 1}. \tag{27}
\]

And by using Lemma 2.1 and expanding \((q + it)^m\) by the binomials, (27) becomes to

\[
I_m(q) = 2\text{Re} \int_0^\infty \left( \sum_{k=0}^m \sum_{h=0}^{\infty} \frac{(-1)^h q^{-2h-1}}{2h + 1} \binom{m}{k} q^{m-k} i^k k+2h+1 \right.
\]

\[
- \sum_{k=0}^m \binom{m}{k} q^{m-k} i^k k+1 \log q

\]

\[
- \frac{1}{2} \sum_{k=0}^m \sum_{h=0}^{\infty} \binom{m}{k} \frac{(-1)^{h-1} i^{k+1}}{h} q^{m-k-2h} i^{k+2h} \binom{m}{k} q^{m-k} i^k \left( \frac{1}{2\pi} \right)^{k+2h+2} \Gamma(k + 2h + 2) \zeta(k + 2h + 1, \frac{1}{2})

\]

\[
- \sum_{k=0}^m \binom{m}{k} q^{m-k} i^k k+1 \left( \frac{1}{2\pi} \right)^{k+1} \Gamma(k + 1) \zeta(k + 1, \frac{1}{2}) \log q

\]

\[
- \frac{1}{2} \sum_{k=0}^m \sum_{h=0}^{\infty} \binom{m}{k} \frac{(-1)^{h-1} i^{k+1}}{h} q^{m-k-2h} \left( \frac{1}{2\pi} \right)^{k+2h+1} \Gamma(k + 2h + 2) \zeta(k + 2h + 1, \frac{1}{2})

\]
in which the last equality follows from (26). Then by applying an elementary calculation procedure for the expression on the right side of (28), we have

\[
I_m(q) = 2 \sum_{k=0}^{m} \sum_{h=0}^{\infty} \binom{m}{k} (-1)^{h+k} q^{m-k-2h-1} \frac{1}{2h+1} \frac{q^{-k}}{(2\pi)^{2h+k+2}} (2h + k + 1)! \zeta(2h + k + 2, \frac{1}{2})
\]

\[
- 2 \sum_{k=0}^{m} \binom{m}{k} \frac{q^{m-k} \log q}{k!} \zeta(k + 1, \frac{1}{2})
\]

\[
- \sum_{k=0}^{m} \sum_{h=1}^{\infty} \binom{m}{k} (-1)^{k+1} q^{m-k-2h} \frac{1}{h} \frac{q^{-k}}{(2\pi)^{2h+k+1}} (2h + k + 1)! \zeta(2h + k + 1, \frac{1}{2})
\].

Now recall the following Euler’s formula for \(\zeta(2k)\) (see [6, p. 266, Theorem 12.17])

\[
\zeta\left(2k, \frac{1}{2}\right) = (2^{2k} - 1)\zeta(2k)
\]

\[
= (2^{2k} - 1)(-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}
\]

\[
= (-1)^{k+1} \frac{2k(2\pi)^{2k}}{2(2k)!} E_{2k-1}(0),
\]

where \(k = 1, 2, 3, \ldots\), and the third equality following from [16, Corollary 3.2]

\[
(2^m - 1)B_m = -\frac{m}{2} E_{m-1}(0), \quad m = 1, 2, 3, \ldots
\]

Then combining (29) with (30), we obtain

\[
I_m(q) = -\frac{1}{2} \sum_{k=0}^{m} \sum_{h=0}^{\infty} \binom{m}{k} \frac{1}{2h+1} E_{2h+k+1}(0) q^{-(2h+k-m-1)}
\]

\[
- \frac{1}{2} \sum_{k=0}^{m} \binom{m}{k} E_k(0) q^{-(k-m)} \log q
\]

\[
+ \frac{1}{2} \sum_{k=0}^{m} \sum_{h=1}^{\infty} \binom{m}{k} \frac{1}{2h} E_{2h+k}(0) q^{-(2h+k-m)}
\].

Finally, after some calculation, (32) becomes to

\[
I_m(q) = \frac{1}{4} (1 + m \log q) q^{-(1-m)} - \sum_{k=1}^{\infty} a_{2k}(m) q^{-(2k-m+1)},
\]

(33)
where the coefficients \(a_{2k}(m)\) are given by

\[
a_{2k}(m) = \begin{cases} \frac{1}{2} E_{2k+1}(0) \left( \frac{m}{2k+1} \log q + \sum_{h=0}^{2k} \binom{m}{h} \frac{(-1)^h}{2k-h+1} \right), & 2k \leq m - 1, \\ \frac{1}{2} E_{2k+1}(0) \sum_{h=0}^{m} \binom{m}{h} \frac{(-1)^h}{2k-h+1}, & 2k \geq m, \end{cases}
\]

and \(E_n(x)\) are the Euler polynomials. This completes the proof. \(\square\)

Finally, by (7) with \(z = -m\) and Lemma 2.2, we get Theorem 1.4.

**Proof of Proposition 1.6.** For \(m = 0, 1, 2, \ldots\), if setting \(z = -m\) in Proposition 1.1 and use (23), then we easily get that

\[
\zeta_E(-m, q) = \frac{1}{2} q^m - 2 \text{Im} \int_0^\infty (q + it)^m \frac{e^{\pi t} dt}{e^{2\pi t} - 1}. \tag{34}
\]

On the right hand side of (34), an application of the binomial identity yields

\[
2 \text{Im} \int_0^\infty (q + it)^m \frac{e^{\pi t} dt}{e^{2\pi t} - 1} = 2 \sum_{k=0}^{m} \binom{m}{k} q^{m-k} (-1)^{k+1} \int_0^\infty t^k \frac{e^{\pi t} dt}{e^{2\pi t} - 1}. \tag{35}
\]

Combining (26), (30), (31), (34), and (35), we obtain

\[
\zeta_E(-m, q) = \frac{1}{2} q^m - 2 \sum_{k=0}^{m} \binom{m}{k} q^{m-k} (-1)^{k+1} \left( \frac{1}{2\pi} \right)^{k+1} \\
\times k!(2k+1)\zeta(k+1) \\
= \frac{1}{2} q^m - \sum_{k=0}^{\left\lfloor \frac{m-1}{2k+1} \right\rfloor} \binom{m}{2k+1} q^{m-2k-1} (2^{2k+2} - 1) B_{2k+2} \frac{2k+2}{2k+1} \tag{36}
\]

\[
= \frac{1}{2} q^m + \frac{1}{2} \sum_{k=0}^{\left\lfloor \frac{m-1}{2k+1} \right\rfloor} \binom{m}{2k+1} q^{m-2k-1} E_{2k+1}(0),
\]

where \(\left\lfloor \cdot \right\rfloor\) denotes the floor function, and the proposition by noticing \(E_{2k}(0) = 0\) for \(k = 1, 2, 3, \ldots\).

**Conflicts of interest**: The author declares no conflict of interest.
References


Min-Soo Kim received Ph.D. degree from Kyungnam University. His main research area is analytic number theory. Recently, his main interests focus on $p$-adic analysis and zeta functions, Bernoulli and Euler numbers and polynomials.

Department of Mathematics Education, Kyungnam University, Changwon, Gyeongnam 51767, Republic of Korea.

e-mail: mskim@kyungnam.ac.kr