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AN ASYMPTOTIC EXPANSION FOR THE FIRST DERIVATIVE OF THE HURWITZ-TYPE EULER ZETA FUNCTION†

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ABSTRACT. The Hurwitz-type Euler zeta function $\zeta_E(z,q)$ is defined by the series

$$\zeta_E(z,q)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+q)^z},$$

for $\mathrm{Re}(z)>0$ and $q\neq 0,-1,-2,\ldots$, and it can be analytic continued to the whole complex plane. An asymptotic expansion for $\zeta_E'(-m,q)$ has been proved based on the calculation of Hermite's integral representation for $\zeta_E(z,q)$.

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1. Introduction

Elizalde [10] gave an asymptotic expansion for the first derivative

$$\zeta'(-n,q)\equiv\frac{\partial}{\partial z}\zeta(z,q)\bigg|_{z=-n},\quad n=0,1,2,\ldots,\tag{1}$$

of the Hurwitz zeta function

$$\zeta(z,q)=\sum_{n=0}^{\infty}\frac{1}{(n+q)^z},\quad \mathrm{Re}(z)>1,\ q\neq 0,-1,-2,\ldots\tag{2}$$

in inverse powers of q. The procedure employed is similar to the standard method: Watson's Lemma and Laplace's method. The Hurwitz zeta function $\zeta(z,q)$ admits an analytic continuation to the entire complex plane except for the simple pole at $z=1$, and the Riemann zeta function $\zeta(z)$ is a special case of $\zeta(z,q)$:

$$\zeta(z,1)=\zeta(z)$$

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(see [6, 10, 17]).

The first order derivative of $\zeta(z,q)$ for z has also been linked to some integrals involving cyclotomic polynomials and iterated logarithms in [2], polygamma functions of negative order in [3], the multiple gamma functions in [4, 5, 7], and a log-gamma integral in [8] and [11]. In [15], Seri obtained an asymptotic formula for higher derivatives of the Hurwitz zeta function $\zeta(z,q)$ with respect to its first argument as $\zeta^{(m)}(z,q)=\partial^m\zeta(z,q)/\partial z^m$.

The Hurwitz-type Euler zeta function (or, equivalently, the alternating Hurwitz zeta function) is defined by the series (see [17, p. 37, (2.2)] and [9, p. 514, (3.1)])

$$\zeta_E(z,q)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+q)^z}, \tag{3}$$

where $\mathrm{Re}(z)>0$ and $q\neq 0,-1,-2,\dots$ (cf. [14, p. 308, (3.4)]). It can be analytic continue as an entire function in the complex plane. The Dirichlet eta function (or, the alternating Riemann zeta function) $\eta(z)$ is a special case of $\zeta_E(z,q)$:

$$\zeta_E(z,1)=\eta(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^z}, \tag{4}$$

where $\mathrm{Re}(z)>0$ (see [9, p. 514, (3.2)]). As in [10], we denote by

$$\zeta_E'(z,q)\equiv\frac{\partial}{\partial z}\zeta_E(z,q). \tag{5}$$

In this note, we shall prove an asymptotic expansion for $\zeta'(-m,q)$ based on the following calculation of Hermite's integral representation for $\zeta_E(z,q)$.

Proposition 1.1 ([17, p. 38, Proposition 1])**.** *For all z and* $\mathrm{Re}(q)>0$

$$\zeta_E(z,q)=\frac{1}{2}q^{-z}+2\int_0^{\infty}(q^2+t^2)^{-z/2}\sin\left[z\tan^{-1}\left(\frac{t}{q}\right)\right]\frac{e^{\pi t}dt}{e^{2\pi t}-1}.$$

Remark 1.1. This expression exhibits the non-singularity structure of $\zeta_E(z,q)$ (and $J(z,q)$ in Williams and Zhang's [17] notation) explicitly, since the integral inside can be analytic continued to all complex numbers $z\in\mathbb{C}$ due to it uniformly convergents for $|z|<R$ with $R>0$.

Proposition 1.2. *For $n=0,1,2,\dots,$ we have an asymptotic expansion*

$$\zeta_E(z,q)=\frac{1}{2}q^{-z}-\frac{1}{2}\sum_{k=0}^{n}\frac{E_{2k+1}(0)\Gamma(2k+z+1)}{(2k+1)!\Gamma(z)}q^{-2k-z-1}+O\left(\frac{1}{q^{2n+z+3}}\right)$$

for $|q|$ tending to ∞, where the gamma function $\Gamma(z)$ is defined by the following Mellin integral

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt$$

and $E_n(x)$ are the Euler polynomials defined by the generating function

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{6}$$

Remark 1.2. This formula was studied by Hu and Kim in [12] from a different method. In particular, Hu and Kim [12] has derived the asymptotic expansions for higher order derivatives $\left(\frac{\partial}{\partial z}\right)^m\zeta_E(z,q)$, where $|q|\to\infty$ and $z\in\mathbb{C}$.

Example 1.3. For $n=0$ and $n=1$, Proposition 1.2 yields the asymptotic:

$$\zeta_E(z,q)=\frac{1}{2}q^{-z}+\frac{1}{4}zq^{-z-1}+O\left(\frac{1}{q^{z+3}}\right),$$
$$\zeta_E(z,q)=\frac{1}{2}q^{-z}+\frac{1}{4}zq^{-z-1}-\frac{1}{48}z(z+1)(z+2)q^{-z-3}+O\left(\frac{1}{q^{z+5}}\right).$$

From this, we immediately get $\zeta_E(0,q)=\frac{1}{2}$.

Now we using the similar method in [10, pp. 348–349, (6)–(17)].

By Proposition 1.1, we arrive to the following expression for the first derivative

$$\zeta_E'(z,q)=-\frac{1}{2}q^{-z}\log q+I_{-z}(q), \tag{7}$$

where

$$\begin{aligned}I_{-z}(q)=2\int_0^\infty(q^2+t^2)^{-\frac{z}{2}}\cos\left(z\tan^{-1}\left(\frac{t}{q}\right)\right)\tan^{-1}\left(\frac{t}{q}\right)\frac{e^{\pi t}dt}{e^{2\pi t}-1}\\-\int_0^\infty(q^2+t^2)^{-\frac{z}{2}}\sin\left(z\tan^{-1}\left(\frac{t}{q}\right)\right)\log(q^2+t^2)\frac{e^{\pi t}dt}{e^{2\pi t}-1}\end{aligned} \tag{8}$$

(cf. [10, (8)]).

Theorem 1.4. *For $m=0,1,2,\ldots$, we have the following asymptotic expansion*

$$\zeta_E'(-m,q)\sim-\frac{1}{2}q^m\log q+\frac{1}{4}(1+m\log q)q^{m-1}-\sum_{k=1}^{\infty}a_{2k}(m)q^{-(2k-m+1)},$$

which is valid for large $|q|$ and $|\arg q|\le\pi-\delta<\pi$ with any fixed $0<\delta\le\pi$, where the coefficients $a_{2k}(m)$ are given by

$$a_{2k}(m)=\begin{cases}\frac{1}{2}E_{2k+1}(0)\left(\binom{m}{2k+1}\log q+\sum_{h=0}^{2k}\binom{m}{h}\frac{(-1)^h}{2k-h+1}\right), & 2k\le m-1,\\ \frac{1}{2}E_{2k+1}(0)\sum_{h=0}^{m}\binom{m}{h}\frac{(-1)^h}{2k-h+1}, & 2k\ge m.\end{cases} \tag{9}$$

Example 1.5. The formula (9) gives a closed form evaluation of the coefficients $a_{2k}(m)$ in terms of the numbers $E_{2k+1}(0)$. For $k=1,2,3,\ldots$, the first few values are

$$a_{2k}(0) = \frac{1}{2}E_{2k+1}(0)\frac{1}{2k+1},$$
$$a_{2k}(1) = \frac{1}{2}E_{2k+1}(0)\left(\frac{1}{2k+1} - \frac{1}{2k}\right),$$
$$a_{2k}(2) = \frac{1}{2}E_{2k+1}(0)\left(\frac{1}{2k+1} - 2\frac{1}{2k} + \frac{1}{2k-1}\right).$$

Therefore we obtain

$$\zeta_E'(0,q) \sim -\frac{1}{2}\log q + \frac{1}{4}q^{-1} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{E_{2k+1}(0)}{2k+1}q^{-(2k+1)}, \tag{10}$$

$$\zeta_E'(-1,q) \sim \frac{1}{4} + \frac{1}{4}\log q - \frac{1}{2}q\log q + \frac{1}{2}\sum_{k=1}^{\infty}\frac{E_{2k+1}(0)}{(2k+1)2k}q^{-2k}, \tag{11}$$

and

$$\zeta_E'(-2,q) \sim \left(\frac{1}{2}\log q + \frac{1}{4}\right)q - \frac{1}{2}q^2\log q - \sum_{k=1}^{\infty}\frac{E_{2k+1}(0)}{(2k+1)2k(2k-1)}q^{-(2k-1)}. \tag{12}$$

In a similar way, we can derive the expansion for the coefficients $a_{2k}(3)$:

$$a_{2k}(3) = \begin{cases} \frac{1}{2}E_{2k+1}(0)\left(\binom{3}{2k+1}\log q + \sum_{h=0}^{2k}\binom{3}{h}\frac{(-1)^h}{2k-h+1}\right), & 2k \le 2, \\ \frac{1}{2}E_{2k+1}(0)\sum_{h=0}^{3}\binom{3}{h}\frac{(-1)^h}{2k-h+1}, & 2k \ge 3. \end{cases} \tag{13}$$

Setting $k = 1$, we have

$$a_2(3) = \frac{1}{8}\log q + \frac{11}{48}, \tag{14}$$

since $E_3(0) = \frac{1}{4}$. And for $k = 2, 3, 4, \ldots$, we get

$$a_{2k}(3) = -\frac{3E_{2k+1}(0)}{(2k+1)2k(2k-1)(2k-2)}. \tag{15}$$

Then substituting these two expressions into Theorem 1.4 with $m = 3$, we get that

$$\begin{aligned}\zeta_E'(-3,q) \sim & -\left(\frac{11}{48} + \frac{1}{8}\log q\right) + \frac{1}{4}(1 + 3\log q)q^2 - \frac{1}{2}q^3\log q \\ & + \sum_{k=2}^{\infty}\frac{3E_{2k+1}(0)}{(2k+1)2k(2k-1)(2k-2)}q^{-(2k-2)}.\end{aligned} \tag{16}$$

Similarly, we also have

$$\begin{aligned}\zeta_E'(-4,q) \sim & -\left(\frac{13}{24} + \frac{1}{2}\log q\right)q + \frac{1}{4}(1 + 4\log q)q^3 - \frac{1}{2}q^4\log q \\ & - \sum_{k=2}^{\infty}\frac{12E_{2k+1}(0)}{(2k+1)2k(2k-1)(2k-2)(2k-3)}q^{-(2k-3)}.\end{aligned} \tag{17}$$

In what follows, we will use the usual convention that an empty sum is taken to be zero. Applying different methods with [17, p. 41, (3.8)], by Proposition 1.1 with $z=-m$ for $m=0,1,2,\ldots$, we get the following proposition.

Proposition 1.6. *For $m=0,1,2,\ldots$, we have*

$$\zeta_E(-m,q)=\frac{1}{2}q^m+\frac{1}{2}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m}{2k+1}q^{m-2k-1}E_{2k+1}(0),$$

where $E_k(x)$ denotes the k-th Euler polynomials and $\lfloor\cdot\rfloor$ denotes the floor function. This implies $\zeta_E(-m,q)=\frac{1}{2}E_m(q)$. In particular, when $m=0$, we obtain $\zeta_E(0,q)=\frac{1}{2}$.

2. Main results

First, we go to the proof Proposition 1.2.

Proof of Proposition 1.2. We rewrite Proposition 1.1 in an equivalent form as (see [17, p. 40, (3.1)])

$$\zeta_E(z,q)=\frac{1}{\Gamma(z)}\int_0^\infty\frac{e^{(1-q)t}t^{z-1}}{1+e^t}dt. \tag{18}$$

And by noticing that

$$\frac{1}{1+e^t}=\frac{1}{2}\sum_{k=0}^{N}\frac{E_k(0)}{k!}t^k+O(t^{N+1}), \tag{19}$$

we get the following integral representation

$$\begin{aligned}\zeta_E(z,q+1)=&\frac{1}{\Gamma(z)}\int_0^\infty e^{-qt}t^{z-1}\left(\frac{1}{1+e^t}-\frac{1}{2}\sum_{k=0}^{N}\frac{E_k(0)}{k!}t^k\right)dt\\&+\frac{1}{2\Gamma(z)}\sum_{k=0}^{N}\frac{E_k(0)}{k!}\int_0^\infty e^{-qt}t^{k+z-1}dt.\end{aligned} \tag{20}$$

Since the term in the bracket equals to $O(t^{N+1})$ as $t\to 0$, the above integral yields a function which is analytic for $\mathrm{Re}(z)>-N-1$. Evaluating the second integral and making use of the functional equation for $\zeta_E(z,q)$

$$\zeta_E(z,q+1)+\zeta_E(z,q)=\frac{1}{q^z},\quad \mathrm{Re}(q)>0,$$

we obtain an asymptotic expansion at infinity

$$\zeta_E(z,q)=\frac{1}{2}q^{-z}-\frac{1}{2}\sum_{k=1}^{N}\frac{E_k(0)\Gamma(k+z)}{k!\Gamma(z)}q^{-k-z}+O\left(\frac{1}{q^{N+z+1}}\right), \tag{21}$$

where $\mathrm{Re}(q) > 0$. By noticing that $E_0(0) = 1$ and $E_{2k}(0) = 0$ for $k = 1, 2, 3, \ldots$, (21) can also be written in an equivalent form

$$\zeta_E(z,q) = \frac{1}{2}q^{-z} - \frac{1}{2}\sum_{k=0}^{n} \frac{E_{2k+1}(0)\Gamma(2k+z+1)}{(2k+1)!\Gamma(z)} q^{-2k-z-1} + O\left(\frac{1}{q^{2n+z+3}}\right), \quad (22)$$

for $n = 0, 1, 2, \ldots$. This completes the proof. $\square$

Proof of Theorem 1.4. The proof of Theorem 1.4 is based on the following two lemmas.

Lemma 2.1 ([10, p. 348, (9)])**.** *We have*

$$\tan^{-1}\left(\frac{t}{q}\right) = \sum_{h=0}^{\infty} \frac{(-1)^h}{2h+1}\left(\frac{t}{q}\right)^{2h+1},$$

$$\log(q^2+t^2) = 2\log q + \sum_{h=1}^{\infty} \frac{(-1)^{h-1}}{h}\left(\frac{t}{q}\right)^{2h}.$$

Lemma 2.2. *For all $m = 0, 1, 2, \ldots,$ we have*

$$I_m(q) = \frac{1}{4}(1 + m\log q)q^{m-1} - \sum_{k=1}^{\infty} a_{2k}(m) q^{-(2k-m+1)},$$

where $a_{2k}(m)$ is defined in (9).

Proof. From Euler's formula, a nonzero complex number $z = q+it$ can be broken down into

$$\begin{aligned} z^m &= (q+it)^m \\ &= \left(\sqrt{q^2+t^2}e^{i\tan^{-1}\left(\frac{t}{q}\right)}\right)^m \\ &= (q^2+t^2)^{\frac{m}{2}}\left(\cos\left(m\tan^{-1}\left(\frac{t}{q}\right)\right) + i\sin\left(m\tan^{-1}\left(\frac{t}{q}\right)\right)\right), \end{aligned} \quad (23)$$

where $m \geq 0$. Thus, we may write

$$\begin{aligned} \mathrm{Re}&\left(\left(\tan^{-1}\left(\frac{t}{q}\right) - \frac{i}{2}\log(q^2+t^2)\right)(q+it)^m\right) \\ &= (q^2+t^2)^{\frac{m}{2}}\cos\left(m\tan^{-1}\left(\frac{t}{q}\right)\right)\tan^{-1}\left(\frac{t}{q}\right) \\ &\quad + \frac{1}{2}(q^2+t^2)^{\frac{m}{2}}\sin\left(m\tan^{-1}\left(\frac{t}{q}\right)\right)\log(q^2+t^2). \end{aligned} \quad (24)$$

Recall Riemann's integral (see [6, p. 251, Theorem 12.2])

$$\Gamma(z)\zeta(z,q) = \int_0^\infty \frac{t^{z-1}e^{(1-q)t}}{e^t-1}dt, \quad \mathrm{Re}(z) > 1, \quad (25)$$

which enables $\zeta(z,q)$ to be analytically continued to the whole complex plane except for a simple pole at $z=1$ with residue 1. If setting $z=\ell+1, q=\frac{1}{2}$ and letting $t\to 2\pi t$ in (25), then we have

$$\begin{aligned}\int_0^\infty \frac{t^\ell e^{\pi t}}{e^{2\pi t}-1}dt &= \int_0^\infty \frac{\left(\frac{t}{2\pi}\right)^\ell e^{\frac{t}{2}}}{e^t-1}\left(\frac{1}{2\pi}\right)dt\\ &= \left(\frac{1}{2\pi}\right)^{\ell+1}\int_0^\infty \frac{t^\ell e^{\frac{t}{2}}}{e^t-1}dt \\ &= \left(\frac{1}{2\pi}\right)^{\ell+1}\Gamma(\ell+1)\zeta\left(\ell+1,\frac{1}{2}\right).\end{aligned} \tag{26}$$

In what follows, we shall prove asymptotic expansions for the integrals on the right hand side of (8). Firstly we immediately see from (8) with $z=-m$ and (24) that

$$I_m(q) = 2\mathrm{Re}\int_0^\infty \left(\tan^{-1}\left(\frac{t}{q}\right)-\frac{i}{2}\log(q^2+t^2)\right)(q+it)^m\frac{e^{\pi t}dt}{e^{2\pi t}-1}. \tag{27}$$

And by using Lemma 2.1 and expanding $(q+it)^m$ by the binomials, (27) becomes to

$$\begin{aligned}I_m(q) &= 2\mathrm{Re}\int_0^\infty \Bigg(\sum_{k=0}^m\sum_{h=0}^\infty \frac{(-1)^h q^{-2h-1}}{2h+1}\binom{m}{k}q^{m-k}i^k t^{k+2h+1}\\ &\quad -\sum_{k=0}^m \binom{m}{k}q^{m-k}i^{k+1}t^k\log q\\ &\quad -\frac{1}{2}\sum_{k=0}^m\sum_{h=1}^\infty \binom{m}{k}\frac{(-1)^{h-1}i^{k+1}}{h}q^{m-k-2h}t^{k+2h}\Bigg)\frac{e^{\pi t}dt}{e^{2\pi t}-1}\\ &= 2\mathrm{Re}\Bigg(\sum_{k=0}^m\sum_{h=0}^\infty \frac{(-1)^h q^{-2h-1}}{2h+1}\binom{m}{k}q^{m-k}i^k\left(\frac{1}{2\pi}\right)^{k+2h+2}\\ &\quad \times\Gamma(k+2h+2)\zeta\left(k+2h+2,\frac{1}{2}\right)\\ &\quad -\sum_{k=0}^m\binom{m}{k}q^{m-k}i^{k+1}\left(\frac{1}{2\pi}\right)^{k+1}\Gamma(k+1)\zeta\left(k+1,\frac{1}{2}\right)\log q\\ &\quad -\frac{1}{2}\sum_{k=0}^m\sum_{h=1}^\infty\binom{m}{k}\frac{(-1)^{h-1}i^{k+1}}{h}q^{m-k-2h}\left(\frac{1}{2\pi}\right)^{k+2h+1}\\ &\quad \times\Gamma(k+2h+1)\zeta\left(k+2h+1,\frac{1}{2}\right)\Bigg),\end{aligned} \tag{28}$$

in which the last equality follows from (26). Then by applying an elementary calculation procedure for the expression on the right side of (28), we have

$$\begin{aligned} I_m(q) = 2 &\sum_{\substack{k=0 \\ k \text{ even}}}^{m} \sum_{h=0}^{\infty} \binom{m}{k} \frac{(-1)^h i^k}{2h+1} \frac{q^{m-k-2h-1}}{(2\pi)^{2h+k+2}} (2h+k+1)! \zeta\left(2h+k+2, \frac{1}{2}\right) \\ &- 2 \sum_{\substack{k=0 \\ k \text{ odd}}}^{m} \binom{m}{k} i^{k+1} \frac{q^{m-k} \log q}{(2\pi)^{k+1}} k! \zeta\left(k+1, \frac{1}{2}\right) \\ &- \sum_{\substack{k=0 \\ k \text{ odd}}}^{m} \sum_{h=1}^{\infty} \binom{m}{k} \frac{(-1)^{h-1} i^{k+1}}{h} \frac{q^{m-k-2h}}{(2\pi)^{2h+k+1}} (2h+k)! \zeta\left(2h+k+1, \frac{1}{2}\right). \end{aligned} \tag{29}$$

Now recall the following Euler's formula for $\zeta(2k)$ (see [6, p. 266, Theorem 12.17])

$$\begin{aligned} \zeta\left(2k, \frac{1}{2}\right) &= (2^{2k}-1)\zeta(2k) \\ &= (2^{2k}-1)(-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k} \\ &= (-1)^k \frac{k(2\pi)^{2k}}{2(2k)!} E_{2k-1}(0), \end{aligned} \tag{30}$$

where $k = 1, 2, 3, \ldots$, and the third equality following from [16, Corollary 3.2]

$$(2^m - 1)B_m = -\frac{m}{2} E_{m-1}(0), \quad m = 1, 2, 3, \ldots. \tag{31}$$

Then combining (29) with (30), we obtain

$$\begin{aligned} I_m(q) = -\frac{1}{2} &\sum_{\substack{k=0 \\ k \text{ even}}}^{m} \sum_{h=0}^{\infty} \binom{m}{k} \frac{1}{2h+1} E_{2h+k+1}(0) q^{-(2h+k-m+1)} \\ &- \frac{1}{2} \sum_{\substack{k=0 \\ k \text{ odd}}}^{m} \binom{m}{k} E_k(0) q^{-(k-m)} \log q \\ &+ \frac{1}{2} \sum_{\substack{k=0 \\ k \text{ odd}}}^{m} \sum_{h=1}^{\infty} \binom{m}{k} \frac{1}{2h} E_{2h+k}(0) q^{-(2h+k-m)}. \end{aligned} \tag{32}$$

Finally, after some calculation, (32) becomes to

$$I_m(q) = \frac{1}{4}(1 + m \log q) q^{-(1-m)} - \sum_{k=1}^{\infty} a_{2k}(m) q^{-(2k-m+1)}, \tag{33}$$

where the coefficients $a_{2k}(m)$ are given by

$$a_{2k}(m)=\begin{cases}\frac{1}{2}E_{2k+1}(0)\left(\binom{m}{2k+1}\log q+\sum_{h=0}^{2k}\binom{m}{h}\frac{(-1)^h}{2k-h+1}\right), & 2k\le m-1,\\ \frac{1}{2}E_{2k+1}(0)\sum_{h=0}^{m}\binom{m}{h}\frac{(-1)^h}{2k-h+1}, & 2k\ge m,\end{cases}$$

and $E_n(x)$ are the Euler polynomials. This completes the proof. □

Finally, by (7) with $z=-m$ and Lemma 2.2, we get Theorem 1.4.

Proof of Proposition 1.6. For $m=0,1,2,\ldots$, if setting $z=-m$ in Proposition 1.1 and use (23), then we easily get that

$$\zeta_E(-m,q)=\frac{1}{2}q^m-2\mathrm{Im}\int_0^\infty(q+it)^m\frac{e^{\pi t}dt}{e^{2\pi t}-1}. \tag{34}$$

On the right hand side of (34), an application of the binomial identity yields

$$2\mathrm{Im}\int_0^\infty(q+it)^m\frac{e^{\pi t}dt}{e^{2\pi t}-1}=2\sum_{\substack{k=0\\ k\text{ odd}}}^{m}\binom{m}{k}q^{m-k}(-1)^{\frac{k-1}{2}}\int_0^\infty\frac{t^ke^{\pi t}dt}{e^{2\pi t}-1}. \tag{35}$$

Combining (26), (30), (31), (34), and (35), we obtain

$$\begin{aligned}\zeta_E(-m,q)&=\frac{1}{2}q^m-2\sum_{\substack{k=0\\ k\text{ odd}}}^{m}\binom{m}{k}q^{m-k}(-1)^{\frac{k-1}{2}}\left(\frac{1}{2\pi}\right)^{k+1}\\ &\quad\times k!(2^{k+1}-1)\zeta(k+1)\\ &=\frac{1}{2}q^m-\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m}{2k+1}q^{m-2k-1}(2^{2k+2}-1)\frac{B_{2k+2}}{2k+2}\\ &=\frac{1}{2}q^m+\frac{1}{2}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m}{2k+1}q^{m-2k-1}E_{2k+1}(0),\end{aligned} \tag{36}$$

where $\lfloor\cdot\rfloor$ denotes the floor function, and the proposition by noticing $E_{2k}(0)=0$ for $k=1,2,3,\ldots$.

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