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AN ADROIT UNRELATED QUESTION RANDOMIZED RESPONSE MODEL WITH SUNDRY STRATEGIES†

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ABSTRACT. When sensitive topics such as gambling habits, drug addiction, alcoholism, tax evasion tendencies, induced abortions, drunk driving, past criminal involvement, and homosexuality are the focus of open or direct surveys, it becomes challenging to obtain accurate information due to non-response bias and response bias. People often hesitate to provide truthful answers. Warner introduced an ingenious method to address this issue. In this study, a new and unrelated randomized response model is proposed to eliminate misleading responses and nonresponses caused by the stigma associated with the attribute being investigated. The proposed randomized response model allows for the estimation of the population percentage with the sensitive characteristic in an unbiased manner. The characteristics and recommendations of the proposed randomized response model are examined, and numerical examples are provided to support the findings of this study.

AMS Mathematics Subject Classification : 62D05.
Key words and phrases : Proportion, sensitive attribute, unrelated attribute.

1. Introduction

Socioeconomic studies frequently touch on certain personal traits that individuals desire to keep private in extensive inquiries; thorough questionnaires contain a large number of items. Information about the majority of them is typically simple to find by just inquiring. A few others, though, could deal with delicate subjects for which individuals are hesitant to provide candid comments. For instance, most people prefer to lie about their savings, the amount of their wealth, their history of intentional tax evasion, and other illegal or unethical practises that result in earnings from clandestine sources, crimes, the trade in

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illicit goods, propensity for intoxication, spending on various addictions, humus-sexuality, and other issues that are generally frowned upon by society. Open or direct queries often fail to yield reliable data on such confidential aspects of human life. Warner [1] developed an alternative survey technique that is known as randomized response (RR) technique. Later various modifications have been given by several researchers see Kim and Warde[2], Mangat and Singh[3]. Kim and Warde[2] have presented mixed randomized response models using simple random sampling with replacement sampling scheme which improves the privacy of respondents. In situations where potentially embarrassing or incriminating response are sought, the randomized response (RR) technique is effective in reducing non – sampling errors in sample surveys. Refusal to respond and lying in surveys are two main sources of such non – sampling errors, as the stigma attached to certain practices (e.g., sexual behaviours and the use of illegal drug) oftentimes leads to discrimination. Warner [1] was first to introduce a randomized response Technique (RRT) model to estimate the proportion for sensitive attributes including homosexuality, drug addiction or abortion. Since the work by Warner [1], a huge literature has emerged on the use and formulation of different randomization device to estimate the population proportion of a sensitive attribute in survey sampling. Mention may be made of the work of Mangat and Singh[3], Naik P.A. [4,5],Singh and Tarray [6,7], Tabbasum et. al. [8], Tarray et al. [9,10,11] , and Tarray and Ganie [12,13] and and the references cited therein.

2. Proposed unrelated randomized response Model

We propose a unrelated question randomized response Technique (RRT) model to estimate the proportion N of persons who possess the stigmatized attribute A under the study when the proportion of unrelated innocuous attribute X is known and unknown, respectively.

2.1. When the Proportion of Unrelated innocuous Attribute is Known. In the proposed two-stage RRT model, each selected respondent in the sample is given two randomization devices (R_1, R_2). The first-stage randomized device R_1 consists of the following statements:
(i) Do you possess sensitive attribute A_i?
(ii) Go to the randomized device R_1
The probabilities are T_i and $(1 - T_i)$, respectively. Second stage randomized device R_2 consists of the following

Statements:
(i) Do you possess sensitive attribute Ai?
(ii) Do you possess non-sensitive attribute Xi?
(iii) Blank Card

The probabilities are p_1, p_2 , and p_3, respectively, such that $\sum_{i=1}^{3} p_i = 1$. If statement (iii) is appeared on card of respondent, then it is required to repeat the process without replacing the card in the second draw. If statement (iii)

is reappeared, then the respondent is suggested Lo report his/her actual status about sensitive attribute A. The whole process of randomization is unseen by the interviewer and does not know whether the respondent's answers come from the first draw or from the second draw. So, the privacy of interviewee(s) is protected and they will respond without any fear.
The probability of getting answer "yes" from the respondent is as follows:

$$\theta_{1i} = T_i\pi_{ai}+(1-T_i)\Big[(\pi_{ai}+(1-\pi_{bi})P_{2i})\Big(1+P_{3i}\frac{k}{k-1}\Big)+\big(1+P_{3i}^2\frac{k}{k-1}\big)\pi_{ai}\Big] \quad (1)$$

where k be the total number of cards in the proposed deck.Solving equation (1) the estimator $\hat{\pi}_{1i}$ is an unbiased for π_{1i} is given as

$$\hat{\pi}_{1i} = \sum_{i=1}^{k} w_i\Bigg[\frac{\hat{\theta}_{2i}-(1-T_i)\big[P_2\big(1+P_{3i}\frac{k}{k-1}\big)\pi_{bi}-P_{2i}\big(1+P_{3i}\frac{k}{k-1}\big)\big]}{T_i+(1-T_i)\big(1+P_{3i}\frac{k}{k-1}+P_{3i}^2\frac{k}{k-1}\big)}\Bigg] \quad (2)$$

where $\hat{\theta}_{1i}$ is the proportion of "yes' answer obtained from the n_{i1} sampled respondents. The random variable $\hat{\theta}_{1i} = n_i/n$ follows the binomial distribution with parameters $(n_i, \theta_{2i})$with variance is given as follows:

$$V(\hat{\pi}_{1i}) = \sum_{i=1}^{k}\frac{w_i^2}{n_i}\Bigg[\pi_{ai}(1-\pi_{ai})+\frac{1-\Big\{T_i+(1-T_i)(\gamma)\Big\}-2(1-T_i)[\zeta]\pi_{bi}}{(T_i+(1-T_i)\delta)}\Bigg]$$

$$+\sum_{i=1}^{k}\frac{w_i^2}{n_i}\Bigg[\frac{\Big(1-T_i)(\psi)\pi_{bi}-\big(P_{2i}(\psi)\big)\Big\}\Big[1-(1-T_i)[\psi]\pi_{bi}-P_{2i}(\psi)\Big]}{(T_i+(1-T_i)\delta)}\Bigg] \quad (3)$$

where ,
$\gamma = \big(1+P_{3i}\frac{k}{k-1}+P_{3i}^2\frac{k}{k-1}\big)$
$\zeta = \big(1+P_{3i}\frac{k}{k-1}+P_{3i}^2\frac{k}{k-1}\big)$
$\zeta = \big[P_2\big(1+P_{3i}\frac{k}{k-1}\big)\pi_{bi}-P_2\big(1+P_{3i}\frac{k}{k-1}\big)\big]$
$\psi = \big[(1+P_{3i}\frac{k}{k-1})\big]\big)$
Lemma 2.1: The unbiased estimate of the variance $V(\hat{\pi}_{k1})$ is given as follows:

$$V(\hat{\pi}_{1i}) = \frac{\hat{\theta}_{2i}(1-\hat{\theta}_{2i})}{n_i\Big[T_i+(1-T_i)\Big(1+P_{3i}\frac{k}{k-1}+P_{3i}^2\frac{k}{k-1}\Big)\Big]^2} \quad (4)$$

2.2. When the Proportion of Unrelated innocuous Attribute is Unknown. To estimate the population proportion of sensitive attribute, π_{ai}, and unrelated innocuous attribute, π_{bi}, simultaneously, we applied a split-sample approach. In this method, a sample size n_i is split into two subsamples of sizes n_{1i} and $n_{2i}(n_{1i} + n_{2i} = n)$ in each subsample, different randomization devices are applied.

The description of randomization devices in two sub sample sizes n_{1i} and n_{2i} is given as follows. The layout of the two-stage randomized response model for n_{1i} is given as follows.
First-stage randomized device R_{11i} consists of the following statements:
(i) Do you possess sensitive attribute A_i?
(ii) Go to the randomized device R_{12i} ,
The probabilities are T_{1i} and $(1 - T_{i1})$, respectively. Second-stage randomized device R_{12i}, consists of the following statements:
(i) Do you possess sensitive attribute A_i?
(ii) Do you possess non-sensitive attributeX_i?
(iii) Blank card
The probabilities are p_1, p_2 , and p_3, respectively, such that $\sum_{i=1}^{3} p_i = 1$. If statement (iii) is appeared on card of respondent, then it is required to repeat the process without replacing the card. In the second draw, if statement (iii) is reappeared, then the respondent is suggested to report his/her actual status about sensitive attribute A_i. The layout of two-stage randomized response model for n_{2i} , is given as follows.
First-stage randomized device R_{21i} consists of the following statements:
(I) Do you possess sensitive attribute A_i?
(ii) Go to the randomized device R_{22i} ,
The probabilities are T_{2i} and $(1 - T_{2i})$, respectively.
Second-stage randomized device (R_{22i}) consists of the following statements:
(I) Do you possess sensitive attribute A_i?
(ii) Do you possess non-sensitive attribute X_i?
(iii) Draw one more card
The probabilities are q_1, q_2 , and q_3, respectively, such that $\sum_{i=1}^{3} q_i = 1$. If statement (iii) is appeared on card of respondent, then it is required to repeat the process without replacing the card In the second draw, if statement (iii) is reappeared, then the respondent is suggested to report his/her actual status about sensitive attribute A_i.
The probabilities of getting answer "yes" from the respondent is as follows:

$$\theta_{3i} = T_{1i}\pi_{ai} + (1 - T_{1i})\left[(\pi_{ai} + (1 - \pi_{bi})p_{2i})\left(1 + p_{3i}\frac{k}{k-1}\right) + \left(1 + P_{3i}^2\frac{k}{k-1}\right)\pi_{ai}\right] \quad (5)$$

$$\theta_{4i} = T_{2i}\pi_{ai}$$

$$+ (1 - T_{2i})\left[(\pi_{ai} + (1 - \pi_{bi})q_{2i})\left(1 + q_{3i}\frac{k}{k-1}\right) + \left(1 + q_{3i}^2\frac{k}{k-1}\right)\pi_{ai}\right] \quad (6)$$

Solving equations (5) and (6) for π_{ai} and π_{bi} , we have estimators for π_{ai} and π_{bi} ,respectively, as follows:

$$\hat{\pi}_{1i} = \sum_{i=1}^{k}\Bigg[\frac{(1 - T_{2i})\big[q_{2i}\big(1 + q_{3i}\frac{k}{k-1}\big)\theta_{4i} - q_{2i}\big(1 + q_{3i}\frac{k}{k-1}\big)\big]}{C_{0i}} - \frac{(1 - T_{2i})\big[p_{2i}\big(1 + p_{3i}\frac{k}{k-1}\big)\theta_{4i} - p_{2i}\big(1 + p_{3i}\frac{k}{k-1}\big)\big]}{C_{0i}}\Bigg] \quad (7)$$

where,

$$\begin{aligned} C_{0i} & \\ &= T_{1i}(1 - T_{2i})\left[q_{2i}\left(1 + q_{3i}\frac{k}{k-1}\hat{Q}_{4i}\right) - q_{2i}\left(1 + q_{3i}\frac{k}{k-1}\right)\right] \\ &- T_{2i}(1 - T_{1i})\left[p_{2i}\left(1 + p_{3i}\frac{k}{k-1}\hat{Q}_{3i}\right) - p_{2i}\left(1 + p_{3i}\frac{k}{k-1}\right)\right] \\ &+ (1 - T_{1i})(1 - T_{2i})\left[(q_{1i} + (1 - q_{1i})p_{2i})\left(1 + p_{3i}\frac{k}{k-1}\right)\left(1 + q_{3i}\frac{k}{k-1}\right)\right] \\ &+ (1 - T_{1i})(1 - T_{2i})(\frac{k}{k-1})\left[p_{3i}^2 q_{2i}\left(1 + q_{3i}\frac{k}{k-1}\right) - q_{3i}^2 p_{2i}\left(1 + p_{3i}\frac{k}{k-1}\right)\right] \\ &\neq 0. \end{aligned} \quad (8)$$

$$\begin{aligned} \hat{\pi}_{pxi} &= \frac{1}{D_{0i}}\left[T_{2i} + (1 - T_{2i})\left(1 + q_{3i}\frac{k}{k-1} + q_{3i}^2\frac{k}{k-1}\right)\right]\hat{\theta}_{3i} \\ &\quad -\left[T_{i1} + (1 - T_{i1})\left(1 + p_{3i}\frac{k}{k-1} + p_{3i}^2\frac{k}{k-1}\right)\right]\hat{\theta}_{4i}. \end{aligned} \quad (9)$$

$$\begin{aligned} D_{0i} &= T_{1j} + (1 - T_{2j})\left[q_{2i}\left(1 + q_{3i}\frac{k}{k-1}\right)\hat{\theta}_{4i} - q_{2i}\left(1 + q_{3i}\frac{k}{k-1}\right)\hat{\theta}_{4i}\right] \\ &\quad -T_{2j} + (1 - T_{1j})\left[p_{2i}\left(1 + p_{3i}\frac{k}{k-1}\right)\hat{\theta}_{4i} - p_{2i}\left(1 + p_{3i}\frac{k}{k-1}\right)\hat{\theta}_{4i}\right] \\ &\quad +(1 - T_{1i})(1 - T_{2j})\left[(q_{2i} + (1 - q_{2i}))\left(1 + q_{3i}\frac{k}{k-1}\right)\left(1 + q_{3i}\frac{k}{k-1}\right)\right] \\ &\quad \neq 0. \end{aligned} \quad (10)$$

where, $\hat{\theta}_{3i}$ and $\hat{\theta}_{4i}$ are the proportion of "yes' answer obtained from n and n sampled respondents, respectively. Since the random variables $\hat{\theta}_{3i}$ and $\hat{\theta}_{4i}$ follow the binomial distribution with parameters (n_{1i}, θ_{3i}) and (n_{2i}, θ_{4i}) respectively, therefore, the properties of estimators $\hat{\pi}_{11i}$ and $\hat{\theta}_{pxi}$, may be summarized in the following theorems.

Theorem 2.1: The estimators $V(\hat{\pi}_{11i})$ and $V(\hat{\pi}_{pxi})$ are unbiased for π_{ai} and π_{bi} with variances, respectively, and are given as follows:

$$V(\hat{\pi}_{11j}) = \frac{d_{2i}^2[c_{3i}^3\pi_{ai}(1-\pi_{ai}) + c_{3i}(1-c_{3i}-2c_{2i}\pi_{ai})\pi_{ai} + c_{2i}\pi_{xi}(1-c_{2i}\pi_{xi})]}{n_{1i}C_{0i}^2} \frac{c_{2i}^2[d_{3i}^3(1-\pi_{ai}) + d_{3i}(1-d_{3i}-2d_{2i}\pi_{ai})\pi_{ai} + d_{2i}\pi_{xi}(1-d_{2i}\pi_{xi})]}{n_{1i}C_{0i}^2} \tag{11}$$

with

$$V(\hat{\pi}_{pxi}) = \frac{d_{3i}^2[c_{3i}^3\pi_{ai}(1-\pi_{ai}) + c_{3i}(1-c_{3i}-2c_{2i}\pi_{ai})\pi_{ai} + c_{2i}\pi_{xi}(1-c_{2i}\pi_{xi})]}{n_{1i}C_{0i}^2} \frac{c_{3i}^2[d_{3i}^3(1-\pi_{ai}) + d_{3i}(1-d_{3i}-2d_{2i}\pi_{ai})\pi_{ai} + d_{2i}\pi_{xi}(1-d_{2i}\pi_{xi})]}{n_{1i}C_{0i}^2} \tag{12}$$

where

$$\begin{aligned} c_{2i} &= (1-T_{1i})\left[P_{2i}\left(1+P_{3i}\frac{k}{k-1}\right)\pi_{bi} - P_{2i}\left(1+P_{3i}\frac{k}{k-1}\right)\right] \\ c_{3i} &= \left[1+(1-T_{1i})\left(1+P_{3i}\frac{k}{k-1}+P_{3i}^2\frac{k}{k-1}\right)\right] \\ d_{2i} &= (1-T_{2i})\left[P_{2i}\left(1+P_{3i}\frac{k}{k-1}\right)\pi_{bi} - P_{2i}\left(1+P_{3i}\frac{k}{k-1}\right)\right] \\ d_{3i} &= \left[1+(1-T_{2i})\left(1+P_{3i}\frac{k}{k-1}+P_{3i}^2\frac{k}{k-1}\right)\right] \end{aligned} \tag{13}$$

3. Relative Efficiency

To show the dominance of the proposed estimators, the empirical comparisons are made over the existing estimators. The relative efficiency of proposed estimators with respect to the existing estimators when the proportion of unrelated innocuous attribute is known and unknown is defined as follows:
$RE_1 = \dfrac{V(\hat{\pi}_a)}{V(\hat{\pi}_{k1})}$ and $RE_2 = \dfrac{V(\hat{\pi}_{p1})}{V(\hat{\pi}_{k1})}$

4. Simulation Study

We perform a simulation study to validate the theoretical results of proposed two-stage unrelated question RRT model. The simulated results of proposed estimators and are presented in Tables 1,2 and 3,respectively. When the proportion of unrelated innocuous attribute is known, the design parameters T and p_1 , $p_2 = (1-p_1)$ and $p_3 = (1-p_1-p_2)$were allowed to vary while $k = 45$ and sample size $n = 1000$ were fixed.

TABLE 1. Simulation results of proposed estimator $\hat{\pi}_{k1}$, when trials =10000, $n = 1000$,$T = 0.3$ at various choices of π_a , π_b , p_1 , $p_2 = (1 - p_1)$, and $p_3 = (1 - p_1 - p_2)$ respectively.

Π_a	P_1	P_2	P_3	Π_b	$V(\hat{\pi}_a)$	$V(\hat{\pi}_{p1})$	$V(\hat{\pi}_{k1})$	PRE_1	PRE_2
	0.9	0.03	0.07	0.1	0.0000949	0.000112	0.0000430	220.77	260.45
	0.9	0.03	0.07	0.3	0.0000991	0.000107	0.0000427	232.22	250.58
0.1	0.9	0.03	0.07	0.5	0.0001032	0.000102	0.0000424	243.73	240.45
	0.9	0.03	0.07	0.7	0.0001073	0.000097	0.0000420	255.28	230.04
	0.9	0.03	0.07	0.9	0.0001113	0.000092	0.0000417	266.89	219.34
	0.9	0.03	0.07	0.1	0.0001672	0.000182	0.0001130	147.95	161.04
	0.9	0.03	0.07	0.3	0.0001703	0.000177	0.0001127	151.18	157.03
	0.9	0.03	0.07	0.5	0.0001734	0.000172	0.0001124	154.37	152.94
	0.9	0.03	0.07	0.7	0.0001765	0.000167	0.0001120	157.54	148.79
	0.9	0.03	0.07	0.9	0.0001795	0.000162	0.0001117	160.68	144.56
	0.8	0.07	0.13	0.1	0.0001773	0.000208	0.0000680	260.70	306.24
	0.8	0.07	0.13	0.3	0.0001847	0.000197	0.0000668	276.53	295.76
0.2	0.8	0.07	0.13	0.5	0.0001917	0.000186	0.0000655	292.58	284.51
	0.8	0.07	0.13	0.7	0.0001985	0.000175	0.0000643	308.87	272.45
	0.8	0.07	0.13	0.9	0.0002050	0.000164	0.0000630	325.42	259.52
	0.7	0.10	0.20	0.1	0.0001914	0.000240	0.0000259	740.14	927.39
	0.7	0.10	0.20	0.3	0.0002044	0.000222	0.0000231	884.54	962.60
	0.7	0.10	0.20	0.5	0.0002167	0.000204	0.0000203	1065.12	1004.62
	0.7	0.10	0.20	0.7	0.0002282	0.000186	0.0000176	1299.34	1056.75
	0.7	0.10	0.20	0.9	0.0002391	0.000166	0.0000148	1618.04	1124.83
	0.9	0.03	0.07	0.1	0.0002194	0.000232	0.0001630	134.63	142.31
	0.9	0.03	0.07	0.3	0.0002216	0.000227	0.0001627	136.21	139.50
	0.9	0.03	0.07	0.5	0.0002237	0.000222	0.0001624	137.76	136.64
	0.9	0.03	0.07	0.7	0.0002257	0.000217	0.0001620	139.29	133.73
	0.9	0.03	0.07	0.9	0.0002277	0.000212	0.0001617	140.78	130.79
	0.8	0.07	0.13	0.1	0.0002330	0.000258	0.0001180	197.42	218.87
	0.8	0.07	0.13	0.3	0.0002381	0.000247	0.0001168	203.85	211.94
0.3	0.8	0.07	0.13	0.5	0.0002428	0.000236	0.0001155	210.21	204.65
	0.8	0.07	0.13	0.7	0.0002474	0.000225	0.0001143	216.48	196.99
	0.8	0.07	0.13	0.9	0.0002516	0.000214	0.0001130	222.67	188.94
	0.7	0.10	0.20	0.1	0.0002519	0.000290	0.0000759	331.99	382.07
	0.7	0.10	0.20	0.3	0.0002611	0.000272	0.0000731	357.08	372.66
	0.7	0.10	0.20	0.5	0.0002695	0.000254	0.0000703	383.17	361.61
	0.7	0.10	0.20	0.7	0.0002773	0.000236	0.0000676	410.41	348.72
	0.7	0.10	0.20	0.9	0.0002843	0.000216	0.0000648	438.96	333.75
	0.6	0.13	0.27	0.1	0.0002778	0.000328	0.0000370	751.37	885.84
	0.6	0.13	0.27	0.3	0.0002932	0.000303	0.0000322	910.35	939.85
	0.6	0.13	0.27	0.5	0.0003070	0.000276	0.0000274	1119.97	1008.61
	0.6	0.13	0.27	0.7	0.0003192	0.000249	0.0000226	1413.46	1101.80
	0.6	0.13	0.27	0.9	0.0003297	0.000220	0.0000177	1861.10	1239.96
	0.9	0.03	0.07	0.1	0.0002517	0.000262	0.0001930	130.42	135.74
	0.9	0.03	0.07	0.3	0.0002528	0.000257	0.0001927	131.22	133.35
	0.9	0.03	0.07	0.5	0.0002539	0.000252	0.0001924	131.98	130.92
	0.9	0.03	0.07	0.7	0.0002549	0.000247	0.0001920	132.73	128.46
	0.9	0.03	0.07	0.9	0.0002558	0.000242	0.0001917	133.45	125.97
	0.8	0.07	0.13	0.1	0.0002687	0.000288	0.0001480	181.50	194.78
	0.8	0.07	0.13	0.3	0.0002714	0.000277	0.0001468	184.94	189.06
0.4	0.8	0.07	0.13	0.5	0.0002740	0.000266	0.0001455	188.27	183.08
	0.8	0.07	0.13	0.7	0.0002762	0.000255	0.0001443	191.48	176.82
	0.8	0.07	0.13	0.9	0.0002782	0.000244	0.0001430	194.56	170.28
	0.7	0.10	0.20	0.1	0.0002923	0.000320	0.0001059	276.11	302.14
	0.7	0.10	0.20	0.3	0.0002977	0.000302	0.0001031	288.73	293.33
	0.7	0.10	0.20	0.5	0.0003024	0.000284	0.0001003	301.36	283.39
	0.7	0.10	0.20	0.7	0.0003064	0.000266	0.0000976	314.01	272.24
	0.7	0.10	0.20	0.9	0.0003096	0.000246	0.0000948	326.69	259.76

Table 1 Cont.

Π_a	P_1	P_2	P_3	Π_b	$V(\hat{\pi}_a)$	$V(\hat{\pi}_{p1})$	$V(\hat{\pi}_{k1})$	PRE_1	PRE_2
	0.6	0.13	0.27	0.1	0.0003248	0.000358	0.0000670	484.94	533.82
	0.6	0.13	0.27	0.3	0.0003345	0.000333	0.0000622	537.69	534.83
	0.6	0.13	0.27	0.5	0.0003426	0.000306	0.0000574	596.72	533.81
	0.6	0.13	0.27	0.7	0.0003491	0.000279	0.0000526	663.86	530.21
	0.6	0.13	0.27	0.9	0.0003539	0.000250	0.0000477	741.71	523.24
	0.9	0.03	0.07	0.1	0.0002640	0.000272	0.0002030	130.04	133.98
	0.9	0.03	0.07	0.3	0.0002641	0.000267	0.0002027	130.29	131.70
	0.9	0.03	0.07	0.5	0.0002641	0.000262	0.0002024	130.51	129.39
	0.9	0.03	0.07	0.7	0.0002641	0.000257	0.0002020	130.71	127.05
	0.9	0.03	0.07	0.9	0.0002640	0.000252	0.0002017	130.88	124.68
	0.8	0.07	0.13	0.1	0.0002843	0.000298	0.0001580	179.93	188.78
	0.8	0.07	0.13	0.3	0.0002848	0.000287	0.0001568	181.69	183.38
0.5	0.8	0.07	0.13	0.5	0.0002851	0.000276	0.0001555	183.32	177.74
	0.8	0.07	0.13	0.7	0.0002851	0.000265	0.0001543	184.81	171.84
	0.8	0.07	0.13	0.9	0.0002848	0.000254	0.0001530	186.16	165.69
	0.7	0.10	0.20	0.1	0.0003127	0.000330	0.0001159	269.92	284.69
	0.7	0.10	0.20	0.3	0.0003144	0.000312	0.0001131	277.92	276.23
	0.7	0.10	0.20	0.5	0.0003153	0.000294	0.0001103	285.70	266.77
	0.7	0.10	0.20	0.7	0.0003154	0.000276	0.0001076	293.25	256.23
	0.7	0.10	0.20	0.9	0.0003149	0.000256	0.0001048	300.55	244.51
	0.6	0.13	0.27	0.1	0.0003517	0.000368	0.0000770	456.99	477.46
	0.6	0.13	0.27	0.3	0.0003558	0.000343	0.0000722	492.71	474.61
	0.6	0.13	0.27	0.5	0.0003582	0.000316	0.0000674	531.33	469.46
	0.6	0.13	0.27	0.7	0.0003590	0.000289	0.0000626	573.59	461.47
	0.6	0.13	0.27	0.9	0.0003581	0.000260	0.0000577	620.48	449.91
	0.9	0.03	0.07	0.1	0.0002562	0.000262	0.0001930	132.77	135.74
	0.9	0.03	0.07	0.3	0.0002553	0.000257	0.0001927	132.51	133.35
	0.9	0.03	0.07	0.5	0.0002543	0.000252	0.0001924	132.21	130.92
	0.9	0.03	0.07	0.7	0.0002533	0.000247	0.0001920	131.89	128.46
	0.9	0.03	0.07	0.9	0.0002522	0.000242	0.0001917	131.54	125.97
	0.8	0.07	0.13	0.1	0.0002800	0.000288	0.0001480	189.16	194.78
	0.8	0.07	0.13	0.3	0.0002782	0.000277	0.0001468	189.57	189.06
0.6	0.8	0.07	0.13	0.5	0.0002762	0.000266	0.0001455	189.82	183.08
	0.8	0.07	0.13	0.7	0.0002740	0.000255	0.0001443	189.91	176.82
	0.8	0.07	0.13	0.9	0.0002714	0.000244	0.0001430	189.81	170.28
	0.7	0.10	0.20	0.1	0.0003132	0.000320	0.0001059	295.82	302.14
	0.7	0.10	0.20	0.3	0.0003110	0.000302	0.0001031	301.62	293.33
	0.7	0.10	0.20	0.5	0.0003081	0.000284	0.0001003	307.06	283.39
	0.7	0.10	0.20	0.7	0.0003045	0.000266	0.0000976	312.11	272.24
	0.7	0.10	0.20	0.9	0.0003002	0.000246	0.0000948	316.73	259.76
	0.6	0.13	0.27	0.1	0.0003587	0.000358	0.0000670	535.65	533.82
	0.6	0.13	0.27	0.3	0.0003571	0.000333	0.0000622	573.98	534.83
	0.6	0.13	0.27	0.5	0.0003538	0.000306	0.0000574	616.20	533.81
	0.6	0.13	0.27	0.7	0.0003489	0.000279	0.0000526	663.47	530.21
	0.6	0.13	0.27	0.9	0.0003423	0.000250	0.0000477	717.42	523.24
	0.9	0.03	0.07	0.1	0.0002285	0.000232	0.0001630	140.20	142.31
	0.9	0.03	0.07	0.3	0.0002265	0.000227	0.0001627	139.27	139.50
	0.9	0.03	0.07	0.5	0.0002245	0.000222	0.0001624	138.30	136.64
	0.9	0.03	0.07	0.7	0.0002225	0.000217	0.0001620	137.29	133.73
	0.9	0.03	0.07	0.9	0.0002203	0.000212	0.0001617	136.25	130.79
	0.8	0.07	0.13	0.1	0.0002557	0.000258	0.0001180	216.62	218.87
	0.8	0.07	0.13	0.3	0.0002516	0.000247	0.0001168	215.49	211.94
0.7	0.8	0.07	0.13	0.5	0.0002474	0.000236	0.0001155	214.13	204.65
	0.8	0.07	0.13	0.7	0.0002428	0.000225	0.0001143	212.52	196.99
	0.8	0.07	0.13	0.9	0.0002381	0.000214	0.0001130	210.65	188.94
	0.7	0.10	0.20	0.1	0.0002936	0.000290	0.0000759	387.02	382.07
	0.7	0.10	0.20	0.3	0.0002876	0.000272	0.0000731	393.45	372.66
	0.7	0.10	0.20	0.5	0.0002810	0.000254	0.0000703	399.43	361.61
	0.7	0.10	0.20	0.7	0.0002736	0.000236	0.0000676	404.91	348.72
	0.7	0.10	0.20	0.9	0.0002655	0.000216	0.0000648	409.83	333.75

Table 1 Cont.

Π_a	P_1	P_2	P_3	Π_b	$V(\hat{\pi}_a)$	$V(\hat{\pi}_{p1})$	$V(\hat{\pi}_{k1})$	PRE_1	PRE_2
	0.6	0.13	0.27	0.1	0.0003457	0.000328	0.0000370	935.07	885.84
	0.6	0.13	0.27	0.3	0.0003383	0.000303	0.0000322	1050.51	939.85
	0.6	0.13	0.27	0.5	0.0003294	0.000276	0.0000274	1201.57	1008.61
	0.6	0.13	0.27	0.7	0.0003188	0.000249	0.0000226	1411.66	1101.80
	0.6	0.13	0.27	0.9	0.0003065	0.000220	0.0000177	1730.26	1239.96
	0.9	0.03	0.07	0.1	0.0001808	0.000182	0.0001130	159.99	161.04
	0.9	0.03	0.07	0.3	0.0001778	0.000177	0.0001127	157.79	157.03
	0.9	0.03	0.07	0.5	0.0001747	0.000172	0.0001124	155.54	152.94
	0.9	0.03	0.07	0.7	0.0001717	0.000167	0.0001120	153.22	148.79
	0.9	0.03	0.07	0.9	0.0001685	0.000162	0.0001117	150.84	144.56
	0.8	0.07	0.13	0.1	0.0002113	0.000208	0.0000680	310.66	306.24
	0.8	0.07	0.13	0.3	0.0002050	0.000197	0.0000668	307.06	295.76
0.8	0.8	0.07	0.13	0.5	0.0001985	0.000186	0.0000655	302.95	284.51
	0.8	0.07	0.13	0.7	0.0001917	0.000175	0.0000643	298.30	272.45
	0.8	0.07	0.13	0.9	0.0001847	0.000164	0.0000630	293.06	259.52
	0.7	0.10	0.20	0.1	0.0002540	0.000240	0.0000259	982.26	927.39
	0.7	0.10	0.20	0.3	0.0002443	0.000222	0.0000231	1057.15	962.60
	0.7	0.10	0.20	0.5	0.0002338	0.000204	0.0000203	1149.46	1004.62
	0.7	0.10	0.20	0.7	0.0002226	0.000186	0.0000176	1267.60	1056.75
	0.7	0.10	0.20	0.9	0.0002107	0.000166	0.0000148	1426.44	1124.83

TABLE 2. Simulation results of proposed estimator $\hat{\pi}_{k1}$, when trials =10000, $n = 1000$,$T = 0.5$ at various choices of π_a , π_b , p_1 , $p_2 = (1 - p_1)$, and $p_3 = (1 - p_1 - p_2)$ respectively.

Π_a	P_1	P_2	P_3	Π_b	$V(\hat{\pi}_a)$	$V(\hat{\pi}_{p1})$	$V(\hat{\pi}_{k1})$	PRE_1	PRE_2
	0.9	0.03	0.07	0.1	0.0000935	0.000106	0.0000559	167.08	188.72
	0.9	0.03	0.07	0.3	0.0000964	0.000102	0.0000557	173.06	183.05
0.1	0.9	0.03	0.07	0.5	0.0000993	0.000098	0.0000555	179.04	177.29
	0.9	0.03	0.07	0.7	0.0001022	0.000095	0.0000553	185.01	171.43
	0.9	0.03	0.07	0.9	0.0001051	0.000091	0.0000550	190.99	165.48
	0.9	0.03	0.07	0.1	0.0001651	0.000176	0.0001259	131.07	139.41
	0.9	0.03	0.07	0.3	0.0001673	0.000172	0.0001257	133.08	136.80
	0.9	0.03	0.07	0.5	0.0001695	0.000168	0.0001255	135.07	134.17
	0.9	0.03	0.07	0.7	0.0001717	0.000165	0.0001253	137.05	131.51
	0.9	0.03	0.07	0.9	0.0001738	0.000161	0.0001250	139.01	128.82
	0.8	0.07	0.13	0.1	0.0001721	0.000194	0.0000923	186.38	210.06
	0.8	0.07	0.13	0.3	0.0001771	0.000186	0.0000914	193.70	203.71
0.2	0.8	0.07	0.13	0.5	0.0001820	0.000178	0.0000905	201.02	197.11
	0.8	0.07	0.13	0.7	0.0001867	0.000171	0.0000896	208.36	190.24
	0.8	0.07	0.13	0.9	0.0001914	0.000162	0.0000887	215.70	183.09
	0.7	0.10	0.20	0.1	0.0001815	0.000215	0.0000597	303.86	360.84
	0.7	0.10	0.20	0.3	0.0001901	0.000203	0.0000577	329.28	351.95
	0.7	0.10	0.20	0.5	0.0001983	0.000190	0.0000557	355.95	341.88
	0.7	0.10	0.20	0.7	0.0002063	0.000178	0.0000537	384.01	330.50
	0.7	0.10	0.20	0.9	0.0002139	0.000164	0.0000517	413.65	317.63
	0.9	0.03	0.07	0.1	0.0002167	0.000226	0.0001759	123.15	128.21
	0.9	0.03	0.07	0.3	0.0002182	0.000222	0.0001757	124.17	126.33
	0.9	0.03	0.07	0.5	0.0002196	0.000218	0.0001755	125.16	124.44
	0.9	0.03	0.07	0.7	0.0002211	0.000215	0.0001753	126.15	122.52
	0.9	0.03	0.07	0.9	0.0002225	0.000211	0.0001750	127.12	120.59
	0.8	0.07	0.13	0.1	0.0002260	0.000244	0.0001423	158.82	171.39
	0.8	0.07	0.13	0.3	0.0002295	0.000236	0.0001414	162.26	167.05
0.3	0.8	0.07	0.13	0.5	0.0002328	0.000228	0.0001405	165.64	162.56
	0.8	0.07	0.13	0.7	0.0002359	0.000221	0.0001396	168.99	157.92
	0.8	0.07	0.13	0.9	0.0002390	0.000212	0.0001387	172.28	153.14

Table 2 Cont.

Π_a	P_1	P_2	P_3	Π_b	$V(\hat{\pi}_a)$	$V(\hat{\pi}_{p1})$	$V(\hat{\pi}_{k1})$	PRE_1	PRE_2
	0.7	0.10	0.20	0.1	0.0002387	0.000265	0.0001097	217.52	241.97
	0.7	0.10	0.20	0.3	0.0002446	0.000253	0.0001077	227.11	235.01
	0.7	0.10	0.20	0.5	0.0002503	0.000240	0.0001057	236.76	227.49
	0.7	0.10	0.20	0.7	0.0002556	0.000228	0.0001037	246.47	219.37
	0.7	0.10	0.20	0.9	0.0002606	0.000214	0.0001017	256.24	210.63
	0.6	0.13	0.27	0.1	0.0002553	0.000291	0.0000785	325.05	370.07
	0.6	0.13	0.27	0.3	0.0002648	0.000273	0.0000750	352.83	363.89
	0.6	0.13	0.27	0.5	0.0002736	0.000255	0.0000715	382.38	356.25
	0.6	0.13	0.27	0.7	0.0002816	0.000236	0.0000680	414.00	346.92
	0.6	0.13	0.27	0.9	0.0002890	0.000216	0.0000645	448.06	335.61
	0.9	0.03	0.07	0.1	0.0002483	0.000256	0.0002059	120.56	124.10
	0.9	0.03	0.07	0.3	0.0002490	0.000252	0.0002057	121.07	122.49
	0.9	0.03	0.07	0.5	0.0002498	0.000248	0.0002055	121.57	120.87
	0.9	0.03	0.07	0.7	0.0002505	0.000245	0.0002053	122.05	119.23
	0.9	0.03	0.07	0.9	0.0002512	0.000241	0.0002050	122.52	117.57
	0.8	0.07	0.13	0.1	0.0002600	0.000274	0.0001723	150.89	158.97
	0.8	0.07	0.13	0.3	0.0002619	0.000266	0.0001714	152.75	155.31
0.4	0.8	0.07	0.13	0.5	0.0002636	0.000258	0.0001705	154.56	151.55
	0.8	0.07	0.13	0.7	0.0002652	0.000251	0.0001696	156.32	147.68
	0.8	0.07	0.13	0.9	0.0002666	0.000242	0.0001687	158.02	143.69
	0.7	0.10	0.20	0.1	0.0002759	0.000295	0.0001397	197.44	211.49
	0.7	0.10	0.20	0.3	0.0002792	0.000283	0.0001377	202.76	205.60
	0.7	0.10	0.20	0.5	0.0002823	0.000270	0.0001357	207.99	199.31
	0.7	0.10	0.20	0.7	0.0002850	0.000258	0.0001337	213.13	192.59
	0.7	0.10	0.20	0.9	0.0002873	0.000244	0.0001317	218.18	185.43
	0.6	0.13	0.27	0.1	0.0002967	0.000321	0.0001085	273.42	295.42
	0.6	0.13	0.27	0.3	0.0003024	0.000303	0.0001050	287.89	288.53
	0.6	0.13	0.27	0.5	0.0003074	0.000285	0.0001015	302.68	280.55
	0.6	0.13	0.27	0.7	0.0003116	0.000266	0.0000980	317.84	271.35
	0.6	0.13	0.27	0.9	0.0003150	0.000246	0.0000945	333.41	260.81
	0.9	0.03	0.07	0.1	0.0002599	0.000266	0.0002159	120.35	122.98
	0.9	0.03	0.07	0.3	0.0002599	0.000262	0.0002157	120.50	121.45
	0.9	0.03	0.07	0.5	0.0002599	0.000258	0.0002155	120.64	119.90
	0.9	0.03	0.07	0.7	0.0002599	0.000255	0.0002153	120.76	118.34
	0.9	0.03	0.07	0.9	0.0002599	0.000251	0.0002150	120.87	116.76
	0.8	0.07	0.13	0.1	0.0002740	0.000284	0.0001823	150.27	155.73
	0.8	0.07	0.13	0.3	0.0002742	0.000276	0.0001814	151.15	152.26
0.5	0.8	0.07	0.13	0.5	0.0002744	0.000268	0.0001805	151.98	148.70
	0.8	0.07	0.13	0.7	0.0002744	0.000261	0.0001796	152.74	145.03
	0.8	0.07	0.13	0.9	0.0002742	0.000252	0.0001787	153.44	141.25
	0.7	0.10	0.20	0.1	0.0002931	0.000305	0.0001497	195.74	204.04
	0.7	0.10	0.20	0.3	0.0002938	0.000293	0.0001477	198.90	198.45
	0.7	0.10	0.20	0.5	0.0002943	0.000280	0.0001457	201.93	192.49
	0.7	0.10	0.20	0.7	0.0002943	0.000268	0.0001437	204.81	186.15
	0.7	0.10	0.20	0.9	0.0002941	0.000254	0.0001417	207.54	179.40
	0.6	0.13	0.27	0.1	0.0003182	0.000331	0.0001185	268.47	278.93
	0.6	0.13	0.27	0.3	0.0003200	0.000313	0.0001150	278.19	272.14
	0.6	0.13	0.27	0.5	0.0003211	0.000295	0.0001115	287.90	264.36
	0.6	0.13	0.27	0.7	0.0003215	0.000276	0.0001080	297.61	255.49
	0.6	0.13	0.27	0.9	0.0003211	0.000256	0.0001045	307.32	245.42
	0.9	0.03	0.07	0.1	0.0002515	0.000256	0.0002059	122.12	124.10
	0.9	0.03	0.07	0.3	0.0002508	0.000252	0.0002057	121.93	122.49
	0.9	0.03	0.07	0.5	0.0002501	0.000248	0.0002055	121.72	120.87
	0.9	0.03	0.07	0.7	0.0002494	0.000245	0.0002053	121.49	119.23
	0.9	0.03	0.07	0.9	0.0002486	0.000241	0.0002050	121.26	117.57
	0.8	0.07	0.13	0.1	0.0002680	0.000274	0.0001723	155.49	158.97
	0.8	0.07	0.13	0.3	0.0002666	0.000266	0.0001714	155.53	155.31
0.6	0.8	0.07	0.13	0.5	0.0002652	0.000258	0.0001705	155.49	151.55
	0.8	0.07	0.13	0.7	0.0002636	0.000251	0.0001696	155.38	147.68
	0.8	0.07	0.13	0.9	0.0002619	0.000242	0.0001687	155.20	143.69

Table 2 Cont.

Π_a	P_1	P_2	P_3	Π_b	$V(\hat{\pi}_a)$	$V(\hat{\pi}_{p1})$	$V(\hat{\pi}_{k1})$	PRE_1	PRE_2
	0.7	0.10	0.20	0.1	0.0002903	0.000295	0.0001397	207.75	211.49
	0.7	0.10	0.20	0.3	0.0002884	0.000283	0.0001377	209.42	205.60
	0.7	0.10	0.20	0.5	0.0002862	0.000270	0.0001357	210.89	199.31
	0.7	0.10	0.20	0.7	0.0002837	0.000258	0.0001337	212.17	192.59
	0.7	0.10	0.20	0.9	0.0002808	0.000244	0.0001317	213.24	185.43
	0.6	0.13	0.27	0.1	0.0003197	0.000321	0.0001085	294.57	295.42
	0.6	0.13	0.27	0.3	0.0003177	0.000303	0.0001050	302.41	288.53
	0.6	0.13	0.27	0.5	0.0003149	0.000285	0.0001015	310.13	280.55
	0.6	0.13	0.27	0.7	0.0003114	0.000266	0.0000980	317.70	271.35
	0.6	0.13	0.27	0.9	0.0003072	0.000246	0.0000945	325.11	260.81
	0.9	0.03	0.07	0.1	0.0002231	0.000226	0.0001759	126.81	128.21
	0.9	0.03	0.07	0.3	0.0002217	0.000222	0.0001757	126.17	126.33
	0.9	0.03	0.07	0.5	0.0002203	0.000218	0.0001755	125.52	124.44
	0.9	0.03	0.07	0.7	0.0002188	0.000215	0.0001753	124.85	122.52
	0.9	0.03	0.07	0.9	0.0002173	0.000211	0.0001750	124.16	120.59
	0.8	0.07	0.13	0.1	0.0002419	0.000244	0.0001423	169.98	171.39
	0.8	0.07	0.13	0.3	0.0002390	0.000236	0.0001414	168.99	167.05
0.7	0.8	0.07	0.13	0.5	0.0002359	0.000228	0.0001405	167.90	162.56
	0.8	0.07	0.13	0.7	0.0002328	0.000221	0.0001396	166.71	157.92
	0.8	0.07	0.13	0.9	0.0002295	0.000212	0.0001387	165.42	153.14
	0.7	0.10	0.20	0.1	0.0002675	0.000265	0.0001097	243.77	241.97
	0.7	0.10	0.20	0.3	0.0002630	0.000253	0.0001077	244.15	235.01
	0.7	0.10	0.20	0.5	0.0002582	0.000240	0.0001057	244.23	227.49
	0.7	0.10	0.20	0.7	0.0002531	0.000228	0.0001037	244.00	219.37
	0.7	0.10	0.20	0.9	0.0002476	0.000214	0.0001017	243.44	210.63
	0.6	0.13	0.27	0.1	0.0003012	0.000291	0.0000785	383.51	370.07
	0.6	0.13	0.27	0.3	0.0002953	0.000273	0.0000750	393.49	363.89
	0.6	0.13	0.27	0.5	0.0002887	0.000255	0.0000715	403.52	356.25
	0.6	0.13	0.27	0.7	0.0002814	0.000236	0.0000680	413.60	346.92
	0.6	0.13	0.27	0.9	0.0002733	0.000216	0.0000645	423.76	335.61
	0.9	0.03	0.07	0.1	0.0001747	0.000176	0.0001259	138.73	139.41
	0.9	0.03	0.07	0.3	0.0001726	0.000172	0.0001257	137.28	136.80
	0.9	0.03	0.07	0.5	0.0001704	0.000168	0.0001255	135.81	134.17
	0.9	0.03	0.07	0.7	0.0001682	0.000165	0.0001253	134.31	131.51
	0.9	0.03	0.07	0.9	0.0001660	0.000161	0.0001250	132.78	128.82
	0.8	0.07	0.13	0.1	0.0001959	0.000194	0.0000923	212.17	210.06
	0.8	0.07	0.13	0.3	0.0001914	0.000186	0.0000914	209.32	203.71
0.8	0.8	0.07	0.13	0.5	0.0001867	0.000178	0.0000905	206.28	197.11
	0.8	0.07	0.13	0.7	0.0001820	0.000171	0.0000896	203.05	190.24
	0.8	0.07	0.13	0.9	0.0001771	0.000162	0.0000887	199.60	183.09
	0.7	0.10	0.20	0.1	0.0002247	0.000215	0.0000597	376.21	360.84
	0.7	0.10	0.20	0.3	0.0002176	0.000203	0.0000577	376.96	351.95
	0.7	0.10	0.20	0.5	0.0002102	0.000190	0.0000557	377.19	341.88
	0.7	0.10	0.20	0.7	0.0002024	0.000178	0.0000537	376.85	330.50
	0.7	0.10	0.20	0.9	0.0001943	0.000164	0.0000517	375.87	317.63

Table 3. Simulation results of proposed estimator $\hat{\pi}_{k1}$, when trials =10000, $n = 1000$,$T = 0.7$ at various choices of π_a , π_b , p_1 , $p_2 = (1 - p_1)$, and $p_3 = (1 - p_1 - p_2)$ respectively.

Π_a	P_1	P_2	P_3	Π_b	$V(\hat{\pi}_a)$	$V(\hat{\pi}_{p1})$	$V(\hat{\pi}_{k1})$	PRE_1	PRE_2
	0.9	0.03	0.07	0.1	0.0000920	0.000099	0.0000693	132.90	143.32
	0.9	0.03	0.07	0.3	0.0000938	0.000097	0.0000691	135.69	140.50
0.1	0.9	0.03	0.07	0.5	0.0000955	0.000095	0.0000690	138.47	137.65
	0.9	0.03	0.07	0.7	0.0000973	0.000093	0.0000689	141.25	134.78
	0.9	0.03	0.07	0.9	0.0000990	0.000091	0.0000687	144.03	131.89
	0.9	0.03	0.07	0.1	0.0001630	0.000169	0.0001393	117.05	121.54
	0.9	0.03	0.07	0.3	0.0001643	0.000167	0.0001391	118.11	120.12
	0.9	0.03	0.07	0.5	0.0001656	0.000165	0.0001390	119.16	118.69
	0.9	0.03	0.07	0.7	0.0001669	0.000163	0.0001389	120.21	117.25
	0.9	0.03	0.07	0.9	0.0001682	0.000161	0.0001387	121.26	115.79
	0.8	0.07	0.13	0.1	0.0001671	0.000180	0.0001181	141.42	152.39
	0.8	0.07	0.13	0.3	0.0001700	0.000175	0.0001176	144.53	149.19
0.2	0.8	0.07	0.13	0.5	0.0001728	0.000171	0.0001170	147.64	145.93
	0.8	0.07	0.13	0.7	0.0001756	0.000166	0.0001165	150.73	142.60
	0.8	0.07	0.13	0.9	0.0001784	0.000161	0.0001160	153.82	139.20
	0.7	0.10	0.20	0.1	0.0001723	0.000192	0.0000969	177.86	198.59
	0.7	0.10	0.20	0.3	0.0001771	0.000185	0.0000957	185.14	193.44
	0.7	0.10	0.20	0.5	0.0001818	0.000178	0.0000945	192.49	188.05
	0.7	0.10	0.20	0.7	0.0001864	0.000170	0.0000932	199.91	182.40
	0.7	0.10	0.20	0.9	0.0001909	0.000162	0.0000920	207.41	176.49
	0.9	0.03	0.07	0.1	0.0002140	0.000219	0.0001893	113.05	115.85
	0.9	0.03	0.07	0.3	0.0002148	0.000217	0.0001891	113.60	114.80
	0.9	0.03	0.07	0.5	0.0002157	0.000215	0.0001890	114.14	113.74
	0.9	0.03	0.07	0.7	0.0002166	0.000213	0.0001889	114.68	112.68
	0.9	0.03	0.07	0.9	0.0002174	0.000211	0.0001887	115.21	111.61
	0.8	0.07	0.13	0.1	0.0002194	0.000230	0.0001681	130.49	136.81
	0.8	0.07	0.13	0.3	0.0002214	0.000225	0.0001676	132.08	134.52
0.3	0.8	0.07	0.13	0.5	0.0002233	0.000221	0.0001670	133.66	132.18
	0.8	0.07	0.13	0.7	0.0002251	0.000216	0.0001665	135.22	129.81
	0.8	0.07	0.13	0.9	0.0002270	0.000211	0.0001660	136.76	127.39
	0.7	0.10	0.20	0.1	0.0002265	0.000242	0.0001469	154.20	165.03
	0.7	0.10	0.20	0.3	0.0002298	0.000235	0.0001457	157.75	161.37
	0.7	0.10	0.20	0.5	0.0002330	0.000228	0.0001445	161.27	157.57
	0.7	0.10	0.20	0.7	0.0002360	0.000220	0.0001432	164.78	153.64
	0.7	0.10	0.20	0.9	0.0002390	0.000212	0.0001420	168.27	149.56
	0.6	0.13	0.27	0.1	0.0002355	0.000257	0.0001257	187.26	203.99
	0.6	0.13	0.27	0.3	0.0002405	0.000246	0.0001236	194.52	199.11
	0.6	0.13	0.27	0.5	0.0002452	0.000236	0.0001215	201.84	193.88
	0.6	0.13	0.27	0.7	0.0002497	0.000225	0.0001193	209.25	188.28
	0.6	0.13	0.27	0.9	0.0002540	0.000214	0.0001172	216.73	182.30
	0.9	0.03	0.07	0.1	0.0002449	0.000249	0.0002193	111.70	113.68
	0.9	0.03	0.07	0.3	0.0002454	0.000247	0.0002191	111.98	112.78
	0.9	0.03	0.07	0.5	0.0002458	0.000245	0.0002190	112.25	111.86
	0.9	0.03	0.07	0.7	0.0002462	0.000243	0.0002189	112.51	110.94
	0.9	0.03	0.07	0.9	0.0002467	0.000241	0.0002187	112.77	110.02
	0.8	0.07	0.13	0.1	0.0002517	0.000260	0.0001981	127.06	131.24
	0.8	0.07	0.13	0.3	0.0002528	0.000255	0.0001976	127.92	129.28

Table 3 Cont.

Π_a	P_1	P_2	P_3	Π_b	$V(\hat{\pi}_a)$	$V(\hat{\pi}_{p1})$	$V(\hat{\pi}_{k1})$	PRE_1	PRE_2
0.4	0.8	0.07	0.13	0.5	0.0002537	0.000251	0.0001970	128.77	127.28
	0.8	0.07	0.13	0.7	0.0002547	0.000246	0.0001965	129.60	125.26
	0.8	0.07	0.13	0.9	0.0002556	0.000241	0.0001960	130.42	123.20
	0.7	0.10	0.20	0.1	0.0002607	0.000272	0.0001769	147.37	154.00
	0.7	0.10	0.20	0.3	0.0002625	0.000265	0.0001757	149.40	150.89
	0.7	0.10	0.20	0.5	0.0002641	0.000258	0.0001745	151.39	147.67
	0.7	0.10	0.20	0.7	0.0002657	0.000250	0.0001732	153.35	144.35
	0.7	0.10	0.20	0.9	0.0002671	0.000242	0.0001720	155.26	140.92
	0.6	0.13	0.27	0.1	0.0002720	0.000287	0.0001557	174.65	183.96
	0.6	0.13	0.27	0.3	0.0002748	0.000276	0.0001536	178.89	179.76
	0.6	0.13	0.27	0.5	0.0002773	0.000266	0.0001515	183.10	175.29
	0.6	0.13	0.27	0.7	0.0002797	0.000255	0.0001493	187.27	170.55
	0.6	0.13	0.27	0.9	0.0002817	0.000244	0.0001472	191.42	165.52
	0.9	0.03	0.07	0.1	0.0002559	0.000259	0.0002293	111.61	113.09
	0.9	0.03	0.07	0.3	0.0002559	0.000257	0.0002291	111.68	112.22
	0.9	0.03	0.07	0.5	0.0002559	0.000255	0.0002290	111.75	111.34
	0.9	0.03	0.07	0.7	0.0002559	0.000253	0.0002289	111.82	110.46
	0.9	0.03	0.07	0.9	0.0002559	0.000251	0.0002287	111.88	109.58
	0.8	0.07	0.13	0.1	0.0002641	0.000270	0.0002081	126.88	129.74
	0.8	0.07	0.13	0.3	0.0002642	0.000265	0.0002076	127.25	127.87
0.5	0.8	0.07	0.13	0.5	0.0002642	0.000261	0.0002070	127.61	125.97
	0.8	0.07	0.13	0.7	0.0002642	0.000256	0.0002065	127.94	124.03
	0.8	0.07	0.13	0.9	0.0002642	0.000251	0.0002060	128.26	122.07
	0.7	0.10	0.20	0.1	0.0002749	0.000282	0.0001869	147.07	151.11
	0.7	0.10	0.20	0.3	0.0002751	0.000275	0.0001857	148.17	148.15
	0.7	0.10	0.20	0.5	0.0002753	0.000268	0.0001845	149.23	145.09
	0.7	0.10	0.20	0.7	0.0002753	0.000260	0.0001832	150.23	141.93
	0.7	0.10	0.20	0.9	0.0002752	0.000252	0.0001820	151.19	138.67
	0.6	0.13	0.27	0.1	0.0002885	0.000297	0.0001657	174.09	178.90
	0.6	0.13	0.27	0.3	0.0002891	0.000286	0.0001636	176.72	174.88
	0.6	0.13	0.27	0.5	0.0002895	0.000276	0.0001615	179.28	170.63
	0.6	0.13	0.27	0.7	0.0002896	0.000265	0.0001593	181.77	166.12
	0.6	0.13	0.27	0.9	0.0002895	0.000254	0.0001572	184.18	161.35
	0.9	0.03	0.07	0.1	0.0002468	0.000249	0.0002193	112.57	113.68
	0.9	0.03	0.07	0.3	0.0002464	0.000247	0.0002191	112.45	112.78
	0.9	0.03	0.07	0.5	0.0002460	0.000245	0.0002190	112.33	111.86
	0.9	0.03	0.07	0.7	0.0002456	0.000243	0.0002189	112.20	110.94
	0.9	0.03	0.07	0.9	0.0002451	0.000241	0.0002187	112.07	110.02
	0.8	0.07	0.13	0.1	0.0002564	0.000260	0.0001981	129.41	131.24
	0.8	0.07	0.13	0.3	0.0002556	0.000255	0.0001976	129.34	129.28
0.6	0.8	0.07	0.13	0.5	0.0002547	0.000251	0.0001970	129.24	127.28
	0.8	0.07	0.13	0.7	0.0002537	0.000246	0.0001965	129.13	125.26
	0.8	0.07	0.13	0.9	0.0002528	0.000241	0.0001960	128.99	123.20
	0.7	0.10	0.20	0.1	0.0002690	0.000272	0.0001769	152.10	154.00
	0.7	0.10	0.20	0.3	0.0002678	0.000265	0.0001757	152.43	150.89
	0.7	0.10	0.20	0.5	0.0002664	0.000258	0.0001745	152.70	147.67
	0.7	0.10	0.20	0.7	0.0002649	0.000250	0.0001732	152.92	144.35
	0.7	0.10	0.20	0.9	0.0002633	0.000242	0.0001720	153.07	140.92
	0.6	0.13	0.27	0.1	0.0002851	0.000287	0.0001557	183.04	183.96
	0.6	0.13	0.27	0.3	0.0002835	0.000276	0.0001536	184.54	179.76
	0.6	0.13	0.27	0.5	0.0002817	0.000266	0.0001515	185.94	175.29
	0.6	0.13	0.27	0.7	0.0002796	0.000255	0.0001493	187.22	170.55
	0.6	0.13	0.27	0.9	0.0002773	0.000244	0.0001472	188.39	165.52
	0.9	0.03	0.07	0.1	0.0002178	0.000219	0.0001893	115.07	115.85
	0.9	0.03	0.07	0.3	0.0002169	0.000217	0.0001891	114.71	114.80
	0.9	0.03	0.07	0.5	0.0002161	0.000215	0.0001890	114.34	113.74
	0.9	0.03	0.07	0.7	0.0002152	0.000213	0.0001889	113.96	112.68
	0.9	0.03	0.07	0.9	0.0002143	0.000211	0.0001887	113.58	111.61
	0.8	0.07	0.13	0.1	0.0002288	0.000230	0.0001681	136.05	136.81
	0.8	0.07	0.13	0.3	0.0002270	0.000225	0.0001676	135.43	134.52

Table 3 Cont.

Π_a	P_1	P_2	P_3	Π_b	$V(\hat{\pi}_a)$	$V(\hat{\pi}_{p1})$	$V(\hat{\pi}_{k1})$	PRE_1	PRE_2
0.7	0.8	0.07	0.13	0.5	0.0002251	0.000221	0.0001670	134.78	132.18
	0.8	0.07	0.13	0.7	0.0002233	0.000216	0.0001665	134.09	129.81
	0.8	0.07	0.13	0.9	0.0002214	0.000211	0.0001660	133.38	127.39
	0.7	0.10	0.20	0.1	0.0002432	0.000242	0.0001469	165.58	165.03
	0.7	0.10	0.20	0.3	0.0002404	0.000235	0.0001457	165.05	161.37
	0.7	0.10	0.20	0.5	0.0002376	0.000228	0.0001445	164.44	157.57
	0.7	0.10	0.20	0.7	0.0002345	0.000220	0.0001432	163.74	153.64
	0.7	0.10	0.20	0.9	0.0002314	0.000212	0.0001420	162.95	149.56
	0.6	0.13	0.27	0.1	0.0002616	0.000257	0.0001257	208.05	203.99
	0.6	0.13	0.27	0.3	0.0002578	0.000246	0.0001236	208.58	199.11
	0.6	0.13	0.27	0.5	0.0002538	0.000236	0.0001215	208.93	193.88
	0.6	0.13	0.27	0.7	0.0002495	0.000225	0.0001193	209.12	188.28
	0.6	0.13	0.27	0.9	0.0002450	0.000214	0.0001172	209.11	182.30
	0.9	0.03	0.07	0.1	0.0001687	0.000169	0.0001393	121.17	121.54
	0.9	0.03	0.07	0.3	0.0001675	0.000167	0.0001391	120.37	120.12
	0.9	0.03	0.07	0.5	0.0001662	0.000165	0.0001390	119.56	118.69
	0.9	0.03	0.07	0.7	0.0001649	0.000163	0.0001389	118.74	117.25
	0.9	0.03	0.07	0.9	0.0001636	0.000161	0.0001387	117.91	115.79
	0.8	0.07	0.13	0.1	0.0001811	0.000180	0.0001181	153.29	152.39
	0.8	0.07	0.13	0.3	0.0001784	0.000175	0.0001176	151.69	149.19
0.8	0.8	0.07	0.13	0.5	0.0001756	0.000171	0.0001170	150.03	145.93
	0.8	0.07	0.13	0.7	0.0001728	0.000166	0.0001165	148.32	142.60
	0.8	0.07	0.13	0.9	0.0001700	0.000161	0.0001160	146.56	139.20
	0.7	0.10	0.20	0.1	0.0001974	0.000192	0.0000969	203.74	198.59
	0.7	0.10	0.20	0.3	0.0001931	0.000185	0.0000957	201.83	193.44
	0.7	0.10	0.20	0.5	0.0001887	0.000178	0.0000945	199.76	188.05
	0.7	0.10	0.20	0.7	0.0001842	0.000170	0.0000932	197.52	182.40
	0.7	0.10	0.20	0.9	0.0001795	0.000162	0.0000920	195.10	176.49

5. Conclusion

According to this research, it is clear from examining the previously provided information that the newly proposed randomized response model offers advantages in terms of relative efficiency when compared to other current models. When managing delicate situations during surveys, it performs better. Because it offers support and advantages in practical settings, the report advises survey practitioners to take into account adopting this novel mathematical model when dealing with sensitive themes.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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