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ON HIGHER-ORDER POLY-TANGENT POLYNOMIALS

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ABSTRACT. In this paper, we construct higher-order poly-tangent polynomials and study several properties, including addition formula and multiplication formula. Finally, we explore the distribution of roots of higher-order poly-tangent polynomials.

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1. Introduction

In [5], we defined the tangent numbers and polynomials. The tangent polynomials are defined as the following generating function

$$\left(\frac{2}{e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}\mathbf{T}_n(x)\frac{t^n}{n!}.$$

In [6], we constructed the poly-tangent numbers and polynomials. A modified poly-tangent numbers and polynomials different from the poly-tangent numbers and polynomials defined in [6] was introduced.

Definition 1.1. For any integer k, the modified poly-tangent polynomials are defined as the following generating function

$$\left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)e^{xt}=\sum_{n=0}^{\infty}T_n^{(k)}(x)\frac{t^n}{n!},$$

where $Li_k(t)=\sum_{n=1}^{\infty}\frac{t^n}{n^k}$ is polylogarithm function.

$T_n^{(k)}=T_n^{(k)}(0)$ are the called poly-tangent numbers when $x=0$. If we set $k=1$ in Definition 1.1, then the poly-tangent polynomials are reduced to classical tangent polynomials because of $Li_1(1-e^{-t})=t$. That is, $T_n^{(1)}(x)=\mathbf{T}_n(x)$.

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2. Some properties of higher-order poly-tangent polynomials

In this section, we define higher-order poly-tangent polynomials and study several properties, including addition formula and multiplication formula.

In [7], we introduced tangent numbers and tangent polynomials of higher order. We also obtain interesting properties of these numbers and polynomials. For a nonnegative integer r, tangent polynomials of higher order are defined as the following generating function

$$\left(\frac{2}{e^{2t}+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} \mathbf{T}_n^{(r)}(x)\frac{t^n}{n!}.$$

Definition 2.1. For any integer k, a nonnegative integer r, higher-order poly-tangent polynomials are defined as the following generating function

$$\left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{xt} = \sum_{n=0}^{\infty} T_n^{(k,r)}(x)\frac{t^n}{n!}.$$

$T_n^{(k,r)} = T_n^{(k,r)}(0)$ are called higher-order poly-tangent numbers when $x = 0$. If we set $k = 1$ and $r = 1$ in Definition 2.1, then the higher-order poly-tangent polynomials are reduced to classical tangent polynomials because of $Li_1(1-e^{-t}) = t$. That is, $T_n^{(1,1)}(x) = \mathbf{T}_n(x)$. Especially when $k = 1$, we get $T_n^{(1,r)}(x) = \mathbf{T}_n^{(r)}(x)$.

Theorem 2.2. For any integer k and a nonnegative integer r, n, and m, we get

$$T_n^{(k,r)}(mx) = \sum_{l=0}^{n} \binom{n}{l} T_l^{(k,r)} m^{n-l} x^{n-l}.$$

Proof. From Definition 1.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n^{(k,r)}(mx)\frac{t^n}{n!} &= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{mxt} \\
&= \left(\sum_{n=0}^{\infty} T_n^{(k,r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(mx)^n\frac{t^n}{n!}\right) \qquad (1)\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} T_l^{(k,r)} m^{n-l}x^{n-l}\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, we finish the proof of Theorem 2.2 by comparing the coefficients of $\frac{t^n}{n!}$. □

If $m = 1$ in Theorem 2.2, then we get the following corollary.

Corollary 2.3. For any integer k and a nonnegative integer r and n, we have

$$T_n^{(k,r)}(x) = \sum_{l=0}^{n} \binom{n}{l} T_l^{(k,r)}\, x^{n-l}.$$

Theorem 2.4. For any integer k and a nonnegative integer r, n, and m, we obtain

$$T_n^{(k,r)}(mx) = \sum_{l=0}^{n} \binom{n}{l} T_l^{(k,r)}(x)\,(m-1)^{n-l} x^{n-l}.$$

Proof. By utlizing Definition 1.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n^{(k,r)}(mx)\frac{t^n}{n!} &= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{xt} e^{(m-1)xt} \\
&= \left(\sum_{n=0}^{\infty} T_n^{(k,r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(m-1)^n x^n \frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} T_l^{(k,r)}(x)\,(m-1)^{n-l}x^{n-l}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2}$$

Therefore, we end the proof by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation (2). $\square$

Theorem 2.5. For any integer k and a nonnegative integer r and n, we get

$$(1)\ \ T_n^{(k,r)}(x+y) = \sum_{l=0}^{n}\binom{n}{l} T_l^{(k,r)}(x) y^{n-l}.$$

$$(2)\ \ T_n^{(k,r+s)}(x+y) = \sum_{l=0}^{n}\binom{n}{l} T_n^{(k,r)}(x) T_n^{(k,s)}(y).$$

Proof. (1) Proof is omitted since it is a similar method of Theorem 2.2.

(2) From Definition 1.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_n^{(k,r+s)}(x+y)\frac{t^n}{n!} \\
&= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)^{r+s} e^{(x+y)t} \\
&= \left(\sum_{n=0}^{\infty} T_n^{(k,r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} T_n^{(k,s)}(y)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} T_l^{(k,r)}(x) T_{n-l}^{(k,s)}(y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{3}$$

Therefore, we end the proof by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation (3). $\square$

Theorem 2.6. For any integer k, a nonnegative integer r, and a positive integer n, we have

$$T_n^{(k,r)}(x+1) - T_n^{(k,r)}(x) = \sum_{l=0}^{n-1} \binom{n}{l} T_l^{(k,r)}(x).$$

Proof. By using Definition 1.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_n^{(k,r)}(x+1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} T_n^{(k,r)}(x)\frac{t^n}{n!} \\
&= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{(x+1)t} - \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{xt} \\
&= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)^r e^{xt}\left(e^t - 1\right) \\
&= \left(\sum_{n=0}^{\infty} T_n^{(k,r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{n!} - 1\right) \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n+1}{l} T_l^{(k,r)}(x)\frac{t^{n+1}}{(n+1)!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{l=0}^{n-1}\binom{n}{l} T_l^{(k,r)}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{4}$$

Then we compare the coefficients of $\frac{t^n}{n!}$ for $n \geq 1$. The reason both sides of the above equation (4) can be compared the coefficients is that $T_0^{(k,r)}(x+1) - T_0^{(k,r)}(x) = 0$. Thus, the proof is done. □

3. Polynomials and numbers related to higher-order poly-tangent polynomials and its symmtric property

In this section, we examine the association between higher-order poly-tangent polynomials and poly-tangent polynomials using Cauchy product. We also explore relation of higher-order poly-tangent polynomials and Stirling numbers of the second kind. Furthermore, we study the symmetry properties of higher-order poly-tangent polynomials.

We recall a multinomial coefficient, which is

$$\binom{n}{m_1, m_2, \cdots, m_l} = \frac{n!}{m_1! m_2! \cdots m_l!}. \tag{5}$$

Let us consider the following equation using the equation (5) above. This equation is an equation expressed by applying Cauchy product continuously.

Theorem 3.1. For any integer k, a nonnegative integer n, and $r \geq 3$, we get

$$\begin{aligned} T_n^{(k,r)}(rx) = & \sum_{m_{r-1}=0}^{n} \sum_{m_{r-2}=0}^{m_{r-1}} \cdots \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} \\ & \times \binom{n}{m_1, m_2 - m_1, \cdots, m_{r-1} - m_{r-2}, n - m_{r-1}} T_{m_1}^{(k)}(x) \\ & \times T_{m_2-m_1}^{(k)}(x) \cdots T_{m_{r-1}-m_{r-2}}^{(k)}(x)\, T_{n-m_{r-1}}^{(k)}(x), \end{aligned}$$

where $\binom{n}{m_1, m_2, \cdots, m_l}$ is multinomial coefficient.

Generating function of the Stirling numbers of the second kind $S_2(n,k)$ is defined as follows:

$$\sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}.$$

Theorem 3.2. For any integer k, a nonnegative integer r and a positive integer n, we obtain

$$T_n^{(k,r)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (x)_m S_2(l,m) T_{n-l}^{(k,r)},$$

where $(x)_m = x(x-1)\cdots(x-m+1)$ is falling factorial.

Proof. From Definition 1.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_n^{(k,r)}(x) \frac{t^n}{n!} &= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)} \right)^r e^{xt} \\ &= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)} \right)^r \sum_{m=0}^{\infty} (x)_m \frac{(e^t-1)^m}{m!} \\ &= \left(\sum_{n=0}^{\infty} T_n^{(k,r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (x)_m\, S_2(n,m) \frac{t^n}{n!} \right) \qquad (6) \\ &= \left(\sum_{n=0}^{\infty} T_n^{(k,r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{m=0}^{n} (x)_m\, S_2(n,m) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (x)_m S_2(l,m) T_{n-l}^{(k,r)} \right) \frac{t^n}{n!}. \end{aligned}$$

Thus, we finish the proof by comparing the coefficients of $\frac{t^n}{n!}$. □

Theorem 3.3. Let r and n be a nonnegative integer and $w_1, w_2 > 0\,(w_1 \neq w_2)$. Then we have

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n}{l}w_1^l w_2^{n-l}T_l^{(k,r)}(w_2x)T_{n-l}^{(k,r)}(w_1x)\\&=\sum_{l=0}^{n}\binom{n}{l}w_2^l w_1^{n-l}T_l^{(k,r)}(w_1x)T_{n-l}^{(k,r)}(w_2x).\end{aligned}$$

Proof. Let us consider the function

$$F(t)=\left(\frac{4Li_k(1-e^{-w_1t})Li_k(1-e^{-w_2t})}{t^2(e^{2w_1t}+1)(e^{2w_2t}+1)}\right)^r e^{2w_1w_2xt}. \tag{7}$$

Then we obtain

$$\begin{aligned}F(t)&=\left(\frac{2Li_k(1-e^{-w_1t})}{t(e^{2w_1t}+1)}\right)^r e^{w_1w_2xt}\left(\frac{2Li_k(1-e^{-w_2t})}{t(e^{2w_2t}+1)}\right)^r e^{w_1w_2xt}\\&=\left(\sum_{n=0}^{\infty}w_1^{n+r}T_n^{(k,r)}(w_2x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}w_2^{n+r}T_n^{(k,r)}(w_1x)\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}w_1^{l+r}w_2^{n-l+r}T_l^{(k,r)}(w_2x)T_{n-l}^{(k,r)}(w_1x)\right)\frac{t^n}{n!}.\end{aligned} \tag{8}$$

By calculating in the same way as the above equation (8), we can get

$$F(t)=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}w_2^{l+r}w_1^{n-l+r}T_l^{(k,r)}(w_1x)T_{n-l}^{(k,r)}(w_2x)\right)\frac{t^n}{n!}. \tag{9}$$

The proof is complete as a result of the equations (8) and (9). □

Let w is an odd number. Then we can easily see

$$\sum_{n=0}^{\infty}\tilde{A}_n(w)\frac{t^n}{n!}=\frac{e^{wt}+1}{e^t+1}, \tag{10}$$

where $\tilde{A}_n(w)=\sum_{l=0}^{w-1}(-1)^l\, l^n$ is called alternating power sum.

Theorem 3.4. Let w_1 and w_2 be an odd number and n be a nonnegative integer. Then we have

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}2^{n-j-l}w_1^{i+l+r}w_2^{2n-2j-i-l+r}T_i^{(k,r)}\\&\qquad\times T_{n-j-i}^{(k,r)}\mathbf{T}_l(w_2x)\tilde{A}_{n-j-l}(w_1)\\&=\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}2^{n-j-l}w_2^{i+l+r}w_1^{2n-2j-i-l+r}T_i^{(k,r)}\\&\qquad\times T_{n-j-i}^{(k,r)}\mathbf{T}_l(w_1x)\tilde{A}_{n-j-l}(w_2).\end{aligned}$$

Proof. First, let us assume that

$$G(t)=2\frac{4^r(Li_k(1-e^{-w_1t}))^r(Li_k(1-e^{-w_2t}))^r(e^{2w_1w_2t}+1)}{t^{2r}\left(e^{2w_1t}+1\right)^r\left(e^{2w_2t}+1\right)^r(e^{2w_1t}+1)(e^{2w_2t}+1)}e^{2w_1w_2xt}. \tag{11}$$

Then we calculate

$$\begin{aligned}
G(t)&=2\left(\frac{2Li_k(1-e^{-w_1t})}{t(e^{2w_1t}+1)}\right)^r\left(\frac{2Li_k(1-e^{-w_2t})}{t(e^{2w_2t}+1)}\right)^r\\
&\quad\times\frac{2}{(e^{2w_1t}+1)}e^{2w_1w_2xt}\frac{e^{2w_1w_2t}+1}{e^{2w_2t}+1}\\
&=\left(\sum_{n=0}^{\infty}w_1^{n+r}T_n^{(k,r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}w_2^{n+r}T_n^{(k.r)}\frac{t^n}{n!}\right)\\
&\quad\times\left(\sum_{n=0}^{\infty}w_1^nT_n(w_2x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}2^nw_2^n\tilde{A}_n(w_1)\frac{t^n}{n!}\right)\\
&=\left(\sum_{n=0}^{\infty}w_1^{n+r}T_n^{(k,r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}w_2^{n+r}T_n^{(k,r)}\frac{t^n}{n!}\right)\\
&\quad\times\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2^{n-l}w_1^lw_2^{n-l}\mathbf{T}_l(w_2x)\tilde{A}_{n-l}(w_1)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\Bigg(\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}2^{n-j-l}w_1^{i+l+r}w_2^{2n-2j-i-l+r}\\
&\qquad\times T_i^{(k,r)}T_{n-j-i}^{(k,r)}\mathbf{T}_l(w_2x)\tilde{A}_{n-j-l}(w_1)\Bigg)\frac{t^n}{n!}.
\end{aligned} \tag{12}$$

In a similar way to the above equation (12), we get

$$\begin{aligned}
G(t)&=\left(\sum_{n=0}^{\infty}w_1^{n+r}T_n^{(k,r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}w_2^{n+r}T_n^{(k,r)}\frac{t^n}{n!}\right)\\
&\quad\times\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2^{n-l}w_2^lw_1^{n-l}\mathbf{T}_l(w_1x)\tilde{A}_{n-l}(w_2)\frac{t^n}{n!}.
\end{aligned} \tag{13}$$

Hence, by using Cauchy product, the proof is complete by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equations (12) and (13). $\square$

Corollary 3.5. Let w_1 and w_2 be an odd number and n be a nonnegative integer. Then we have

$$\sum_{l=0}^{n}\binom{n}{l}2^{n-l}w_1^lw_2^{n-l}\mathbf{T}_l(w_2x)\tilde{A}_{n-l}(w_1)=\sum_{l=0}^{n}\binom{n}{l}2^{n-l}w_2^lw_1^{n-l}\mathbf{T}_l(w_1x)\tilde{A}_{n-l}(w_2).$$

4. Distribution of zeros of the higher-order poly-tangent polynomials

Using generating functions, the generalized forms of known polynomials such as the Bernoulli, Euler, falling factorial and tangent polynomials are studied. In particular, various properties of these polynomials were investigated through numerical experiments, see for example [1] , [2], [4], [5], [6], [7], [8], [9], [10].

In this section, we discover new interesting pattern of the zeros of the ghigher-order poly-Euler polynomials $T_n^{(k,r)}(x)$. We propose some conjectures by numerical experiments. The higher-order poly-tangent polynomials $T_n^{(k,r)}(x)$ can be determined explicitly.

A few of them are

$$
\begin{aligned}
T_0^{(k,r)}(x) &= 1,\\
T_1^{(k,r)}(x) &= -\frac{3r}{2} + 2^{-k}r + x,\\
T_2^{(k,r)}(x) &= -\frac{11r}{12} - 2^{2-k}r - 2^{-2k}r + 3\cdot 2^{-k}r + 2\cdot 3^{-k}r\\
&\quad + \frac{9r^2}{4} + 2^{-2k}r^2 - 3\cdot 2^{-k}r^2 - 3rx + 2^{1-k}rx + x^2,\\
T_3^{(k,r)}(x) &= 2^{1-3k}r + 3\cdot 2^{1-2k}r + 3\cdot 2^{2-2k}r - 9\cdot 2^{1-k}r - 2^{2-k}r - 9\cdot 2^{-2k}r\\
&\quad + 23\cdot 2^{-k}r - 5\cdot 3^{1-k}r + 3^{2-k}r - 6^{1-k}r + \frac{33r^2}{8} - 2^{1-3k}r^2\\
&\quad + 9\cdot 2^{-1-2k}r^2 - 3\cdot 2^{2-2k}r^2 - 27\cdot 2^{-2-k}r^2 - 27\cdot 2^{-1-k}r^2\\
&\quad + 9\cdot 2^{1-k}r^2 + 2^{2-k}r^2 - 2^{-3k}r^2 + 9\cdot 2^{-2k}r^2 - 3^{2-k}r^2\\
&\quad + 6^{1-k}r^2 - \frac{27r^3}{8} - 9\cdot 2^{-1-2k}r^3 + 27\cdot 2^{-2-k}r^3 + 2^{-3k}r^3\\
&\quad - \frac{11rx}{4} - 3\cdot 2^{2-k}rx - 3\cdot 2^{-2k}rx + 9\cdot 2^{-k}rx + 2\cdot 3^{1-k}rx\\
&\quad + \frac{27r^2x}{4} + 3\cdot 2^{-2k}r^2x - 9\cdot 2^{-k}r^2x - \frac{9rx^2}{2} + 3\cdot 2^{-k}rx^2 + x^3.
\end{aligned}
$$

We investigate the beautiful zeros of the higher-order poly-tangent polynomials $T_n^{(k,r)}(x)$ by using a computer. We plot the zeros of the ghigher-order poly-tangent polynomials $T_n^{(k,r)}(x)$ for $n = 50, r = 3, k = -2, -1, 1, 2$ and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 50$ and $k = -2$. In Figure 1(top-right), we choose $n = 50$ and $k = -1$. In Figure 1(bottom-left), we choose $n = 50$, and $k = 1$. In Figure 1(bottom-right), we choose $n = 50$ and $k = 2$.

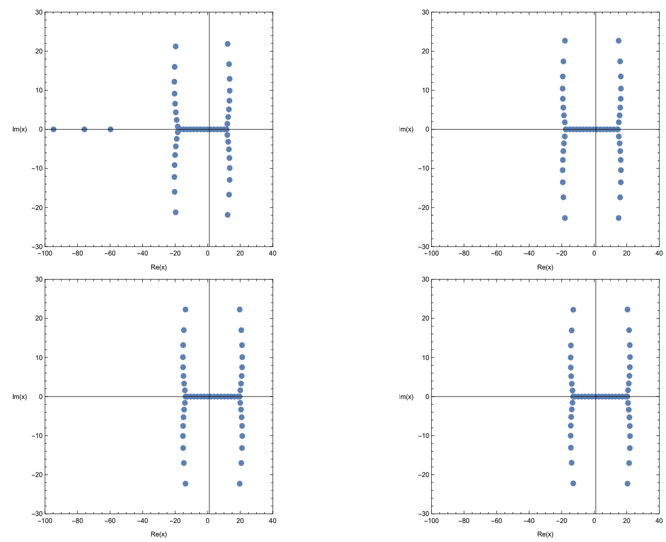

FIGURE 1. Zeros of $T_n^{(k,r)}(x)$

Stacks of zeros of $T_n^{(k,r)}(x)$ for $1 \leq n \leq 50$ from a 3-D structure are presented(Figure 2).

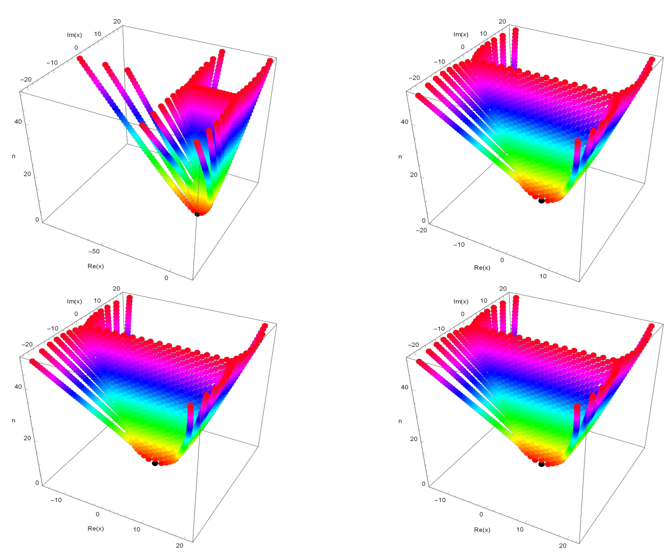

FIGURE 2. Stacks of zeros of $T_n^{(k,r)}(x)$ for $1 \leq n \leq 50$

In Figure 2(top-left), we choose $k = -2$ and $r = 3$. In Figure 2(top-right), we choose $k = -1$ and $r = 3$. In Figure 2(bottom-left), we choose $k = 1$ and $r = 3$. In Figure 2(bottom-right), we choose $k = 2$ and $r = 3$.

We plot the real zeros of the higher-order poly-tangent polynomials $T_n^{(k,r)}(x)$.with variable a for $a = 4$ and $x \in \mathbb{C}$(Figure 3).

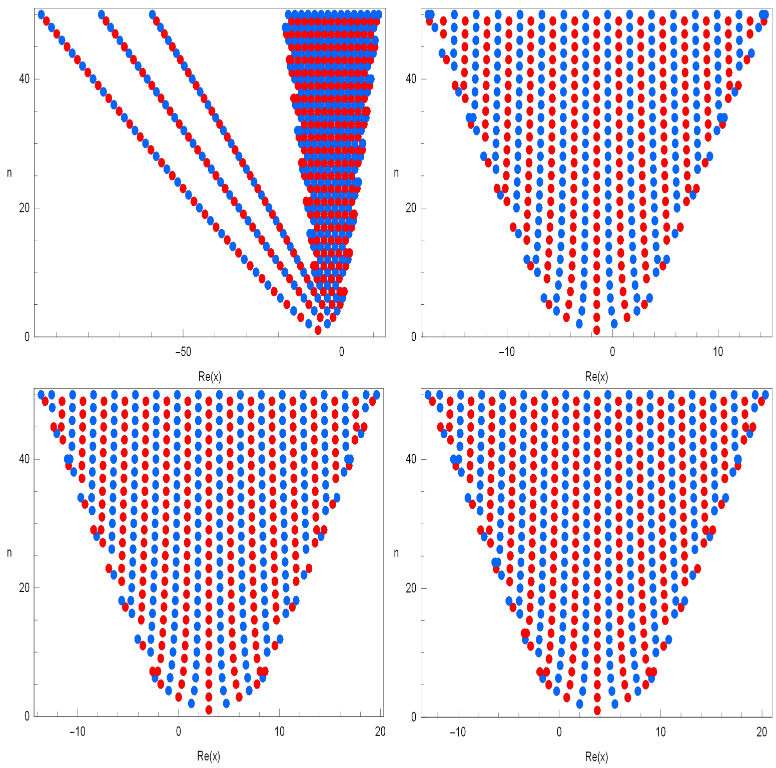

FIGURE 3. Real zeros of $T_n^{(k,r)}(x)$ for $1 \leq n \leq 50$

In Figure 3(top-left), we choose $k = -2$ and $r = 3$. In Figure 3(top-right), we choose $k = -1$ and $r = 3$. In Figure 3(bottom-left), we choose $k = 1$ and $r = 3$. In Figure 3(bottom-right), we choose $k = 2$ and $r = 3$.

Next, we calculated an approximate solution satisfying higher-order poly-tangent polynomials $T_n^{(k,r)}(x)$ for $x \in \mathbb{R}$. The results are given in Table 1 and Table 2.

Table 1. Approximate solutions of $T_n^{(k,r)}(x) = 0, k = -2, r = 3$

degree n	x
1	-7.5000
2	-10.458, -4.5420
3	-12.937, -6.8013, -2.7617
4	-15.194, -8.6881, -4.6321, -1.4859
5	-17.322, -10.385, -6.2187, -3.0715, -0.50314
6	-19.365, -11.977, -7.5932, -4.5455, -1.7777, 0.25835
7	-21.349, -13.509, -8.8139, -5.8451, -3.1839, -0.59737, 0.79828
8	-23.287, -15.009, -9.9300, -6.9667, -4.5063, -1.9418

Table 2. Approximate solutions of $T_n^{(k,r)}(x) = 0, k = 2, r = 3$

degree n	x
1	3.7500
2	2.0119, 5.4881
3	0.73788, 3.7534, 6.7587
4	-0.26000, 2.3120, 5.1969, 7.7511
5	-1.0470, 1.0576, 3.7557, 6.4540, 8.5297
6	-1.6346, -0.083322, 2.4436, 5.0695, 7.6025, 9.1023
7	-1.9217, -1.2552, 1.2301, 3.7572, 6.2850, 8.8345, 9.3201
8	0.085506, 2.5170, 4.9987, 7.4315

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

References

1. R. Ayoub, *Euler and zeta function*, Amer. Math. Monthly **81** (1974), 1067-1086.
2. N.S. Jung, C.S. Ryoo, *Identities involving q-analogue of modified tangent polynomials*, J. Appl. Math. & Informatics **39** (2021), 643-654. https://doi.org/10.14317/jami.2021.643
3. N.S. Jung, C.S. Ryoo, *Numerical investigation of zeros of the fully q-poly-tangent numbers and polynomials of the second type*, J. Appl. & Pure Math. **3** (2021), 137-150.
4. Jung Yoog Kang, *Some properties involving (p,q)-Hermite polynomials arising from differential equations*, J. Appl. & Pure Math. **4** (2022), 221-231. https://doi.org/10.23091/japm.2022.221
5. C.S. Ryoo, *A note on the tangent numbers and polynomials*, Adv. Studies Theor. Phys. **7** (2013), 447-454.
6. C.S. Ryoo, R.P. Agarwal, *Some identities involving q-poly-tangent numbers and polynomials and distribution of their zeros*, Advances in Difference Equations **2017** (2017), 2017:213. DOI 10.1186/s13662-017-1275-2
7. C.S. Ryoo, *Some properties of poly-cosine tangent and poly-sine tangent polynomials* , Adv. Studies Theor. Phys. **8** (2014), 457-462.
8. C.S. Ryoo, *Multiple tangent zeta Function and tangent polynomials of higher order*, J. Appl. Math. & Informatics **40** (2022), 371-391. https://doi.org/10.14317/jami.2022.371
9. C.S. Ryoo, J.Y. Kang, *Properties of q-differential equations of higher order and visualization of fractal using q-Bernoulli polynomials*, Fractal Fract. **2022** (2022), 296. https://doi.org/10.3390/fractalfract6060296
10. H. Shin, J. Zeng, *The q-tangent and q-secant numbers via continued fractions*, European J. Combin. **31** (2010), 1689-1705.

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