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ASYMPTOTIC PROPERTIES OF THE CONDITIONAL HAZARD FUNCTION ESTIMATE BY THE LOCAL LINEAR METHOD FOR FUNCTIONAL ERGODIC DATA

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Abstract. This article introduces a method for estimating the conditional hazard function of a real-valued response variable based on a functional variable. The method uses local linear estimation of the conditional density and cumulative distribution function and is applied to a functional stationary ergodic process where the explanatory variable is in a semi-metric space and the response is a scalar value. We also examine the uniform almost complete convergence of this estimation technique.

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1. Introduction

The hazard function is an important statistical tool used in risk analysis and survival analysis. In nonparametric estimation, its accuracy is of great interest, especially when dealing with high-dimensional data. Functional Data Analysis (FDA) is a subfield that deals with such complex data, including curves and surfaces. Pioneering work on in this field has been done by [7], as well as [8].

The conditional hazard rate is a vital component of statistics, with many applications in various fields. A lot of research has been done on the estimation of the hazard function for both independent and dependent mixing data, with studies like [11]-[10]. [6] introduced a kernel estimator for the conditional hazard rate in infinite-dimensional space for functional covariates and established various asymptotic properties.

The relationship between a variable of interest and a functional covariate is

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critical in nonparametric statistical issues. [15] provided and examined the estimator of the conditional density and distribution when data is functional and studied its almost complete convergence using the local linear method. In recent years, a lot of research has been done on the local linear method, including the exploration of the quadratic error of the local linear estimator of the conditional density in [21] and the local linear modelization of the conditional density and distribution function for functional ergodic data in [26]-[25].

The functional ergodic processes have the property that one sample of the process reflects the entire set. [22] researched the rates of high consistencies of the regression function estimator, while [5] investigated the almost entire rate convergence of the functional recursive kernel of the conditional quantile. In this work, we focus on the almost-complete convergence with rates of the local linear estimator of the conditional hazard function. We assume that the data are samples from a stable ergodic process, and the covariate takes its values in an infinite-dimensional space.

We present the estimator of the conditional hazard function and the some notations used in Section 2, list some assumptions in Section 3, and provide our asymptotic properties results with proofs in Section 4, In Section 5 and the last we present a numerical simulation using python.

2. Model Formulation

We consider a sequence $(X_i, Y_i)_{i=1,\ldots,n}$ of strictly stationary ergodic processes defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, where $X_i \in \mathcal{F}$, a semi-metric space with semi-metric d, and $Y_i \in \mathbb{R}$. The definition proposed by [22] is used in this context. Our goal is to establish asymptotic conclusions on the concentration qualities of the probability measure of the functional variable within small balls, which will provide good mathematical properties of functional nonparametric approaches. Let C, C', and C_1, C_2, ... be positive constants, x be a fixed point in $\mathcal{F}$, N_x be a fixed neighborhood of x, and $\mathcal{S}$ be a fixed compact subset of R. We aim to estimate the conditional hazard function h^x, where F^x denotes the conditional distribution function of Y given $X = x$. We assume that F^x has a continuous density f^x with respect to the Lebesgue measure on R. The hazard function h^x for $y \in$ R and $F^x(y) < 1$ is defined by:

$$h^x(y) = \frac{f^x(y)}{1 - F^x(y)}.$$

To estimate the conditional distribution function F^x, we develop the following local linear method:

$$\widehat{F}^x(y) = \frac{\sum_{i=1}^n \Gamma_i(x) K(h_K^{-1} d(x, X_j)) H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^n \Gamma_i(x) K(h_K^{-1} d(x, X_j))} = \frac{\sum_{i=1}^n \Gamma_i K_i H_i}{\sum_{i=1}^n \Gamma_i K_i}, \quad (1)$$

where K is the kernel, H is a distribution function and $h_K = h_{K,n}$ (respectively. $h_H = h_{H,n}$) are a sequence of positive real numbers.

whith

$$\Gamma_i(x) = \sum_{j=1}^{n} \rho_j^2(x) K_j(x) - (\sum_{j=1}^{n} \rho_j(x) K_j(x)) \rho_i,$$

where $\rho_j(x) = \rho(X_j, x)$, $K_j(x) = K(h_K^{-1} d(x, X_j))$, and $H_j(y) = H(h_H^{-1}(y - Y_i))$. Thus, the conditional density estimator, noted $\widehat{f}^x$, using the local linear method is defined by:

$$\widehat{f}^x(y) = \frac{h_H^{-1} \sum_{i,j=1}^n W_{ij}(x) H(h_H^{-1}(y - Y_i))}{\sum_{i,j=1}^n W_{ij}(x)} = \frac{h_H^{-1} \sum_{i=1}^n \Gamma_i K_i H_i}{\sum_{i=1}^n \Gamma_i K_i}. \tag{2}$$

Where

$$W_{ij}(x) = \rho(X_i, x)(\rho(X_i, x) - \rho(X_j, x)) K(h_K^{-1} d(x, X_i)) K(h_K^{-1} d(x, X_j)),$$

whith

$$\Gamma_i(x) = K_i^{-1}(\sum_{j=1}^{n} W_{ij}) = \rho_j^2 K_j - (\sum_{j=1}^{n} \rho_j K_j) \rho_i,$$

where $\rho_i = \rho(X_i, x)$, $K_i = K(h_K^{-1} d(x, X_i))$, and $H_i = H(h_H^{-1}(y - Y_i))$.
Basing on the equation (1) and (2), the natural estimator of the conditional hazard function, is the random variable defined by:

$$\widehat{h}^x(y) = \frac{h_H^{-1} \sum_{i=1}^n \Gamma_i K_i H_i}{\sum_{i=1}^n \Gamma_i K_i - \sum_{i=1}^n \Gamma_i K_i H_i}. \tag{3}$$

3. Assumptions and Notations

We introduce some notations to express our outcomes. Suppose that for $i = 1, ..., n$, the s-field generated by $((X_1, Y_1), ..., (X_i, Y_i))$ is denoted by $\mathfrak{F}_i$, and the s-field generated by ((X1,Y1),..., (Xi,Yi),Xi+1) is denoted by $\mathfrak{G}_i$. We also assume that the strictly stationary ergodic process $(X_i, Y_i)_{i \in \mathbb{N}}$ satisfies some conditions. In addition, let $\phi_x(h_1, h_2) = \mathbb{P}(h_2 \leq \delta(X, x) \leq h_1)$ represent the probability function of a small ball. Our consistency results are outlined in Theorem 4.1 and rely on the following five assumptions:

(H1) (i) The function $\phi(x, h) := \mathbb{P}(X \in B(x, h)) > 0$, $\forall\ h > 0$, where $B(x, h) = x' \in \mathcal{F} / d(x', x) < h$.
(ii) For all $i = 1, ..., n$ there exist a deterministic function $\phi_i(x, .)$ such that almost surely:
$0 < \mathbb{P}(X_i \in B(x, h) | \mathcal{F}_{i-1}) \leq \phi_i(x, h)$, $\forall\ h > 0$; and $\phi_i(x, h) \to 0$ as $h \to 0$.
(iii) For any $r > 0$, $\dfrac{1}{n\phi_x(h)} \sum_{i=1}^{n} \phi_{i,x}(r) \to 1$; and $n\phi_x(h) \to \infty$ as $r \to 0$.

(H2) (i) The conditional distribution function F^x is such that, $\forall y \in \mathcal{S},\ \exists \beta > 0,\ \inf_{y\in\mathcal{S}}(1 - F^x(y)) > \beta$, $\forall\ (y_1,y_2) \in \mathcal{S}\times\mathcal{S}$, $\forall\ (x_1,x_2) \in N_x \times N_x$:

$$|F^{x_1}(y_1) - F^{x_2}(y_2)| \leq C_1(d(x_1,x_2)^{\beta_1} + |y_1 - y_2|^{\beta_2}); \quad \beta_1 > 0,\ \beta_2 > 0.$$

(ii) The density f^x is such that,$\forall y \in \mathcal{S},\ \exists \beta > 0$, $f^x(y) < \beta$, $\forall\ (y_1,y_2) \in \mathcal{S}\times\mathcal{S}$, $\forall\ (x_1,x_2) \in N_x \times N_x$,

$$|f^{x_1}(y_1) - f^{x_2}(y_2)| \leq C_2(d(x_1,x_2)^{\beta_1} + |y_1 - y_2|^{\beta_2}); \quad \beta_1 > 0,\ \beta_2 > 0.$$

(H3) The function ρ satisfies the following condition:

$$\forall\ z \in \mathcal{F},\ C_1|d(x,z)| \leq |\rho(x,z)| \leq C_2|d(x,z)|.$$

(H4) (i)The kernel K is a a nonnegative function on its support $[-1,1]$ such that:

$$0 < C_1\mathbb{I}_{[-1,1]} < K(t) < C_2\mathbb{I}_{[-1,1]} < \infty,$$

where $\mathbb{I}_A$ is the indicator function.

(ii)The kernel function H is a positive, bounded, Lipschitzian continuous function such that:

$$\int |t|^{\beta_2} H^{(1)}(t)dt < \infty \text{ and } \int {H^{(1)}}^2(t)dt < \infty\ .$$

(iii) On the distribution function H:

$$\mathbb{E}(H_i(y)|\mathfrak{G}_{i-1}) = \mathbb{E}(H_i(y)|X_i).$$

(H5) (i)The bandwidths h_K are such that: $\exists n_0 > 0$ for which:

$$-\frac{1}{\phi(h_K)}\int_{-1}^{1}\phi(zh_K,h_K)\frac{d}{dz}(z^2K(z))dz > C\ ,\ \ \forall n > n_0.$$

(ii)

$$h_K\int_{B(x,h_K)}\rho(\mu,x)d\mathbb{P}(\mu) = 0(\int_{B(x,h_K)}\rho^2(\mu,x)d\mathbb{P}(\mu)),$$

where $d\mathbb{P}(x)$ is the cumulative distribution of X.

(iii)

$$\lim_{n\to\infty} h_K = 0,\ \lim_{n\to\infty} h_H = 0, \text{ and } \lim_{n\to\infty}\frac{\log n}{n\phi(h_K)} = 0,$$

and

$$\lim_{n\to\infty} n^{\lambda}h_H = \infty,\text{and } \lim_{n\to+\infty}\frac{\log(n)}{nh_H\phi_x(h^{K=})} = 0,\ \ \forall\ \lambda > 0.$$

The hypotheses for this paper are as follows: The first assumption (H1) involves the use of small ball techniques, which are discussed in detail. The second assumption (H2) relates to the Lipschitz's condition for the conditional distribution and density functions, which are assumed to be continuous with respect to each variable. This enables us to evaluate the bias concept without relying on differentiability. The third assumption (H3) is considered to be unrestrictive. The fourth assumption (H4)(i) places regularity conditions on the kernel K used for the estimates. Finally, assumptions (H5)(i) through (ii) are concerned with the specific behavior of the smoothing parameter h_k and its relationship with the small ball probabilities and the kernel K, as well as controlling the local behavior of ρ to simulate the local shape of the model. (H5)(iii) are technical conditions. The remaining assumptions (H4)(ii) and (iii) are also a technical conditions imposed for the sake of brevity in the proofs.

4. Main results

In this section, we aim to introduce our principal result which is the asymptotics properties of the estimator $\widehat{h}^x(y)$.

Theorem 4.1. *Once the hypothesis (H1)-(H5) are met, we have*

$$\sup_{y\in\mathcal{S}}|\widehat{h}^x(y)-h^x(y)| = O\left(h_K^{b_1}+h_H^{b_2}\right)+O\left(\sqrt{\frac{\log n}{nh_H\phi_x}(h_K)}\right) \ a.co. \tag{4}$$

Proof. Before starting the proof of our main result, it is useful to note the following decomposition and subadditivity results on which the theorem 4.1 is based:

$$\begin{aligned}\widehat{h}^x(y)-h^x(y) &= \frac{\widehat{f}^x(y)}{1-\widehat{F}^x(y)}-\frac{f^x(y)}{1-F^x(y)}\\ &= \frac{\widehat{f}^x(y)-\widehat{f}^x(y)F^x(y)-f^x(y)+f^x(y)\widehat{F}^x(y)}{\left(1-\widehat{F}^x(y)\right)(1-F^x(y))}\\ &= \frac{1}{1-\widehat{F}^x(y)}\left[\left(\widehat{f}^x(y)-f^x(y)\right)+\frac{f^x(y)}{1-F^x(y)}\left(\widehat{F}^x(y)-F^x(y)\right)\right].\end{aligned}$$

□

Theorem 4.2. *Under assumptions (H1),(H2)(i) and (H3)-(H5), we have*

$$\sup_{y\in\mathcal{S}}|\widehat{F}^x(y)-F^x(y)| = O(h_K^{b_1})+O(h_H^{b_2})+O\left(\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right), \ a.co. \tag{5}$$

Theorem 4.3. *Under assumptions (H1),(H2)(ii) and (H3)-(H7), we have*

$$\sup_{y\in\mathcal{S}}|\widehat{f}^x(y)-f^x(y)| = O(h_K^{b_1})+O(h_H^{b_2})+O\left(\sqrt{\frac{\varphi(h_K)\log n}{n^2h_H\phi_x^2(h_K)}}\right), \ a.co. \tag{6}$$

Theorem 4.4. *Under assumptions of theorem 4.2, we have*

$$\exists\ \delta>0 \ \ \textit{such that} \ \sum_{n=1}^{\infty}\mathbb{P}\left\{\inf_{y\in\mathcal{S}}|1-\widehat{F}^x(y)|\le\delta\right\}<\infty. \tag{7}$$

We introduce some additional notation:

$$\widehat{F}_N^x(y)=\frac{1}{n\mathbb{E}(\Gamma_1K_1)}\sum_{i=1}^n\Gamma_iK_iH_i, \qquad \overline{F}_N^x(y)=\frac{1}{n\mathbb{E}(\Gamma_1K_1)}\sum_{i=1}^n\mathbb{E}(\Gamma_iK_iH_i|\mathfrak{F}_{i-1}),$$

$$\widehat{F}_D(x)=\frac{1}{n\mathbb{E}(\Gamma_1K_1)}\sum_{i=1}^n\Gamma_iK_i, \qquad \overline{F}_D(x)=\frac{1}{n\mathbb{E}(\Gamma_1K_1)}\sum_{i=1}^n\mathbb{E}(\Gamma_iK_i|\mathfrak{F}_{i-1}),$$

$$\widehat{f}_k^x(y) = \frac{1}{nh_H^k \mathbb{E}(\Gamma_1 K_1)} \sum_{i=1}^{n} \Gamma_i K_i H_i^k, \quad \overline{f}_k^x(y) = \frac{1}{nh_H^k \mathbb{E}(\Gamma_1 K_1)} \sum_{i=1}^{n} \mathbb{E}(\Gamma_i K_i H_i^k | \mathfrak{F}_{i-1}).$$

We can write the following decomposition:

$$\widehat{F}^x(y) - F^x(y) = \widehat{B}_n(x,y) + \frac{1}{\widehat{F}_D(x)} \left[(\widehat{B}_n(x,y) + F^x(y)) \widehat{Q}_n(x,y) + \widehat{R}_n(x,y) \right], \quad (8)$$

where
$\widehat{B}_n(x,y) = \dfrac{\overline{F}_N^x(y)}{\overline{F}_D(x)} - F^x(y), \qquad \widehat{Q}_n(x,y) = \overline{F}_D(x) - \widehat{F}_D^(x),$
$\widehat{R}_n(x,y) = \widehat{F}_N^x(y) - \overline{F}_N^x(y)$, and:

$$\begin{aligned} \widehat{f}^x(y) - f^x(y) \quad &= \left(\frac{\overline{f}_1^x(y)}{\overline{f}_0^x(y)} - f^x(y) \right) + \frac{1}{\widehat{f}_0^x} [\left(\frac{\overline{f}_1^x(y)}{\overline{f}_0^x(y)} - f^x(y) \right) \left(\overline{f}_0^x(y) - \widehat{f}_0^x(y) \right) (9) \\ &+ \left((\widehat{f}_1^x(y) - \overline{f}_1^x(y)) - f^x(y) (\widehat{f}_0^x(y) - \overline{f}_0^x(y)) \right)] \end{aligned}$$

Thus, we show the previous theorems 4.2 and 4.3 by using the following intermediate lemmas:

Lemma 4.5. *Under assumptions (H1),(H2)(i),and (H3)-(H5), we have that*

$$\sup_{y \in \mathcal{S}} \left| \widehat{B}_n(x,y) \right| = O\left(h_K^{b_1} + h_H^{b_2} \right) \ a.co.$$

Lemma 4.6. *Under assumptions (H1),(H2)(i) and (H3)-(H5), we obtain*

$$\sup_{y \in \mathcal{S}} \left| \widehat{R}_n(x,y) \right| = O\left(\sqrt{\frac{\log(n)}{n\phi(h_K)}} \right) \ a.co.$$

Lemma 4.7. *Under assumptions (H1)-(H4)(i),and (H5), we get*

$$\sup_{y \in \mathcal{S}} \left| \widehat{Q}_n(x,y) \right| = O\left(\sqrt{\frac{\log(n)}{n\phi(h_K)}} \right) \ a.co,$$

Corollary 4.8. *Under the assumptions of Lemma 4.7, we have*

$$\sum_{n=1}^{\infty} \mathbb{P}(\inf_{x \in \mathcal{C}_{\mathcal{F}}} \widehat{F}_D(x) < \frac{1}{2}) < \infty.$$

Lemma 4.9. *Under the assumptions (H1),(H2)(ii),(H3), and (H4), we have*

$$(i) \quad \sup_{y \in \mathcal{S}} \left| (\frac{\overline{f}_1^x(y)}{\overline{f}_0^x(y)} - f^x(y)) \right| = O(1).$$

$$(ii) \quad \sup_{y \in \mathcal{S}} \left| (\frac{\overline{f}_1^x(y)}{\overline{f}_0^x(y)} - f^x(y)) \right| = O\left(h_K^{b_1} \right) + O\left(h_H^{b_2} \right).$$

Lemma 4.10. *Under the assumptions (H1),(H3)-(H5), we have*

$$\widehat{f}_0^x(y) - \overline{f}_0^x(y) = O\left(\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2\phi_x^2(h_K)}}\right), \; a.co.$$

Lemma 4.11. *Under the assumptions of lemma 4.10, we have*

$$\exists C > 0 \quad such\ that \quad \sum_{n=1}^{\infty} \mathbb{P}\left(\widehat{f}_0^x(y) < C\right) < \infty.$$

Lemma 4.12. *Under the assumptions (H1),(H2)(i),(H3)-(H5), we have*

$$\sup_{y\in\wp}\left|\widehat{f}_1^x(y) - \overline{f}_1^x(y)\right| = O\left(\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2 h_H\phi_x^2(h_K)}}\right), \; a.co.$$

First, we provide the following initial technical lemmas that are necessary to support our asymptotic properties results.

Lemma 4.13. *Under the assumptions (H1),(H3) and (H4)(i), we have:* $\forall(k,l) \in \mathbb{N}^* \times \mathbb{N}$

$$\begin{aligned}
(i) \quad & \mathbb{E}(K_i^k|\rho_i|^l|\mathfrak{F}_{i-1}) \le Ch_K^l\phi_{i,x}(h_K),\\
(ii) \quad & \mathbb{E}(\Gamma_i K_i|\mathfrak{F}_{i-1}) = O\left(nh_K^2\phi_{i,x}(h_K)\right),\\
(iii) \quad & \mathbb{E}(\Gamma_l K_l) = O\left(nh_K^2\phi_x(h_K)\right).
\end{aligned}$$

Proof. For this last lemma 4.13: Part(i); On starts by using (H3) followed by using (H4), we get

$$K_i^k|\phi_i|^l h_K^{-1} \le CK_i^k|d(X_i,x)|^l h_K^{-1} \le C|d(X_i,x)|^l h_K^{-1}\mathbb{I}_{[-1,1]}\left(d(X_i,x)\right),$$

and thereby, $\mathbb{E}\left(K_i^k|\phi_i|^l h_K^{-1}|\mathfrak{F}_{i-1}\right) \le C\mathbb{P}\left(X_i \in B(x,h_k)|\mathfrak{F}_{i-1}\right) \le C\phi_{i,x}(h_K)$, which is the alleged result.

Part(ii); Recalling that because the Kernel K is bounded on $[-1,1]$ and under (H3), we get

$$|\Gamma_i| \le nCh_K^2 + nCh_K|\rho_i|.$$

by using (i), we find

$$\mathbb{E}(\Gamma_i K_i|\mathfrak{F}_{i-1}) \le nC_1h_K^2\phi_{i,x}(h_K) + nC_2h_K^2\phi_{i,x}(h_K) \le nCh_K^2\phi_{i,x}(h_K).$$

Part(iii) of this lemma is directly verified by combining it with (H)(iii) and by treating $\mathfrak{F}_i$ as the trivial $\sigma-$ filed. ☐

Lemma 4.14. *Under the assumptions (H1),(H3),and (H4)(i), we have:* $\forall(k,l) \in \mathbb{N}^* \times \mathbb{N}$

$$\lim_{n\to\infty} \overline{f}_0^x(y) = O(1).$$

Proof. We begin by using parts (ii) and (iii) of Lemma 4.13 to obtain

$$\lim_{n\to\infty} \overline{f}_0^x(y) = O(1) \lim_{n\to\infty} \frac{1}{n\phi_x(h_K)} \sum_{i=1}^{n} \phi_{i,x}(h_K).$$

Finally, the applied part(iii)of (H1) to get the desirable results. □

Lemma 4.15. *Under the assumptions (H1),(H2)(i), (H3), and (H4), we have*

$$(i) \sup_{x\in\mathcal{S}} \overline{F}_D(x) = O(1), \qquad (ii) \inf_{x\in\mathcal{S}} \overline{F}_D(x) = O(1).$$

Proof. Before beginning the proof of (i), it is evident that using Lemma A from [14], we obtain the following:

$$nC_1 h_K^2 \phi(h_K) \leq \mathbb{E}(\Gamma_1(x)K_1(x)) \leq nC_2 h_K^2 \phi(h_K). \tag{10}$$

Then, by considering Lemma 4.13 , we get

$$\sup_{x\in\mathcal{S}} \overline{F}_D(x) = O(1) \sup_{x\in\mathcal{S}} \frac{1}{n\phi(h_K)} \sum_{i=1}^{n} \phi_i(h_K).$$

Therefore, the (H1)(iii) is a consequence of the asserted result (i) of this lemma. The proof of (ii) resembles that of (i). □

Lemma 4.16. *Under the assumptions (H1),(H2)(i),and (H3)-(H5), we have*

$$(i) \quad h_K \mathbb{E}(\rho_i K_i^a | \mathfrak{F}_{i-1}) = O(h_K^2 \phi_i(h_K)), \qquad \forall a > 0.$$

$$(ii) \quad \frac{1}{n\phi(h_K)} \sum_{i=1}^{n} \mathbb{E}(K_i^c | \mathfrak{F}_{i-1}) = M_c + O(1), \qquad \text{for } c = 1, 2.$$

$$\begin{aligned}(iii) \quad & \frac{1}{n\phi(h_K)} \sum_{i=1}^{n} \mathbb{E}(\Gamma_i^2 K_i^2 | \mathfrak{F}_{i-1}) \\ & = (n-1)^2 (N(1,2))^2 h_K^4 \phi^2(h_K) M_2 + O\left(h_K^4 \phi^2(h_K)\right).\end{aligned}$$

Proof. The proof of (i) and (ii) are analogous to the proof of (a) and (b) of lemma A.1 in [30].To prove (iii), We make advantage of the conditional variance's definition, to prove (iii) Consequently,

$$\frac{1}{n\phi(h_K)} \sum_{i=1}^{n} \mathbb{E}(\Gamma_i^2 K_i^2 | \mathfrak{F}_{i-1}) = \frac{1}{n\phi(h_K)} \sum_{i=1}^{n} \left(Var(\Gamma_i K_i | \mathfrak{F}_{i-1}) + (\mathbb{E}(\Gamma_i K_i | \mathfrak{F}_{i-1}))^2\right). \tag{11}$$

It is still necessary to study each concept of 11. For the first concept in this equation's right-hand side, we find

$$\begin{aligned}& Var(\Gamma_i K_i | \mathfrak{F}_{i-1}) \\ & = (n-1)\left(Var(\rho_1^2(x) K_1(x) K_i | \mathfrak{F}_{i-1}) + Var(\rho_1(x) K_1(x) \rho_i K_i | \mathfrak{F}_{i-1})\right)\end{aligned}$$

$$= (n-1)(\underbrace{\mathbb{E}(\rho_1^4(x)K_1^2(x))\mathbb{E}(K_i^2|\mathfrak{F}_{i-1})}_{T_1} - \underbrace{\big(\mathbb{E}(\rho_1^2(x)K_1(x))\mathbb{E}(K_i|\mathfrak{F}_{i-1})\big)^2}_{T_2}$$
$$+ \underbrace{\mathbb{E}(\rho_1^2(x)K_1^2(x))\mathbb{E}(\rho_i^2K_i^2|\mathfrak{F}_{i-1})}_{T_3} - \underbrace{(\mathbb{E}(\rho_1(x)K_1(x))\mathbb{E}(\rho_iK_i|\mathfrak{F}_{i-1}))^2}_{T_4}).$$

Then, by using (i) of lemma 4.13, we get : $\frac{n-1}{n\phi(h_K)}\sum_{i=1}^{n} T_i = O((n-1)h_K^4\phi(h_K))$ for $i = 1, 2, 3, 4$.

It follows that $\frac{1}{n\phi(h_K)}\sum_{i=1}^{n} Var(\Gamma_iK_i|\mathfrak{F}_{i-1}) \to 0, \qquad \text{as } n \to \infty.$

On the other hand, we must investigate the first concept on the right hand side of Equality 11 to finish the proof of (iii) in this lemma. We write

$$\frac{1}{n\phi(h_K)}\sum_{i=1}^{n}\left(\mathbb{E}(\Gamma_iK_i|\mathfrak{F}_{i-1})\right)^2$$
$$= \frac{1}{n\phi(h_K)}\sum_{i=1}^{n}(\mathbb{E}(\sum_{j=1}^{n}\rho_j^2(x)K_j(x)K_i - \sum_{j=1}^{n}\rho_j(x)K_j(x)\rho_iK_i|\mathfrak{F}_{i-1}))^2$$
$$= \gamma_{n1} + \gamma_{n2} + \gamma_{n3},$$

where

$$\gamma_{n1} = \frac{(n-1)^2}{n\phi(h_K)}(\mathbb{E}(\rho_1^2(x)K_1(x)))^2\sum_{i=1}^{n}\left(\mathbb{E}(K_i|\mathfrak{F}_{i-1})\right)^2,$$
$$\gamma_{n2} = \frac{(n-1)^2}{n\phi(h_K)}\left(\mathbb{E}(\rho_1(x)K_1(x))\right)^2\sum_{i=1}^{n}\left(\mathbb{E}(\rho_iK_i|\mathfrak{F}_{i-1})\right)^2,$$
$$\gamma_{n3} = -\frac{2(n-1)^2}{n\phi(h_K)}\mathbb{E}(\rho_1^2(x)K_1(x))\mathbb{E}(\rho_1(x)K_1(x))\sum_{i=1}^{n}\mathbb{E}\left(K_i|\mathfrak{F}_{i-1}\right)\mathbb{E}\left(\rho_iK_i|\mathfrak{F}_{i-1}\right).$$

By using Jensen's inequality in relation to the concept γ_{n1}, we obtain

$$\gamma_{n1} \leq \frac{(n-1)^2}{n\phi(h_K)}\left(\mathbb{E}(\rho_1^2(x)K_1(x))\right)^2\sum_{i=1}^{n}\mathbb{E}(K_i^2|\mathfrak{F}_{i-1}),$$

We use (c) of lemma A.2 in [30] and (ii) of lemma 4.16 to get

$$\gamma_{n1} = (n-1)^2\left((N(1,2))^2h_K^4\phi^2(h_K)M_2 + O(h_K^4\phi^2(h_K))\right). \qquad (12)$$

Concerning the concept γ_{n2}, we use (b) of lemma A.1 in [30] and (i) of lemma 4.16 to produce

$$\gamma_{n2} = O\left((n-1)^2h_K^4\phi(h_K)\right). \qquad (13)$$

For the concept γ_{n3}, we use (i) of lemma A in [14], (i) of lemma 4.13, and (i) of lemma 4.16 to obtain

$$\gamma_{n3} = O\left((n-1)^2 h_K^5 \phi(h_K)\right). \tag{14}$$

Combining 12, 13, and 14 enables one to reach the desired result. □

The next, by Following the above technical lemmas and Asymptotic properties, we aim to proof our main results:
We prove the first lemma 4.5;

Proof. We begin by composing

$$\sup_{y\in\mathcal{S}}\left|\widehat{B}_n(x,y)\right| = \sup_{y\in\mathcal{S}}\left|\widehat{B}_n(x,y)\right| / \inf_{y\in\mathcal{S}}\left|\overline{F}_D(x)\right|,$$

where $\widetilde{B}_n(x,y) = \overline{F}_N^x(y) - F^x(y)\overline{F}_D(x)$. First, observe that $\widetilde{B}_n(x,y)$ can be written as

$$\begin{aligned}
\widetilde{B}_n(x,y) &= \frac{1}{n\mathbb{E}\left(\Gamma_1 K_1\right)}\sum_{i=1}^{n}\mathbb{E}\left(\Gamma_i K_i H_i|\mathfrak{F}_{i-1}\right) - F^x(y)\mathbb{E}\left(\Gamma_i K_i|\mathfrak{F}_{i-1}\right)\\
&= \frac{1}{n\mathbb{E}\left(\Gamma_1 K_1\right)}\sum_{i=1}^{n}\mathbb{E}\left(\Gamma_i K_i|\mathfrak{G}_{i-1}|\mathfrak{F}_{i-1}\right) - F^x(y)\mathbb{E}\left(\Gamma_i K_i|\mathfrak{F}_{i-1}\right)\\
&\leq \frac{1}{n\mathbb{E}(\Gamma_1 K_1)}\sum_{i=1}^{n}\{\mathbb{E}\left(\Gamma_i K_i|\mathbb{E}[H_i|X_i] - F^x(y)|\mathfrak{F}_{i-1}\right)\}. \qquad (15)
\end{aligned}$$

The assumption (H4)(iii) is used to obtain the last inequality.
The next step is the integration by parts and changing of variables

$$\mathbb{E}(H_i|X_i) = \int_{\mathbb{R}} H^{(1)}(t)F^x(y - h_H t)dt.$$

Remark 4.1. $H^{(1)}$ is probability density function.

Hence, $|\mathbb{E}(H_i|X_i) - F^x(y)| \leq \int_{\mathbb{R}} H^{(1)}(t)|F^x(y-h_H t) - F^x(y)|dt.$
Moreover, it follows by assumptions (H2)(i) and (H4)(i) that

$$\mathbb{I}_{B(x,h_K)}(X_i)|\mathbb{E}(H_i|X_i) - F^x(y)| \leq \int_{\mathbb{R}} H^{(1)}(t)(h_K^{b_1} + |t|^{b_2}h_H^{b_2})dt.$$

Under the assumption (H4)(ii), we discover that:

$$\mathbb{I}_{B(x,h_K)}(X_i)|\mathbb{E}(H_i|X_i) - F^x(y)| \leq C(h_K^{b_1} + h_H^{b_2}). \tag{16}$$

By Combining the inequality 15 with (i) of Lemma 4.15, we obtain

$$\sup_{y\in\mathcal{S}}\left|\widehat{B}_n(x,y)\right| = O\left(h_K^{b_1} + h_H^{b_2}\right)\sup_{y\in\mathcal{S}}\overline{F}_D(x),$$

Which gives the result. □

The Proof of lemma 4.6;

Proof. Firstly, since the Kernel k is constrained to $[-1,1]$ by using (H3), it is clear that:

$$|\Gamma_i(x)| \le nCh_K^2 + nCh_K|\rho_i(x)|. \tag{17}$$

Second, for all $y \in \mathcal{S}$, we have

$$\sup_{y\in\mathcal{S}}\left|\widehat{R}_n(x,y)\right|$$
$$\le \underbrace{\sup_{y\in\mathcal{S}}\left|\widehat{F}_N^x(y)-\widehat{F}_N^{x_{k(x)}}(y)\right|}_{R_1}+\underbrace{\sup_{y\in\mathcal{S}}\left|\widehat{F}_N^{x_{k(x)}}(y)-\overline{F}_N^{x_{k(x)}}(y)\right|}_{R_2}+\underbrace{\sup_{y\in\mathcal{S}}\left|\overline{F}_N^{x_{k(x)}}(y)-\overline{F}_N^{x}(y)\right|}_{R_3}.$$

with $k(x) = arg\min_{k\in 1,2,\ldots,d_n}|d(x,x_k)|$

The three concepts that make up this decomposition will now each be discussed separately. We define by the consistency of the concept R_1.
The boundeness on K and H, and By using the inequality 10,we have

$$R_1 \le \sup_{y\in\mathcal{S}}\frac{1}{n}\sum_{i=1}^{n}|H_i(y)||\frac{1}{\mathbb{E}(\Gamma_1(x)K_1(x))}\Gamma_i(x)K_i(x)\mathbb{I}_{B(x,h_K)}(X_i)$$
$$-\frac{1}{\mathbb{E}\left(\Gamma_1(x_{k(x)})K_1(x_{k(x)})\right)}\Gamma_i(x_{k(x)})K_i(x_{k(x)})\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)|$$
$$\le\left(\frac{C}{n^2h_K^2\phi(h_K)}\sup_{y\in\mathcal{S}}\sum_{i=1}^{n}|\Gamma_i(x)\mathbb{I}_{B(x,h_K)}(X_i)|\times|K_i(x)-K_i(x_{k(x)})\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)|\right)$$
$$+\left(\frac{C}{n^2h_K^2\phi(h_K)}\sup_{y\in\mathcal{S}}\sum_{i=1}^{n}|K_i(x_{k(x)})\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)|\times|\Gamma_i(x)\mathbb{I}_{B(x,h_K)}(X_i)-\Gamma_i(x_{k(x)})|\right)$$
$$: F_1+F_2.$$

Let's start by discussing the concept F_1.By using inequality 17 and the fact that Kernel k satisfies the Lipschitz condition, we can write

$$|\Gamma_i(x)\mathbb{I}_{B(x,h_K)}(X_i)||K_i(x)-K_i(x_{k(x)})\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)|$$
$$\le nCh_K^2\left(\frac{r_n}{h_K}\mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)+\mathbb{I}_{B(x,h_K)\cap\overline{(x_{k(x)},h_K)}}(X_i)\right),$$

which implies that:

$$F_1 \le \frac{Cr_n}{nh_K\phi(h_K)}\sup_{y\in\mathcal{S}}\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)$$
$$+\frac{C}{n\phi(h_K)}\sup_{y\in\mathcal{S}}\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{B(x,h_K)\cap\overline{(x_{k(x)},h_K)}}(X_i).$$

With regards to the concept F_2, we get that

$$\mathbb{I}_{B(x_{k(x)},h_K)}(X_i)|\Gamma_i(x)\mathbb{I}_{B(x,h_K)}-\Gamma_i(x_{k(x)})|$$

$$\leq \underbrace{\mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)|\Gamma_i(x)-\Gamma_i(x_{k(x)})|}_{A} + \underbrace{nCh_K^2\mathbb{I}_{B(x_{k(x)},h_K)\cap\overline{(x,h_K)}}(X_i)}_{B}.$$

Next, we determine the first element of this inequality's right side

$$\begin{aligned}A &= \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)|\sum_{i=1}^{n}\rho_i^2(x)K_i(x)-\rho_i^2(x_{k(x)})K_i(x_{k(x)})\\ &- \left(\left(\sum_{i=1}^{n}\rho_i(x)K_i(x)\right)\rho_j(x)\right)-\left(\left(\sum_{i=1}^{n}\rho_i(x_{k(x)})K_i(x_{k(x)})\right)\rho_j(x_{k(x)})\right)|\\ &\leq A_1+A_2\end{aligned}$$

where

$$A_1 = \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)|\sum_{i=1}^{n}\rho_i^2(x)K_i(x)-\rho_i^2(x_{k(x)})K_i(x_{k(x)})|$$

$$\begin{aligned}A_2 = \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)|&\left(\sum_{i=1}^{n}\rho_i(x)K_i(x)\right)\rho_j(x)\\ &-\left(\sum_{i=1}^{n}\rho_i(x_{k(x)})K_i(x_{k(x)})\right)\rho_j(x_{k(x)})|\end{aligned}$$

For examine the concepts A_1 and A_2, we put

$$\begin{aligned}T^{k,l} &= \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)|\left(\sum_{i=1}^{n}\rho_i^k(x)K_i(x)\right)\rho_j^l(x)\\ &-\left(\sum_{i=1}^{n}\rho_i^k(x_{k(x)})K_i(x_{k(x)})\right)\rho_j^l(x_{k(x)})| \quad \text{with } k=1,2 \text{ and } l=0,1.\end{aligned}$$

Therefore, $\qquad T^{k,l}\leq T_1^{k,l}+T_2^{k,l}$; with

$$T_1^{k,l} = \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)\left(\sum_{i=1}^{n}|\rho_i^k(x)|K_i(x)\times|\rho_j^l(x)-\rho_j^l(x_{k(x)})|\right),$$

$$\begin{aligned}T_2^{k,l} &= \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)\\ &\times\left(|\rho_j^l(x_{k(x)})|\times|\sum_{i=1}^{n}\rho_i^k(x)K_i(x)-\rho_i^k(x_{k(x)})K_i(x_{k(x)})|\right).\end{aligned}$$

Assuming $l=1$ and by the hypothesis (H3)(ii), we have

$$\mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)|\rho_j(x)-\rho_j(x_{k(x)})|\leq Cr_n\mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i).$$

So, for $l=0, k=2$

$$T_1^{k,l}=0 \tag{18}$$

and for $l=1, k=1$

$$T_1^{k,l} = \leq nCr_n h_K \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i). \tag{19}$$

For the concept $T_2^{k,l}$

$$T_2^{k,l} = \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)\left(\sum_{i=1}^{n}|\rho_j^l(x_{k(x)})|K_i(x)\times|\rho_i^k(x)-\rho_i^k(x_{k(x)})|\right)$$
$$+\mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)\left(\sum_{i=1}^{n}|\rho_j^l(x_{k(x)})||\rho_i^k(x_{k(x)})||K_i(x)-K_i(x_{k(x)})|\right)$$

Observe that

$$\mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)|\rho_j^2(x)-\rho_j^2(x_{k(x)})| \leq Cr_n h_K \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i),$$

which implies that for $k=1,2$

$$\mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)|\rho_j^k(x)-\rho_j^k(x_{k(x)})| \leq Cr_n h_K^{k-1} \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i).$$

Therefore, for $l=0$, and $k=2$

$$T_2^{k,l} \leq nCr_n h_K \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i). \tag{20}$$

and for $l=1$, and $k=1$

$$T_2^{k,l} \leq nCr_n h_K \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i). \tag{21}$$

Then, by combining 18 with 20, we find that:
$A_1 \leq nCr_n h_K \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)$, by combining inequalities 19 and 21, we have: $A_2 \leq nCr_n h_K \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)$. which implies that:
$A \leq nCr_n h_K \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)$. Thus,

$$F_2 \leq \frac{Cr_n}{nh_K\phi(h_K)} \sup_{y\in\mathcal{S}} \sum_{i=1}^{n} \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)$$
$$+\frac{C}{n\phi(h_K)} \sup_{y\in\mathcal{S}} \sum_{i=1}^{n} \mathbb{I}_{B(x_{k(x)},h_K)\cap\overline{(x,h_K)}}(X_i).$$

Consequently, we obtain: $R_1 \leq C \sup_{y\in\mathcal{S}}(R_{1.1}+R_{1.2}+R_{1.3})$, where

$$R_{1.1} = \frac{C}{n\phi(h_K)} \sum_{i=1}^{n} \mathbb{I}_{B(x_{k(x)},h_K)\cap\overline{(x,h_K)}}(X_i),$$
$$R_{1.2} = \frac{Cr_n}{n\phi(h_K)} \sum_{i=1}^{n} \mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i),$$
$$R_{1.3} = \frac{C}{n\phi(h_K)} \sum_{i=1}^{n} \mathbb{I}_{B(x,h_K)\cap\overline{(x_{k(x)},h_K)}}(X_i).$$

Then, using the standard inequality for sums of bounded random variables with Z_i identified, we evaluate those final concepts so that

$$Z_i = \begin{cases} \frac{1}{\phi(h_K)}[\mathbb{I}_{B(x_{k(x)},h_K)\cap\overline{(x,h_K)}}(X_i)], & \text{for } R_{1.1}; \\ \frac{r_n}{h_K\phi(h_K)}[\mathbb{I}_{B(x,h_K)\cap(x_{k(x)},h_K)}(X_i)], & \text{for } R_{1.2}; \\ \frac{1}{\phi(h_K)}[\mathbb{I}_{B(x,h_K)\cap\overline{(x_{k(x)},h_K)}}(X_i)], & \text{for } R_{1.3};. \end{cases}$$

It is clear that for $R_{1.1}$ and $R_{1.3}$, we have

$$Z_i = O\left(\frac{1}{\phi(h_K)}\right), \qquad \mathbb{E}[Z_i] = O\left(\frac{r_n}{\phi(h_K)}\right), \qquad \mathbb{E}[Z_i^2] = O\left(\frac{r_n}{\phi(h_K)^2}\right).$$

Therefore, $\quad R_{1.1} = O\left(\frac{r_n}{\phi(h_K)}\right) + O\left(\sqrt{r_n \log(n)/n\phi(h_K)^2}\right) \quad a.co.$
In a similar manner, the assumption (H5) permits getting, for $R_{1.2}$:

$$Z_i = O\left(\frac{r_n}{h_K\phi(h_K)}\right), \qquad \mathbb{E}[Z_i] = O\left(\frac{r_n}{h_K}\right), \qquad \mathbb{E}[Z_i^2] = O\left(\frac{r_n^2}{h_K^2\phi(h_K)}\right),$$

which implies that $\quad R_{1.2} = O\left(\sqrt{\log(d_n)/n\phi(h_K)}\right) \quad a.co.$
We must combine all of the intermediate results to get:
$R_1 = O\left(\sqrt{\log(d_n)/n\phi(h_K)}\right) \quad a.co.$
Conversely, the concept R_2, we have $\forall\epsilon > 0$,

$$\begin{aligned} \mathbb{P}\left(R_2 > \epsilon\sqrt{\log(d_n)/n\phi(h_K)}\right) &= \mathbb{P}\left(\max_{k\in 1,\dots n} |\widehat{F}_N^{x_{k(x)}}(y) - \overline{F}_N^{x_{k(x)}}(y)| > \epsilon\right) \\ &\le d_n \max_{k\in 1,\dots d_n} \mathbb{P}\left(|\widehat{F}_N^{x_{k(x)}}(y) - \overline{F}_N^{x_{k(x)}}(y)| > \epsilon\sqrt{\log(d_n)/n\phi(h_K)}\right). \end{aligned}$$

Let $\quad |\widehat{F}_N^{x_{k(x)}}(y) - \overline{F}_N^{x_{k(x)}}(y)| = \frac{1}{\mathbb{E}(\Gamma_1 K_1)\sum_{i=1}^{n} S_i},$

with $\quad S_i = \Gamma_i(x_{k(x)})K_i(x_{k(x)})H_i(y) - \mathbb{E}\left(\Gamma_i(x_{k(x)})K_i(x_{k(x)})H_i(y)|\mathfrak{F}_{i-1}\right)$. We obtain

$$\mathbb{E}(S_i^2|\mathfrak{F}_{i-1}) = \mathbb{E}\left((\Gamma_i K_i)^2 H_i^2|\mathfrak{F}_{i-1}\right) - \mathbb{E}\left((\Gamma_i K_i)H_i|\mathfrak{F}_{i-1}\right)^2 \le \mathbb{E}\left((\Gamma_i K_i)^2 H_i^2|\mathfrak{F}_{i-1}\right).$$

As $H_i \le 1$, we deduce that $\quad \mathbb{E}(S_i^2|\mathfrak{F}_{i-1}) \le \mathbb{E}(\Gamma_i^2 K_i^2|\mathfrak{F}_{i-1})$. By using the equation 17, we get $\quad \mathbb{E}(S_i^2|\mathfrak{F}_{i-1}) \le 2Cn^2 h_K^4(h_K)$.

Remark 4.2. S_i is an array of triangles representing the bounded martingale difference relative to the sequence of σ-fields $(\mathfrak{F}_{i-1})_{i\ge 1}$.

Next we get for every $\epsilon > 0$, using the exponential inequality of lemma 1 of [17] (with $d_i^2 = Cn^2 h_K^4(h_K)$),

$$\mathbb{P}\left(|\widehat{F}_N^{x_{k(x)}}(y) - \overline{F}_N^{x_{k(x)}}| > \epsilon\sqrt{\log(d_n)/n\phi(h_K)}\right)$$

$$= \mathbb{P}\left(|\frac{1}{n\mathbb{E}(\Gamma_1 K_i)}\sum_{i=1}^{n} S_i| > \epsilon\sqrt{\log(d_n)/n\phi(h_K)}\right)$$
$$\leq \exp -C\epsilon^2 \log(d_n).$$

Thus, by choosing ϵ such that $C\epsilon^2 = \zeta$, we obtain

$$d_n \max_{k\in 1,\ldots d_n} \mathbb{P}\left(|\widehat{F}_N^{x_{k(x)}}(y) - \overline{F}_N^{x_{k(x)}}(y)| > \epsilon\sqrt{\log(d_n)/n\phi(h_K)}\right) \leq Cd_n^{1-\zeta}.$$

Since $\sum_{n=1}^{\infty} d_n^{1-\zeta} < \infty$, we get that $\quad R_2 = O\left(\sqrt{\log(d_n)/n\phi(h_K)}\right) \quad a.co.$

For the concept R_3, we have $\quad R_3 \leq \mathbb{E}\left(\sup_{y\in\mathcal{S}} |\widehat{F}_N^{x_{k(x)}}(y) - \overline{F}_N^{x_{k(x)}}(y)||\mathfrak{F}_{\mathrm{i-1}}\right).$

We follow the same procedures as when studying the concept R_1 to determine

$$R_3 = O\left(\sqrt{\log(d_n)/n\phi(h_K)}\right) \quad a.co.$$

This is sufficient to finish the proof of Lemma 4.6. □

We Proof of lemma 4.7;

Proof. Lemma 4.6 can be used to determine this result by taking $H_i = 1$. (H4)(ii) and (H4) (iii) are not needed in this situation.

For corollary 4.8 it is clear that $\inf_{x\in\mathcal{C}_\mathcal{F}} |\widehat{F}_D(x)| \leq \frac{1}{2}$, $\exists x \in \mathcal{C}_\mathcal{F}$ so that:

$$1 - \widehat{F}_D(x) \geq \frac{1}{2} \Rightarrow \sup_{x\in\mathcal{C}_\mathcal{F}} |1 - \widehat{F}_D(x)| \geq \frac{1}{2}.$$

In accordance with this Lemma 4.7, we obtain

$$\mathbb{P}\left(\inf_{x\in\mathcal{C}_\mathcal{F}} |\widehat{F}_D(x)| \leq \frac{1}{2}\right) \leq \mathbb{P}\left(\sup_{x\in\mathcal{C}_\mathcal{F}} |1 - \widehat{F}_D(x)| \geq \frac{1}{2}\right).$$

Consequently, $\quad \mathbb{P}\left(\inf_{x\in\mathcal{C}_\mathcal{F}} |\widehat{F}_D(x)| \leq \frac{1}{2}\right) < \infty. \quad$ which end the proof. □

The Proof of lemma 4.9;

Proof. Note that

$$\frac{\overline{f}_1^x(y)}{\overline{f}_0^x(y)} - f^x(y)$$
$$= \frac{1}{nh_H\mathbb{E}(\Gamma_1 K_1)\overline{f}_0^x(y)}\sum_{i=1}^{n} \mathbb{E}(\Gamma_i K_i H_i|\mathfrak{F}_{i-1}) - h_H f^x(y)\mathbb{E}(\Gamma_i K_i|\mathfrak{F}_{i-1})$$
$$= \frac{1}{nh_H\mathbb{E}(\Gamma_1 K_1)\overline{f}_0^x(y)}\sum_{i=1}^{n} \mathbb{E}\left(\Gamma_i K_i \mathbb{E}(H_i|\mathfrak{G}_{i-1})|\mathfrak{F}_{i-1}\right) - h_H f^x(y)\mathbb{E}(\Gamma_i K_i|\mathfrak{F}_{i-1})$$

$$\leq \frac{1}{nh_H \mathbb{E}(\Gamma_1 K_1)\overline{f}_0^x(y)} \sum_{i=1}^{n} \mathbb{E}\left(\Gamma_i K_i |\mathbb{E}[H_i|X_i] - h_H f^x(y)|\mathfrak{F}_{i-1}\right).$$

Following, a integration by parts and a change of variable allows for

$$\mathbb{E}(H_i|X_i) = \int_{\mathbb{R}} H(t) f^x(y - h_H t)dt, \tag{22}$$

Thus, we have $|\mathbb{E}[H_i|X_i] - h_H f^x(y)| \leq h_H \int_{\mathbb{R}} H(t)|f^x(y - h_H t) - f^x(y)|dt$.
On one hand, we get the part (i) of Lemma 4.9, if we adopt the hypothesis (H2)(i), followed by (H4)(ii), and Lemma 4.14.
On the other hand, if we use the hypothesis (H2)(i), we obtain

$$\mathbb{I}_{B(x,h_K)}(X_i)|\mathbb{E}[H_i|X_i] - h_H f^x(y)| \leq h_H \int_{\mathbb{R}} H(t)(h_K^{b_1} + |t|^{b_2} h_H^{b_2})dt.$$

$$\begin{aligned} \overline{f}_1^x(y) - f^x(y)\overline{f}_0^x(y) &= (O(h_K^{b_1}) + O(h_H^{b_2})) \times \frac{1}{n\mathbb{E}(\Gamma_1 K_1)} \sum_{i=1}^{n} \mathbb{E}(\Gamma_i K_i|\mathfrak{F}_{i-1}) \\ &= \left(O(h_K^{b_1}) + O(h_H^{b_2})\right) \times \overline{f}_0^x(y). \end{aligned}$$

The part(ii) of Lemma 4.9 can be obtained by using Lemma 4.14. □

Now we Proof of lemma 4.10;

Proof. Let us first write

$$\begin{aligned} \widehat{f}_k^x(y) - \overline{f}_k^x(y) &= \frac{1}{nh_H^k \mathbb{E}(\Gamma_1 K_1)} \sum_{i=1}^{n} (\Gamma_i K_i H_i^k - \mathbb{E}(\Gamma_i K_i H_i^k|\mathfrak{F}_{i-1})) \\ &= \frac{1}{nh_H^k \mathbb{E}(\Gamma_1 K_1)} \sum_{i=1}^{n} T_i, \qquad \text{with } k = 0, 1, \end{aligned}$$

Given that $\mathbb{E}(\Gamma_i K_i H_i^k|\mathfrak{F}_{i-1})$ is F_{i-1} measurable, it follows that

$$\begin{aligned} \mathbb{E}(T_i^2|\mathfrak{F}_{i-1}) &= \mathbb{E}\left((\Gamma_i K_i)^2 H_i^{2k}|\mathfrak{F}_{i-1}\right) - \mathbb{E}\left((\Gamma_i K_i H_i^k|\mathfrak{F}_{i-1})^2\right) \\ &\leq \mathbb{E}\left((\Gamma_i K_i)^2 \mathbb{E}(H_i^{2k}|\mathfrak{G}_{i-1})|\mathfrak{F}_{i-1}\right) \leq \mathbb{E}\left((\Gamma_i K_i)^2 \mathbb{E}(H_i^{2k}|X_i)|\mathfrak{F}_{i-1}\right). \end{aligned}$$

Next, utilizing 22 and the hypotheses (H2)(i), we obtain $\mathbb{E}(H_i^{2k}|X_i) = O(H_i^k)$.
So, $\mathbb{E}(T_i^2|\mathfrak{F}_{i-1}) \leq Ch_H^k \mathbb{E}(\Gamma_i^2 K_i^2|\mathfrak{F}_{i-1})$;

$$\begin{aligned} \mathbb{E}(T_i^2|\mathfrak{F}_{i-1}) &\leq 2Ch_H^k \left(\mathbb{E}((\sum_{j=1}^{n} \rho_j^2 K_j)^2 K_i^2|\mathfrak{F}_{i-1}) + \mathbb{E}((\sum_{j=1}^{n} |\rho_j|K_j)^2 \rho_i^2 K_i^2|\mathfrak{F}_{i-1})\right). \\ &\leq 2Ch_H^k \left(Cn^2 h_K^4 \mathbb{E}(K_i^2|\mathfrak{F}_{i-1}) + Cn^2 h_K^2 \mathbb{E}(\rho_i^2 K_i^2|\mathfrak{F}_{i-1})\right). \end{aligned}$$

This end inequality results from (H3) and (H4)(i).
Then, by using Lemma 4.16 (i) we can obtain $\mathbb{E}(T_i^2|\mathfrak{F}_{i-1}) \leq 2C'n^2 h_H^k h_K^4 \phi_{i,x}(h_K)$.

We now use the exponentially inequality of Lemma 1 in [22] (with $d_i^2 = C' n^2 h_H^k h_K^4 \phi_{i,x}(h_K)$) to obtain for all $\epsilon > 0$:

$$\mathbb{P}\left(|\widehat{f}_k^x(y) - \overline{f}_k^x(y)| > \epsilon\right)$$
$$= \mathbb{P}\left(|\frac{1}{nh_H^k \mathbb{E}(\Gamma_1 K_1)}\sum_{i=1}^n T_i| > \epsilon\right) \leq 2\exp\left[-\frac{\epsilon^2 n^2 h_H^{2k}(\mathbb{E}(\Gamma_1 K_1))^2}{2\left(D_n + C\epsilon n h_H^k \mathbb{E}(\Gamma_1 K_1)\right)}\right].$$

Taking $\epsilon = \epsilon_0 \sqrt{\frac{\varphi_x(h_k)\log(n)}{n^2 h_H^k \phi_x^2(h_K)}}$, then

$$\mathbb{P}\left(|\widehat{f}_k^x(y) - \overline{f}_k^x(y)| > \epsilon_0 \sqrt{\frac{\varphi_x(h_k)\log(n)}{n^2 h_H^k \phi_x^2(h_K)}}\right)$$
$$\leq 2\exp\left[-\frac{n^2 h_H^{2k}(\mathbb{E}(\Gamma_1 K_1))^2 \epsilon_0^2 \frac{\varphi_x(h_k)\log(n)}{n^2 h_H^k \phi_x^2(h_K)}}{2\left(D_n + Cnh_H^k \mathbb{E}(\Gamma_1 K_1)\epsilon_0 \sqrt{\frac{\varphi_x(h_k)\log(n)}{n^2 h_H^k \phi_x^2(h_K)}}\right)}\right].$$

By using Lemma 4.13(iii), enables us to compose

$$\mathbb{P}\left(\widehat{f}_k^x(y) - \overline{f}_k^x(y) > \epsilon_0 \sqrt{\frac{\varphi_x(h_k)\log(n)}{n^2 h_H^k \phi_x^2(h_K)}}\right)$$
$$\leq \quad 2\exp\left[-\frac{n^2 h_H^{2k}(O(nh_K^2\phi_x(h_K)))^2 \epsilon_0^2 \frac{\varphi_x(h_k)\log(n)}{n^2 h_H^k \phi_x^2(h_K)}}{2nh_H^k h_K^2 \varphi_x(h_K)\left(C' n h_K^2 + O(n\phi_x(h_K))\epsilon_0 \sqrt{\frac{\log(n)}{n^2 h_H^k \phi_x^2(h_K)\varphi_x(h_K)}}\right))}\right]$$
$$\leq \quad 2\exp\left[-\frac{n^2 h_H^{2k}(O(nh_K^2\phi_x(h_K)))^2 \epsilon_0^2 \frac{\varphi_x(h_k)\log(n)}{n^2 h_H^k \phi_x^2(h_K)}}{2nh_H^k h_K^2 \varphi_x(h_K)\left(C' n h_K^2 + O(1)\epsilon_0 \sqrt{\frac{\log(n)}{h_H^k \varphi_x(h_K)}}\right))}\right].$$

Then let's use the fact that for any n, under (H1)(ii) and (iii), we have $\varphi_x(h_K) \geq Cn\phi_x(h_K)$, which suggests that

$$\frac{\log(n)}{h_H^k \varphi_x(h_K)} \leq C' \frac{\varphi_x(h_K)\log(n)}{n^2 h_H^k \phi_x^2(h_K)}.$$

Therefore, under (H5), we have $\lim_{n\to\infty} \frac{\log(n)}{h_H^k \varphi_x(h_K)} = 0$.

From the above, we obtain $\mathbb{P}\left(\widehat{f}_k^x(y) - \overline{f}_k^x(y) > \epsilon_0 \sqrt{\frac{\varphi_x(h_k)\log(n)}{n^2 h_H^k \phi_x^2(h_K)}}\right) \leq 2n^{-C_0\epsilon_0^2}$;

As a result, by applying Borel-Lemma Cantelli's and making ϵ_0 sufficiently large, we can conclude that

$$\widehat{f}_k^x(y) - \overline{f}_k^x(y) = \left(\sqrt{\frac{\varphi_x(h_k)\log(n)}{n^2 h_H^k \phi_x^2(h_K)}}\right) \ a.co. \tag{23}$$

This last result completes the proof of Lemma 4.10 by setting $k = 0$. □

The Proof of lemma 4.11;

Proof. In accordance with the hypotheses (H1)(iii) and (H4), we get

$$0 < \frac{C}{n\phi_x(h_K)} \sum_{i=1}^{n} \mathbb{P}\left(X_i \in B(x, h_k)|\mathfrak{F}_{i-1}\right) \leq \overline{f}_0^x(y) \leq |\widehat{f}_0^x(y) - \overline{f}_0^x(y)| + \widehat{f}_0^x(y).$$

Therefore,

$$\begin{aligned} & \mathbb{P}\left(\widehat{f}_0^x(y) \leq \frac{C}{2}\right) \\ & \leq \mathbb{P}\left(\frac{C}{n\phi_x(h_K)} \sum_{i=1}^{n} \mathbb{P}(X_i \in B(x, h_k)|\mathfrak{F}_{i-1}) < \frac{C}{2} + |\widehat{f}_0^x(y) - \overline{f}_0^x(y)|\right) \\ & \leq \mathbb{P}\left(|\frac{C}{n\phi_x(h_K)} \sum_{i=1}^{n} \mathbb{P}(X_i \in B(x, h_k)|\mathfrak{F}_{i-1}) - |\widehat{f}_0^x(y) - \overline{f}_0^x(y)| - C| > \frac{C}{2}\right). \end{aligned}$$

Under the Lemma 4.10 and (H1)(iii) we receipt that

$$\sum_n \mathbb{P}\left(|\frac{C}{n\phi_x(h_K)} \sum_{i=1}^{n} \mathbb{P}(X_i \in B(x, h_k)|\mathfrak{F}_{i-1}) - |\widehat{f}_0^x(y) - \overline{f}_0^x(y)| - C| > \frac{C}{2}\right) < \infty.$$

It produces the result. □

We Proof of lemma 4.12;

Proof. We can infer from the compactness of $\wp$ that there is a sequence of real numbers $(y_k)_{k=1,\dots,d_n}$ such that

$\wp \subset \bigcup_{k=1}^{d_n} \wp_k$, where $\wp_k = (y_k - l_n, y_k + l_n)$, and $l_n = n^{-1-\alpha}$ and $d_n = O(l_n - 1)$.

With the following decomposition, we begin our proof:

$$\begin{aligned} & \sup_{y\in\wp} \left|\widehat{f}_1^x(y) - \overline{f}_1^x(y)\right| \\ & \leq \underbrace{\sup_{y\in\wp} \left|\widehat{f}_1^x(y) - \widehat{f}_1^x(z)\right|}_{S_1} + \underbrace{\sup_{y\in\wp} \left|\widehat{f}_1^x(z) - \overline{f}_1^x(z)\right|}_{S_2} + \underbrace{\sup_{y\in\wp} \left|\overline{f}_1^x(z) - \overline{f}_1^x(y)\right|}_{S_3}. \end{aligned}$$

We now define the three concepts.
On the one hand, for the concept S_1, by utilizing hypothesis (H5), we have

$$\begin{aligned} S_1 &\leq \sup_{y\in\wp} \left| \frac{1}{nh_H \mathbb{E}(\Gamma_1 K_1)} \sum_{i=1}^{n} \Gamma_i K_i |H_i(y) - H_i(z)| \right| \\ &\leq \sup_{y\in\wp} \frac{C|y-z|}{h_H} \left(\left| \frac{1}{nh_H \mathbb{E}(\Gamma_1 K_1)} \sum_{i=1}^{n} \Gamma_i K_i \right| \right) \leq C \frac{l_n}{h_H^2} |\widehat{f}_0^x(y)|. \end{aligned}$$

Thus, using Lemma 4.11, we get $S_1 \leq C\frac{l_n}{h_H^2}$.
Since $l_n = n^{-1-\alpha}$, we obtain $\frac{l_n}{h_H^2} = O\left(\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2 h_H \phi_x^2(h_K)}}\right)$.
Consequently, for n big enough, we discover a $\eta > 0$ such that

$$\mathbb{P}\left(S_1 > \eta\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2 h_H \phi_x^2(h_K)}}\right) = 0. \tag{24}$$

Results similar to the concept of S_3 were : $S_3 \leq C\frac{l_n}{h_H^2}|\overline{f}_0^x(y)|$.
Therefore, Lemma 4.14 allows us to wtite: $S_3 \leq C\frac{l_n}{h_H^2}$.
We can find the following for similar arguments as S_1:

$$\mathbb{P}\left(S_3 > \eta\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2 h_H \phi_x^2(h_K)}}\right) = 0. \tag{25}$$

On the other hand, in order to finish the proof of this Lemma, we must demonstrate that

$$S_2 = O\left(\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2 h_H \phi_x^2(h_K)}}\right) \quad a.co.$$

By using 23 for $k = 1$, we get for $\eta > 0$ and for all $z \in \wp_k$:

$$\mathbb{P}\left(\left|\widehat{f}_1^x(z) - \overline{f}_1^x(z)\right| > \eta\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2 h_H \phi_x^2(h_K)}}\right) \leq C' n^{-C_0\eta^2}.$$

Thus, we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{y\in\wp}\left|\widehat{f}_1^x(z) - \overline{f}_1^x(z)\right| > \eta\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2 h_H \phi_x^2(h_K)}}\right)\\
&\leq \mathbb{P}\left(\max_{z\in\wp}\left|\widehat{f}_1^x(z) - \overline{f}_1^x(z)\right| > \eta\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2 h_H \phi_x^2(h_K)}}\right)\\
&\leq 2d_n \max_{z\in\wp}\mathbb{P}\left(\left|\widehat{f}_1^x(z) - \overline{f}_1^x(z)\right| > \eta\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2 h_H \phi_x^2(h_K)}}\right)\\
&\leq C' n^{-C_0\eta^2+1+\alpha}.
\end{aligned}$$

Therefore, by choosing η such that $C_0\eta^2 = 2 + 2\alpha$, we find that

$$\mathbb{P}\left(\sup_{y\in\wp}\left|\widehat{f}_1^x(z) - \overline{f}_1^x(z)\right| > \eta\sqrt{\frac{\varphi_x(h_K)\log(n)}{n^2 h_H \phi_x^2(h_K)}}\right) \leq C' n^{-1-\alpha}. \tag{26}$$

Last but not least, Lemma 4.12 is directly extract from formulae 24, 25 and 26. □

The Proof of theorem 4.4;

Proof. Directly from Theorem 4.2, which we have previously proved, that

$$\sum_{n=1}^{\infty} \mathbb{P}\{|\widehat{F}^x(y) - F^x(y)| > \varepsilon\} < \infty.$$

On the other hand, under the condition $\inf_{y\in\mathcal{S}}(1 - F^x(y)) > \beta$, we choose $\delta = \frac{\beta}{2}$, to show

$$\mathbb{P}\{\inf_{y\in\mathcal{S}} |1 - \widehat{F}^x(y)| \leq \delta\} < \infty.$$

□

and the proof of Theorem 4.1 is now completed.

5. Numerical Simulation

In this section, we conduct several numerical experiments on the conditional hazard function in a prediction context. In another sense, we use the mean square error (MSE) as a criterion to demonstrate the performance and superiority of our estimator . In order to do this, we compare the (MSE) of the local linear approch (L.L), which is the subject of this study, to the kernel method (K.M), where the data in the following simulation are of a functional ergodic type. For that, we define the two models using the formula below:

$$\widehat{h}^x_{L.L}(y) = \frac{h_H^{-1}\sum_{i=1}^n \Gamma_i K_i H_i}{\sum_{i=1}^n \Gamma_i K_i - \sum_{i=1}^n \Gamma_i K_i H_i}, \qquad \widehat{h}^x_{K.M}(y) = \frac{h_H^{-1}\sum_{i=1}^n K_i H_i}{\sum_{i=1}^n K_i - \sum_{i=1}^n K_i H_i}.$$

To do this, we construct the random variables $(X_i, Y_i)_{i=1,\ldots,1000}$ using the following regression model:

$$Y_i = r(X_i) + \varepsilon_i,$$

where $\varepsilon_i \sim \mathcal{N}(0, 0.3)$, and r is a operator is given by:

$$r(X_i) = \frac{1}{5(\int_0^1 X_i(t)dt)^2}$$

For any $t \in [0, 1]$, the process $X_i(t)$ is defined as follows:

$$X_i(t) = a_i t + \cos(\zeta_i - t)$$

where a_i (respectively,ζ_i)is uniformly distributed $U[0, 1]$ (respectively, normally distributed $\mathcal{N}(0, 0.3)$).
Our method for calculating the empirical mean square error is:

$$MSE(K.M) = \frac{1}{n}\sum_{i=1}^n (Y_i - \widehat{h}^x_{K.M}(y))^2 \quad and \quad MSE(L.L) = \frac{1}{n}\sum_{i=1}^n (Y_i - \widehat{h}^x_{L.L}(y))^2$$

We obtain the curve which represents the conditional hazard function and its estimator by the method of kernels and by the linear local method with the smoothing parameter $h = 0.1$(see figure 1), then with the smoothing parameter $h = 0.2$ in figure 2.

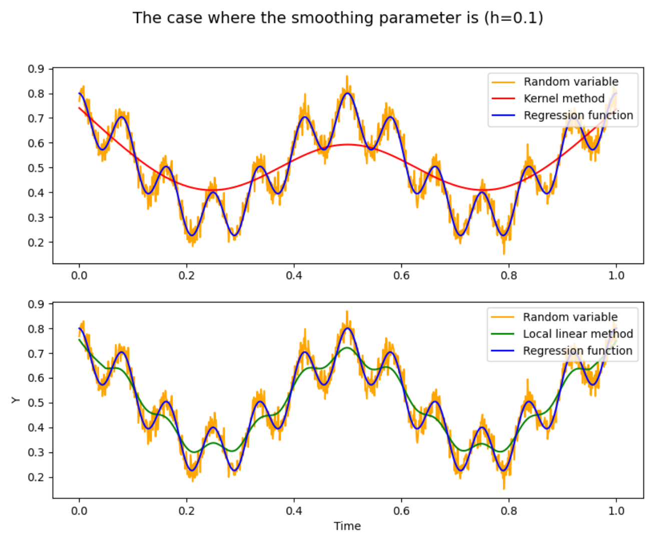

FIGURE 1. The conditional hazard function and its estimator by the kernel method and by the local linear method for h =0.1

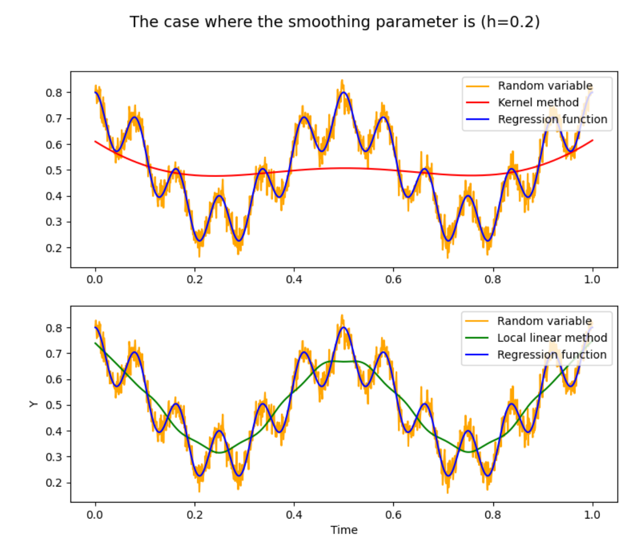

FIGURE 2. The conditional hazard function and its estimator by the kernel method and by the local linear method for h =0.2

The comparison in the two previous figures indicates that the linear local approach (L.L) gives better results than the classical kernel method(K.M). This is confirmed by the mean square error (MSE) for deferent smoothing parameter values "h" as shown in the following table 1:

TABLE 1. The empirical mean square error results of the two estimation methods for deferent smoothing parameter

The parameter (h)	0.15	0.20	0.25	0.30
$MSE(K.M)$	0.0149	0.0193	0.0220	0.0235
$MSE(L.L)$	0.0053	0.0055	0.0056	0.0060

6. Conclusion

In this study, we have treated the nonparametric estimation techniques of the conditional hazard function. We used the local linear method to estimate the hazard function. This method is mainly used to evaluate the performance of the estimator. The main results we have established are the almost complete convergence speed of the conditional hazard function, the empirical density function, and the distribution function by specifying its convergence speed for functional ergodic data. We also presented the asymptotic properties of each functional model. In addition, we have built a local linear estimator of the hazard function and we carry out simulations using the python software which allows us to observe the influence of the parameter of smoothing (h) on the method of kernels and local linear.
Note that MSE(L.L) is less than MSE(K.M). This result indicates the efficacy of our approach.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : The data that were utilized in this work are included in the article.

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