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A STUDY ON DEGENERATE q-BERNOULLI POLYNOMIALS AND NUMBERS†

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Abstract. In this paper, the degenerate q-Bernoulli polynomials are defined by generalizing it more, and various properties of these polynomials are introduced. To do this, we define generating functions of them and use the definition to introduce some interesting properties. Finally, we observe the structure of the roots for the degenerate q-Bernoulli polynomials.

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1. Introduction

Many researchers have used the generative function $\frac{t}{e^t-1}e^{xt}$ and $\frac{t}{e^t-1}$ to define classical Bernoulli polynomials and numbers and reveal their properties. Their interesting properties are the relationship between polynomials and numbers, the sum of numbers, and symmetry properties. However, given a change in the generative function, new similar Bernoulli polynomials and numbers can be found, and similar new properties can be found. In this paper, exponential function e^x is more generalized and defined, and Bernoulli polynomials and numbers are defined using this definition. This allows us to observe more generalized Bernoulli polynomials and numbers.

As already well known, the classical Bernoulli polynomials $B_n(x)$ are given by the generating function as follows:

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \quad \text{(cf. [1-13])}. \tag{1.1}$$

When $x=0, B_n=B_n(0)$ are called Bernoulli numbers.

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The Bernoulli polynomials		The degenerated Bernoulli polynomials
$\frac{t}{e^t-1}e^{xt}$	$\xleftarrow[\lambda\to 0]{}$	$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}$
$q\to 1 \uparrow$	$_{h=0} \nwarrow ^{q\to 0}$	$q\to 1 \uparrow$
$\frac{t}{e_q(t)-1}e_q(t)$	$\xleftarrow[h=0]{}$	$\frac{t}{e_{q,h}(1:t)-1}e_{q,h}(x:t)$
The q-Bernoulli polynomials		The degenerated q-Bernoulli polynomials

FIGURE 1. The various generating functions and Bernoulli polynomial research flow chart (see, [2,9,11]).

In Figure 1, we can observe the relationship between the various generating functions and the Burnoulli polynomials. The generating function at the bottom right can be seen as a generalization of the other three generating functions. In this paper, we define generalized Bernoulli polynomials and numbers define and investigate their properties.

The main symbols used in this paper are as follows. $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}_+$ be the set of nonnegative integers, $\mathbb{Z}$ be the set of integers, $\mathbb{R}$ be the set of real numbers and $\mathbb{C}$ be the set of complex numbers, respectively.

We first introduce the basic concepts and sevaral exponential functions.

Let $n, q \in \mathbb{R}$ and $q \neq 1$. Jackson defined the q-number as below:

$$[n]_q = \frac{1-q^n}{1-q}$$

and $\lim_{q\to 1}[n]_q = n$.

For $[n]_q! = [n]_q[n-1]_q\cdots[2]_q[1]_q$ and $[0]_q = 1$, q-binomial is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[n-r]_q![r]_q!}.$$

Many mathematicians have studied various exponential functions. First of all, let's start with the basic functions we know and find out what functions are there.

1. e^t is well known exponential function. Also, since e^t is an analytic function on complex number field, e^t is capable of series expansion as belows:

$$e^t = \sum_{n=0}^{\infty}\frac{t^n}{n!}.$$

2. $e_q(t)$ is a q-exponential function defiened as

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}.$$

For real $q > 1$, the function $e_q(t)$ is an entire function of t, For $0 < q < 1$, $e_q(t)$ is regular in the disk $|z| < 1/(1-q)$. Also, the inverse of $e_q(t)$ is $e_{q^{-1}}(-t)$, i.e., $e_q(t)e_{q^{-1}}(-t) = 1$.

3. $e_{p,q}(t)$ is a (p,q)-exponential function defiened as

$$e_{p,q}(t) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}} t^n}{[n]_{p,q}!}, \quad E_{p,q}(t) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{[n]_{p,q}!}.$$

Here, $[n]_{p,q} = \frac{p^n - q^n}{p-q}$.

4. $(1+\lambda t)^{\frac{1}{\lambda}}$ is a degenerated exponetial function and $\lim_{\lambda \to 0}(1+\lambda t)^{\frac{1}{\lambda}} = e^t$.

More recently, Burcu Silindir and Ahmet Yantir [9] have generalized exponential function with Definition 1.1 and 1.2 in the following way.

Definition 1.1. For $0 < q < 1$, one define the generalized quantum binomial, (q,h)-analogue of $(x - x_0)^n$, as the polynomial

$$(x - x_0)^n_{q,h} = \begin{cases} 1 & if \quad n = 0, \\ \prod\limits_{i=1}^{n} (x - (q^{i-1}x_0 + [i-1]_q h)) & if \quad n > 0, \end{cases}$$

where $x_0 \in \mathbb{R}$ (see [9]).

When $q \to 1$ and $h \to 0$, the generalized quantum binomial approximates the ordinary binomial as belows:

$$\lim_{(q,h)\to(1,0)} (x - x_0)^n_{(q,h)} = (x - x_0)^n.$$

In this paper, $(x-0)^n_{(q,h)}$ is denoted as $(x)^n_{q,h}$ for convenience.
That is, $(x-0)^n_{(q,h)} = (x)^n_{q,h} = x(x-[1]_q h)(x-[2]_q h)\cdots(x-[n-1]_q h)$.

Burcu Silindir and Ahmet Yantir have already defined the exponential function as following Definition 1.2 and we note that we have used different type to avoid confusion.

Definition 1.2. For $0 < q < 1$, one define the degenerated q-exponential function, denote by

$$e_{q,h}(x:t) = \sum_{n=0}^{\infty} \frac{(x)^n_{q,h}}{[n]_q!} t^n,$$

where $(0)^0_{q,h} = 1$, for $n \in \mathbb{N}$ and $e_{q,h}(0,t) = 1$.

From Definition 1.1, we get

$$
\begin{aligned}
(1-0)^n_{q,h} &= \prod_{i=1}^{n}(1-[i-1]_q h) \\
&= (-1)^{n-1}\prod_{i=1}^{n-1}\frac{1}{[i]_q}\left(h-\frac{1}{[i]_q}\right) \\
&= \frac{1}{\prod_{i=1}^{n-1}[i]_q}\sum_{j=0}^{n-1}(-h)^{n-1}(-h^{-1})^j A_{n-1,j}(\frac{1}{[i]_q}), \qquad (1.2)
\end{aligned}
$$

where $A_{n,k}(\frac{1}{[i]_q})$ is sum of all k selectable out of n's $\frac{1}{[i]_q}$.

By1.2, we obtain

$$
\begin{aligned}
e_{q,h}(1:t) &= \sum_{m=0}^{\infty}(1-0)^n_{q,h}\frac{t^n}{[n]_q} \\
&= \sum_{n=0}^{\infty}(-1)^{n-1}\frac{1}{\prod_{i=1}^{n-1}[i]_q}\sum_{j=0}^{n-1}(-1)^j h^{n-j-1}A_{n-1,j}(\frac{1}{[i]_q})\frac{t^n}{[n]_q!}.
\end{aligned}
$$

2. A degenerate q-Bernoulli polynomial and it's properties

Definition 2.1. For $0<q<1$ and $t\in\mathbb{R}$, we define a degenerate q-Bernoulli polynomials and numbers by the following generating function

$$\frac{t}{e_{q,h}(1:t)-1}e_{q,h}(x:t)=\sum_{n=0}^{\infty}\beta_{n,q}(x:h)\frac{t^n}{[n]_q!}$$

and

$$\frac{t}{e_{q,h}(1:t)-1}=\sum_{n=0}^{\infty}\beta_{n,q}(h)\frac{t^n}{[n]_q!}.$$

When $q\to 1$ and $h=0$ it is equal to the classical Bernoulli polynomial.

A two-parameters time scale $T_{q,h}$ is introduced as follows:

$$T_{q,h}:=\{q^n x+[n]_q h|x\in\mathbb{R},n\in\mathbb{Z},h,q\in\mathbb{R}^+,q\neq 1\}\cup\{\frac{h}{1-q}\}.$$

Definition 2.2. Let $f:T_{q,h}\to\mathbb{R}$ be any function. Then, the delta (q,h)-derivative of f, $D_{q,h}(f)$ is defined by

$$D_{q,h}f(x)=\frac{f(qx+h)-f(x)}{qx+h-x}.$$

From the above Definition 2.2, we can get several properties as belows:

$$D_{q,h}(x)_{q,h}^{n} = [n]_q (x)_{q,h}^{n-1},$$
$$D_{q,h} e_{q,h}(x:t) = t e_{q,h}(x:t).$$

By the Definition 2.1 and $(\alpha x)_{q,h}^{n} = \alpha^n (x)_{q,\frac{h}{\alpha}}^{n}$, we get the following identity.

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_{n,q}(\alpha x : h)\frac{t^n}{[n]_q!} &= \frac{t}{e_{q,h}(1:t)-1} e_{q,h}(\alpha x : t) \\
&= \sum_{n=0}^{\infty} \beta_{n,q}(h)\frac{t^n}{[n]_q!} \times \sum_{n=0}^{\infty} (\alpha x)_{q,h}^{n}\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}(h)(\alpha x)_{q,h}^{k}\right)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}(h)\alpha^k (x)_{q,\frac{h}{\alpha}}^{k}\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficient on both sides, we obtain the desired result.

Theorem 2.1. *Let n be a nonnegative integers, $\alpha, h \in \mathbb{R}$ and $0 < q < 1$. We have*

$$\begin{aligned}
\beta_{n,q}(x:h) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}(h)(x)_{q,h}^{k} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \alpha^k \beta_{n-k,q}(h)(x)_{q,\frac{h}{\alpha}}^{k}.
\end{aligned}$$

Theorem 2.2. *Let n be a nonnegative integers, $k \in \mathbb{Z}$ and $0 < q < 1$. We have*

$$\begin{cases} \beta_{1,q}(1:h) - \beta_{n,q}(h) = 1 & \text{if} \quad n = 1, \\ \beta_{n,q}(1:h) - \beta_{n,q}(h) = 0 & \text{if} \quad n \geq 2. \end{cases}$$

Proof. It is easily checked that

$$\begin{aligned}
t &= \frac{t}{e_{q,h}(1:t)-1}(e_{q,h}(1:t)-1) \\
&= \frac{t}{e_{q,h}(1:t)-1} e_{q,h}(1:t) - \frac{t}{e_{q,h}(1:t)-1} \\
&= \sum_{n=0}^{\infty} \beta_{n,q}(1:h)\frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} \beta_{n,q}(h)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left(\beta_{n,q}(1:h) - \beta_{n,q}(h)\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficient on both sides, we obtain the desired result.

Definition 2.1 is represented as below:

$$\frac{t}{e_{q,h}(1:t)-1}e_{q,h}(x:t)=\sum_{n=0}^{\infty}\beta_{n,q}(x:h)\frac{t^n}{[n]_q!}=e_q(\beta_q(x:h)t).$$

Also, above identity is represented as below:

$$te_{q,h}(x:t)=e_q(\beta_q(x:h)t)e_{q,h}(1:t)-e_q(\beta_q(x:h)t).$$

Above identity is equal to

$$t\sum_{n=0}^{\infty}(x)_{q,h}^n\frac{t^n}{[n]_q!}=\sum_{n,q}^{\infty}\beta_{n,q}(x:h)\frac{t^n}{[n]_q!}\times\sum_{n=0}^{\infty}(1)_{q,h}^n\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}\beta_{n,q}(x:h)\frac{t^n}{[n]_q}.$$

Above identity is equal to

$$\sum_{n=0}^{\infty}[n]_q(x)_{q,h}^{n-1}\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\beta_{k,q}(x:h)(1)_{q,h}^{n-k}-\beta_{n,q}(x:h)\right)\frac{t^n}{[n]_q!}.$$

Comparing the coefficient on both sides and by equation (1.2), we obtain the desired result as following.

Theorem 2.3. *Let n be a nonnegative integer, $k\in\mathbb{Z}$ and $0<q<1$. We have*

$$\begin{aligned}[n]_q(x)_{q,h}^{n-1}&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\beta_{k,q}(x:h)(1)_{q,h}^{n-k}-\beta_{n,q}(x:h)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\beta_{k,q}(x:h)\frac{1}{\prod_{i=1}^{n-1}[i]_q}\sum_{j=0}^{n-k-1}(-h)^{n-k-1}(-h^{-1})^jA_{n-k-1,j}(\frac{1}{[i]_q})\\&\quad-\beta_{n,q}(x:h).\end{aligned}$$

Theorem 2.4. *Let n be a nonnegative integer, $h,q\in\mathbb{R}$ and $0<q<1$. Then*

$$D_{q,h}(\beta_{n,q}(x:h))=[n]_q\beta_{n-1,q}(x:h).$$

Proof. From the Definition 2.2, we get the relation of (q,h)-derivative for the degenerate q-Bernoulli polynomials as belows. It is easily checked that

$$\begin{aligned}D_{q,h}(\beta_{n,q}(x:h))&=D_{q,h}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\beta_{n-k,q}(h)(x)_{q,h}^k\right)\\&=\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\beta_{n-k,q}(h)D_{q,h}(x)_{q,h}^k\\&=\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\beta_{n-k,q}(h)[k]_q(x)_{q,h}^{k-1}\\&=[n]_q\sum_{s=0}^{n-1}\begin{bmatrix}n-1\\s\end{bmatrix}_q\beta_{n-1-s,q}(h)(x)_{q,h}^s\\&=[n]_q\beta_{n-1,q}(x:h).\end{aligned}$$

Thus, $\lim_{q\to 1} D_{q,h}(\beta_{n,q}(x:h)) = D_h(\beta_n(x:h)) = n\beta_{n-1}(x:h)$ and $\lim_{q\to 1}\beta_{n-k,q}(x:h) = \beta_{n-k}(x:h) = \frac{(n-k)!}{n!}D_h^{(k)}\beta_n(x:h)$,

$\lim_{h\to 0} D_{q,h}\beta_{n,q}(x:h) = [n]_q\beta_{n-1,q}(x)$ and $\lim_{h\to 0}\beta_{n-k,q}(x:h) = \beta_{n-k,q}(x) = \frac{[n-k]!}{[n]!}D_q^{(k)}\beta_{n,q}(x)$.

Theorem 2.5. *Let $0<q<1$ and $h\in\mathbb{N}$. Then, we get*

$$\beta_{n-k,q}(x:h) = \frac{[n-k]_q!}{[n]_q!}D_{q,h}^{(k)}\beta_{n,q}(x:h).$$

Proof Let's consider below. From the Theorem 2.4

$$\begin{aligned}
D_{q,h}^{(k)}\beta_{n,q}(x:h) &= D_{q,h}^{(k-1)}[n]_q\beta_{n-1,q}(x:h)\\
&= D_{q,h}^{(k-2)}[n]_q[n-1]_q\beta_{n-2,q}(x:h)\\
&\quad\vdots\\
&\quad\vdots\\
&= [n]_q[n-1]_q\cdots[n-k+1]_q\beta_{n-k,q}(x:h)\\
&= \frac{[n]_q!}{[n-k]_q!}\beta_{n-k,q}(x:h).
\end{aligned}$$

From the Definition 2.1 and Theorem 2.5, we get a differential equation as below:

Theorem 2.6. *Let $0<q<1$ and n be a nonnegative integer. Then, we get*

$$\sum_{k=0}^{n}\frac{(1)_{q,h}^{k}}{[k]_q!}\left(D_{q,h}^{(k)}\beta_{n,q}(x:h)\right) - \beta_{n,q}(x:h) - [n]_q(x)_{q,h}^{n-1} = 0.$$

From the Definition 2.1,

$$\begin{aligned}
\sum_{n=0}^{\infty}\beta_{n,q}(x:h)\frac{t^{n+1}}{[n]_q!} &= \left(\frac{t}{e_{q,h}(1:t)-1}e_{q,h}(1:t) - \frac{t}{e_{q,h}(1:t)-1}\right)\frac{te_{q,h}(x:t)}{e_{q,h}(1:t)-1}\\
&= \sum_{n=0}^{\infty}\left(\beta_{n,q}(1:h)-\beta_{n,q}(h)\right)\frac{t^n}{[n]_q!}\times\sum_{n=0}^{\infty}\beta_{n,q}(x:h)\frac{t^n}{[n]_q!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\left(\beta_{n-k,q}(1:h)-\beta_{n-k,q}(h)\right)\beta_{k,q}(x:h)\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

If we rearrange the left side of the above expression, then

$$\begin{aligned}
&\sum_{n=1}^{\infty}[n]_q\beta_{n-1,q}(x:h)\frac{t^n}{[n]_q!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q(\beta_{n-k,q}(1:h)-\beta_{n-k,q}(h))\beta_{k,q}(x:h)\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Hence, comparing both sides for t, we get the following theorem:

Theorem 2.7. *For $n \in \mathbb{N}$ and $0 < q < 1$*

$$[n]_q \beta_{n-1,q}(x:h) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (\beta_{n-k,q}(1:h) - \beta_{n-k,q}(h))\beta_{k,q}(x:h).$$

Corollary 2.7.1. *For $n \in \mathbb{N}$ and $0 < q < 1$,*

$$\sum_{k=0}^{n} \frac{\beta_{k,a}(1:h) - \beta_{k,q}(h)}{[k]_q!} D_{q,h}^{(n-k)} \beta_{n,q}(x:h) - [n]_q \beta_{n-1,q}(x:h) = 0.$$

From Definition 2.1, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_{n,q}(\alpha x) \frac{t^n}{[n]_q!} &= \frac{t}{e_{qh}(1:h) - 1} e_{q,h}(\alpha x : t) \\
&= \sum_{n=0}^{\infty} \beta_{n,q}(h) \frac{t^n}{[n]_q!} \times \sum_{n=0}^{\infty} (\alpha x)_{q,h}^n \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}(h)(\alpha x)_{q,h}^k \right) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \alpha^k \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}(h)(\alpha x)_{q,h\alpha^{-1}}^k \right) \frac{t^n}{[n]_q!}.
\end{aligned}$$

Hence, comparing both sides for t, we get the following theorem:

Theorem 2.8. *For $n \in \mathbb{N}$, $\alpha, q \in \mathbb{R}$ and $0 < q < 1$,*

$$\begin{aligned}
\beta_{n,q}(\alpha x) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}(h)(\alpha x)_{q,h}^k \\
&= \sum_{k=0}^{n} \alpha^k \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}(h)(\alpha x)_{q,h\alpha^{-1}}^k.
\end{aligned}$$

From Definition 2.1, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \beta_{n,q}(qx) \frac{t^{n+1}}{[n]_q!} = \frac{t}{e_{qh}(1:h) - 1} e_{q,h}(qx : t) \\
&= \frac{1}{q} \left(\frac{qt}{e_{q,q^{-1}h}(1;qt) - 1} e_{q,q^{-1}h}(1:qt) - \frac{qt}{e_{q,q^{-1}h}(1:qt) - 1} \right) \frac{t}{e_{q,h}(1:t) - 1} e_{q,h}(qx:t) \\
&= \sum_{n=0}^{\infty} q^{n-1} \left(\beta_{n,q}(1:q^{-1}h) - \beta_{n,q}(q^{-1}h) \frac{t^n}{[n]_q} \right) \sum_{n=0}^{\infty} \beta_{n,q}(qx:h) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k-1} \left(\beta_{k,q}(1:q^{-1}h) - \beta_{k,q}(q^{-1}h) \right) \beta_{n-k,q}(qx:h) \frac{t^n}{[n]_q!}.
\end{aligned}$$

If we rearrange the left side of the above expression, then

$$\sum_{n=1}^{\infty} n\beta_{n-1,q}(qx:h)\frac{t^n}{[n]_q!}$$
$$=\sum_{n=0}^{\infty} q^{k-1}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\beta_{k,q}(1:q^{-1}h)-\beta_{k,q}(q^{-1}h)\right)\beta_{n-k,q}(qx:h)\frac{t^n}{[n]_q!}.$$

Hence, comparing both sides for t, we get the following Theorem 2.9.

Theorem 2.9. *For $n \in \mathbb{N}$ and $0 < q < 1$,*

$$\beta_{n-1,q}(qx:h)$$
$$=\frac{1}{n}\sum_{n=0}^{n} q^{k-1}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\beta_{k,q}(1:q^{-1}h)-\beta_{k,q}(q^{-1}h)\right)\beta_{n-k,q}(qx:h).$$

3. Computational values and graphical representations of degenerate q-Bernoulli polynomials

In this section, computational values and graphical representations of degenerate q-Bernoulli polynomials $\beta_{n,q}(x:h)$ are shown.

A few of them are

$$\beta_{0,q}(x:h)=1,$$
$$\beta_{1,q}(x:h)=x-\frac{1}{[2]_q!}+\frac{h}{[2]_q!},$$
$$\begin{aligned}\beta_{2,q}(x:h)=&-x+x^2+\frac{1}{[2]_q!}-\frac{2h}{[2]_q!}+\frac{h^2}{[2]_q!}-\frac{[2]_q!}{[3]_q!}+\frac{2h[2]_q!}{[3]_q!}-\frac{h^2[2]_q!}{[3]_q!}\\&+\frac{hq[2]_q!}{[3]_q!}-\frac{h^2q[2]_q!}{[3]_q!},\end{aligned}$$
$$\begin{aligned}\beta_{3,q}(x:h)=&-x+2hx+hqx-2hx^2-hqx^2+x^3+\frac{2}{[2]_q!}-\frac{6h}{[2]_q!}+\frac{6h^2}{[2]_q!}\frac{2h^3}{[2]_q!}\\&-\frac{2hq}{[2]_q!}+\frac{4h^2q}{[2]_q!}-\frac{2h^3q}{[2]_q!}-\frac{[3]_q!}{([2]_q!)^3}+\frac{3h[3]_q!}{([2]_q!)^3}-\frac{3h^2[3]_q!}{([2]_q!)^3}\\&+\frac{h^3[3]_q!}{([2]_q!)^3}+\frac{x[3]_q!}{([2]_q!)^2}-\frac{hx[3]_q!}{([2]_q!)^2}-\frac{x^2[3]_q!}{([2]_q!)^2}+\frac{hx^2[3]_q!}{([2]_q!)^2}-\frac{[3]_q!}{[4]_q!}\\&+\frac{3h[3]_q!}{[4]_q!}-\frac{3h^2[3]_q!}{[4]_q!}+\frac{h^3[3]_q!}{[4]_q!}+\frac{2hq[3]_q!}{[4]_q!}-\frac{4h^2q[3]_q!}{[4]_q!}+\frac{2h^3q[3]_q!}{[4]_q!}\\&+\frac{hq^2[3]_q!}{[4]_q!}-\frac{3h^2q^2[3]_q!}{[4]_q!}+\frac{2h^3q^2[3]_q!}{[4]_q!}-\frac{h^2q^3[3]_q!}{[4]_q!}+\frac{h^3q^3[3]_q!}{[4]_q!}.\end{aligned}$$

We investigate the beautiful zeros of the degenerate q-Bernoulli polynomials $\beta_{n,q}(x:h)$ by using a Mathematica 12.1. We plot the zeros of the degenerate q-Bernoulli polynomials $\beta_{n,q}(x:h)=0$ for $n=30$ (Figure 2).

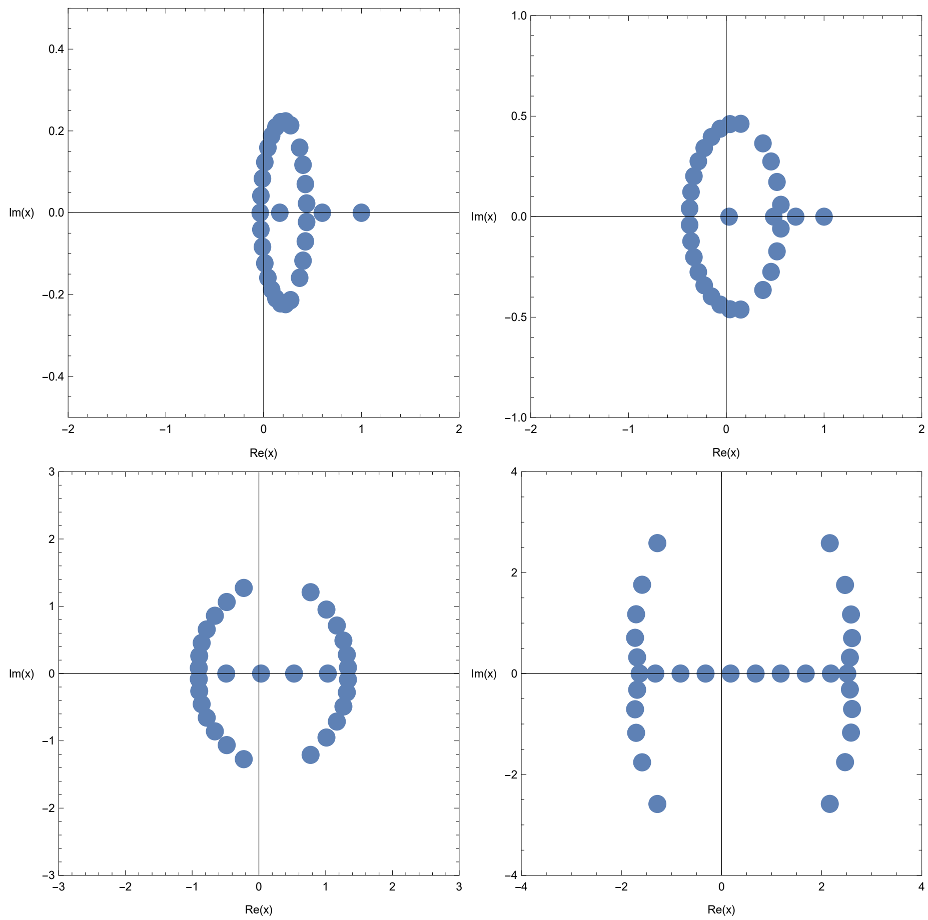

FIGURE 2. Zeros of $\beta_{n,q}(x:h)=0$

In Figure 2(top-left), we choose $q=\frac{5}{10}$ and $h=\frac{1}{10}$. In Figure 2(top-right), we choose $q=\frac{7}{10}$ and $h=\frac{1}{100}$. In Figure 2(bottom-left), we choose $q=\frac{9}{10}$ and $h=\frac{1}{1000}$. In Figure 2(bottom-right), we choose $q=\frac{99}{100}$ and $h=\frac{1}{10000}$.

Stacks of zeros of the degenerate q-Bernoulli polynomials $\beta_{n,q}(x:h)=0$ for $3 \leq n \leq 30$, forming a 3D structure, are presented (Figure 3).

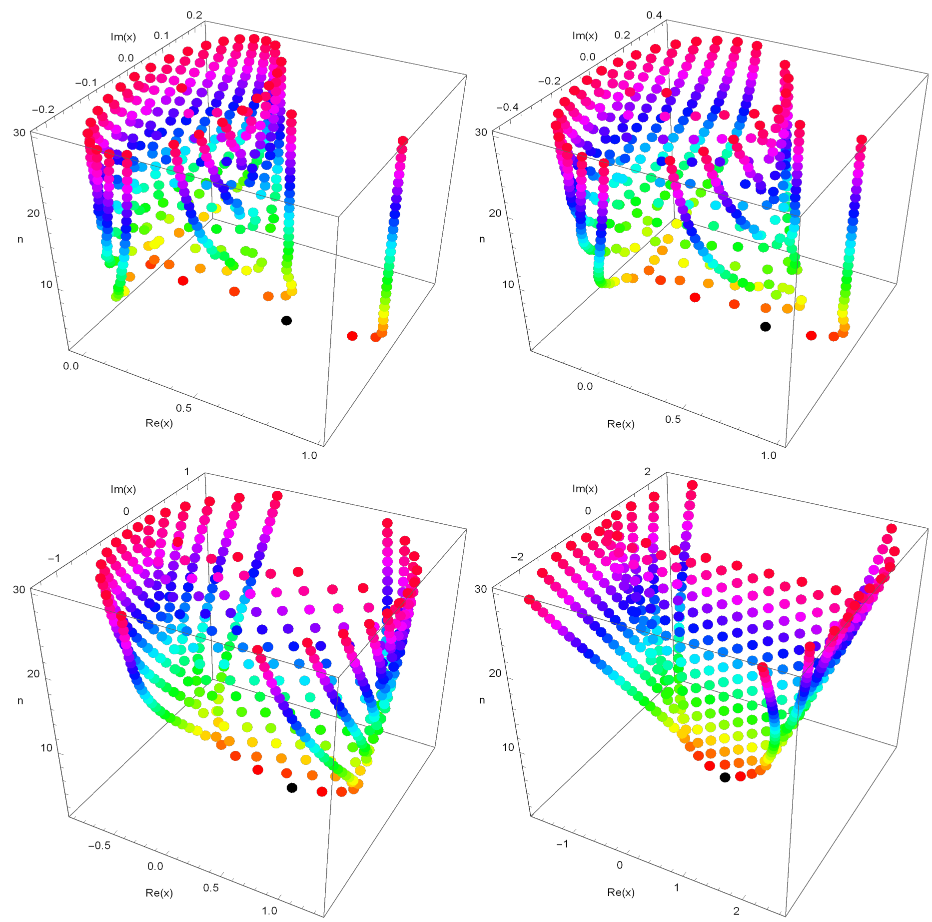

FIGURE 3. Zeros of $\beta_{n,q}(x:h)=0$

In Figure 3(top-left), we choose $q=\frac{5}{10}$ and $h=\frac{1}{10}$. In Figure 3(top-right), we choose $q=\frac{7}{10}$ and $h=\frac{1}{100}$. In Figure 3(bottom-left), we choose $q=\frac{9}{10}$ and $h=\frac{1}{1000}$. In Figure 3(bottom-right), we choose $q=\frac{99}{100}$ and $h=\frac{1}{10000}$.

Next, we calculated an approximate solution satisfying the degenerate q-Bernoulli polynomials $\beta_{n,q}(x:h)=0$ for $q=\frac{9}{10}, h=\frac{1}{10}$. The results are given in Table 1.

Table 1. Approximate solutions of $\beta_{n,q}(x:h)=0$

n	x
1	0.60000
2	0.11641, 0.88359
3	−0.040625, 0.36233, 0.97830
4	−0.020225, 0.052324, 0.51673, 1.0012
5	$-0.013119 \pm 0.079112i$ 0.20481, 0.59408, 1.0024
6	$0.039761 \pm 0.130296i$, −0.043048, 0.33060, 0.61976, 1.0007
7	$0.0005079 \pm 0.0066442i$, $0.07819 \pm 0.14745i$, 0.41936, 0.61698, 1.0000
8	$-0.026494 \pm 0.058655i$, $0.09391 \pm 0.17187i$, 0.18635, 0.47211, 0.60363, 0.99994
9	$0.003958 \pm 0.103753i$, $0.12310 \pm 0.19300i$, 0.30560, −0.042553, 0.48684, 0.59444, 0.99998
10	$-0.011499 \pm 0.026214i$, $0.034779 \pm 0.121788i$, $0.14739 \pm 0.19954i$, 0.41655, 0.44598, 0.59535, 1.0000
11	$-0.029523 \pm 0.046394i$, $0.04120 \pm 0.14218i$, $0.44649 \pm 0.06323i$, $0.16107 \pm 0.20553i$, 0.16220 0.59893, 1.0000

In conclusion, the degenerated q-Bernouulli polynomials's generating function can be seen as a degenerated generating funtion of the Bernoulli polynomials, the q-Bernoulli polynomial and the degenerated Bernoulli polynomials.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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