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A NOVEL WEIBULL MARSHALL-OLKIN POWER LOMAX DISTRIBUTION: PROPERTIES AND APPLICATIONS TO MEDICINE AND ENGINEERING

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Abstract. This paper introduced the Weibull Marshall-Olkin Power Lomax (WMOPL) distribution. The statistical aspects of the proposed model are presented, such as the quantiles function, moments, mean residual life and mean deviations, variance, skewness, kurtosis, and reliability measures like the residual life function, and stress-strength reliability. The parameters of the new model are estimated using six different methods, and simulation research is illustrated to compare the six estimation methods. In the end, two real data sets show that the Weibull Marshall-Olkin Power Lomax distribution is flexible and suitable for modeling data.

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1. Introduction

When describing real-world phenomena, statistical distributions are frequently used. Several developed distributions for modeling lifetime data have been presented. The fact that these generalized distributions have more parameters is a typical feature. The Lomax (1954) distribution, sometimes known as the Pareto II distribution, is used in many applications. Hassan and Al-Ghamdi (2009) used it for dependability modeling and life testing. Atkinson and Harrison (1978) modeled company failure data using Lomax distribution, and Corbellini et al. (2010) modeled firm size and queuing difficulties using Lomax distribution. It has also been utilized in the biological sciences for modeling the size distribution of computer data on servers, (Holland et al., 2006). Some writers, such

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as Bryson (1974) recommended this distribution as an alternative to the exponential distribution for heavy-tailed data. Several generalizations exist based on the Lomax distribution. Punathumparambath (2011) introduced the double-Lomax distribution and applied it to IQ data. Ahsanullah (1991) has explored the record statistics of Lomax distribution. Myhre and Saunders (1982); Cramer and Schmiedt (2011), and others examined the ramifications of various forms of right-truncation and right-censorship for Lomax distribution.

On the other hand, several extensions of the Lomax distribution are available in the literature, including the Beta–Lomax (BL) (Rajab et al., 2013); Topp Leone Kumaraswamy Lomax (TLKwL) (Ibrahim et al., 2021); McDonald-Lomax (McL) (Lemonte and Cordeiro, 2013); Exponentiated Lomax (EL) (Abdul-Moniem, 2012); the Marshall–Olkin extended-Lomax (MOEL) (Ghitany et al., 2007; Gupta et al., 2010); transmuted exponentiated Lomax (TEL) (Ashour and Eltehiwy, 2013) and the Gamma Lomax (GL) (Cordeiro et al., 2015). The Power Lomax distribution, an extension of the Lomax distribution and one of the most widely used distributions was first introduced by Rady et al. (2016). It has many applications in various fields, such as medical and biological sciences, engineering, finance, actuarial science, lifetime, and reliability modeling.

Some extended generalization forms of the Power Lomax distributions are listed:

Marshall-Olkin Power Lomax (Ul-Haq et al., 2020); the Marshall–Olkin extended Power Lomax (Gillariose and Tomy, 2020); the exponentiated Power Lomax (El-Monsef et al., 2021); the Marshall-Olkin alpha Power Lomax (Almongy et al., 2021); weighted Power Lomax (Hassan et al., 2021); the Weibull Power Lomax (Hussain et al., 2020); the transmuted Power Lomax (Moltok et al., 2019); modified Power Lomax (Okorie et al., 2017); inverse Power Lomax (Hassan and Ab-Allah, 2019); Kumaraswamy Generalized Power Lomax (Nagarjuna et al., 2021); type II Topp-Leone Power Lomax (Marzouki et al., 2019); truncated Power Lomax (Hassan et al., 2020); truncated Weibull Power Lomax (al-Marzouki et al., 2019); truncated Cauchy Power Lomax (Almarashi et al., 2020); Harris extended Power Lomax (Ogunde et al., 2021); sine Power Lomax (Nadajuna et al., 2021) and Alpha Power Lomax (Bulut et al., 2021).

Based on the $T-X$ generator, Korkmaz et al., (2019) introduced the Weibull Marshall–Olkin–G($WMO-G$) family of generators. Let $G(x;\mathfrak{A})$ be the cumulative distribution function (CDF) of any random variable X as baseline distribution with parameter vector $\mathfrak{A}$ and $\mathfrak{r}(t)$ be the probability density function (PDF) and $R(t)$ be the CDF of a continuous random variable T defined as $T\in[p,q]$ for $-\infty< p < q <\infty$. Let $\mathfrak{V}[G(x;\mathfrak{A})]$ be a function of $G(x;\mathfrak{A})$ satisfy:

- $\mathfrak{V}[G(x;\mathfrak{A})]\in[p,q]$,
- $\mathfrak{V}G[(x;\mathfrak{A})]\to p$ if x tends to $-\infty$ and $\mathfrak{V}[G(x;\mathfrak{A})]\to q$ if x tends to ∞, and
- $\mathfrak{V}[G(x;\mathfrak{A})]$ does not decrease monotonically.

The $T-X$ generator's CDF is provided by

$$F\left(x\right)=\int_{0}^{\mathfrak{V}(G(x;\mathfrak{A}))}\mathfrak{r}\left(t\right)dt=\mathcal{R}(\mathfrak{V}(G(x;\mathfrak{A})) \tag{1}$$

Consider the Weibull random variable T with $\beta>0$ as a shape parameter and PDF as

$$\mathfrak{r}\left(t\right)=\beta t^{\beta-1}e^{-t^{\beta}}\quad t>0,\quad \beta>0, \tag{2}$$

assume

$$\mathfrak{V}\left[G\left(x,\mathfrak{A}\right)\right]=-\ln\left[\frac{\alpha\overline{G}\left(x,\mathfrak{A}\right)}{G\left(x,\mathfrak{A}\right)+\alpha\overline{G}\left(x,\mathfrak{A}\right)}\right]=-\ln\left[\frac{\alpha\overline{G}\left(x,\mathfrak{A}\right)}{1+\overline{\alpha}\overline{G}\left(x,\mathfrak{A}\right)}\right] \tag{3}$$

by substituting Equations (2) and (3) to Equation (1), we obtained CDF of the Weibull Marshall-Olkin-G family as

$$F\left(x,\alpha,\beta,\psi\right)=1-\exp\left(-\left\{-\ln\left[\frac{\alpha\overline{G}\left(x,\mathfrak{A}\right)}{G\left(x,\mathfrak{A}\right)+\alpha\overline{G}\left(x,\mathfrak{A}\right)}\right]\right\}^{\beta}\right), \tag{4}$$

the probability density function (PDF) corresponding to Equation (4) is given by

$$\begin{aligned}f\left(x,\alpha,\beta,\psi\right)=&\frac{\beta g(x,\mathfrak{A})}{\overline{G}(x,\mathfrak{A})\left[1-\overline{\alpha}\overline{G}\left(x,\mathfrak{A}\right)\right]}\left\{-\ln\left[\frac{\alpha\overline{G}\left(x,\mathfrak{A}\right)}{G\left(x,\mathfrak{A}\right)+\alpha\overline{G}\left(x,\mathfrak{A}\right)}\right]\right\}^{\beta-1}\\&\times\exp\left(-\left\{-\ln\left[\frac{\alpha\overline{G}\left(x,\mathfrak{A}\right)}{G\left(x,\mathfrak{A}\right)+\alpha\overline{G}\left(x,\mathfrak{A}\right)}\right]\right\}^{\beta}\right),\end{aligned} \tag{5}$$

where the baseline PDF is denoted by $g(x,\mathfrak{A})$, $\overline{\alpha}=1-\alpha$, and $\alpha>0$ and $\beta>0$ are shape parameters. The survival function (SF) and hazard rate function (HRF) of the WMO-G family are, respectively, given by

$$S(x)=\exp\left(-\left\{-\ln\left[\frac{\alpha\overline{G}(x,\mathfrak{A})}{G(x,\mathfrak{A})+\alpha\overline{G}(x,\mathfrak{A})}\right]\right\}^{\beta}\right)$$

$$h(x;\alpha,\beta,\mathfrak{A})=\frac{\beta\mathfrak{v}(x;\mathfrak{A})}{\left[1-\overline{\alpha}\overline{G}(x,\mathfrak{A})\right]}\left\{-\ln\left[\frac{\alpha\overline{G}(x,\mathfrak{A})}{G(x,\mathfrak{A})+\overline{\alpha}\overline{G}(x,\mathfrak{A})}\right]\right\}^{\beta-1},$$

where $\mathfrak{v}\left(x;\mathfrak{A}\right)$ is the baseline HRF and $\mathfrak{v}\left(x;\mathfrak{A}\right)=\left(g(x,\mathfrak{A})\overline{G}\left(x,\mathfrak{A}\right)\right)$.

For α=1, the Weibull-X family is obtained as a particular case of the WMO-G family (Alzaatreh et al., 2013; Cordeiro et al., 2015). We have the MO-G family for β=1 (Marshall and Olkin, 1997). It follows the baseline distribution when α=β=1.

The primary rationale for utilizing the Weibull Marshall-Olkin-G family is to improve the flexibility of the kurtosis compared to the baseline model. In general, the purpose of developing novel distributions is to produce mathematical models that are adaptable. Adding more parameters such as location, shape, and scale may easily attain this flexibility. This paper attempts to suggest a new distribution by combining the PL distribution with the Weibull Marshall-Olkin- G family of generic distributions. The proposed distribution is denoted by the WMOPL, which stands for the Weibull Marshall-Olkin Power Lomax distribution. Examining novel adaptive family distributions can result in desirable expansions of the Power Lomax distributions, with the possibility of new perspectives. The WMOPL distribution can generalize several well-established models from the literature. Its probability density function can take on a variety of shapes depending on the additional shape parameter. Its kurtosis is more flexible than the Power Lomax model and better than some generalized distributions of the Power Lomax baseline.

The remainder of this paper is organized in the following manner. The probability density function (PDF), cumulative distribution function (CDF), survival, and hazard rate functions, and two linear expansions of the WMOPL distribution demonstrate in Section 2. Section 3 discusses the mathematical and statistical features of the WMOPL distribution. Section 4 describes the WMOPL distribution reliability measures. Section 5 presents six different methods to estimate the parameters for the WMOPL distribution. In Section 6, the proposed WMOPL is applied to some real-world data, and the new distribution flexibility is demonstrated using two real-world data sets. Finally, the paper is concluded in Section 7.

2. Proposed model

Rady et al. (2016) introduced an extension of the Lomax distribution proposed by considering the power transformation$X = T^{\frac{1}{\gamma}}$, where the random variable T follows the Lomax distribution. This extension is known as the Power Lomax distribution.

Consider the Power Lomax (PL) random variable X with parameters $(\lambda, \gamma, \sigma)$ and corresponding CDF

$$G(x;\lambda,\gamma,\sigma) = 1-\lambda^{\sigma}(\lambda+x^{\gamma})^{-\sigma}; \quad x > 0; \quad \lambda,\gamma,\sigma > 0,$$

we present the Weibull Marshall-Olkin Power Lomax (WMOPL) distribution, a novel five-parameter model. By configuring the Power Lomax CDF from Equation (4), we obtain

$$F(x;\alpha,\beta,\lambda,\gamma,\sigma) = 1 - \exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+x^{\gamma})^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}\right]\right)^{\beta}\right\} \tag{6}$$

the pdf corresponding to Equation (6) is

$$f(x;\alpha,\beta,\lambda,\gamma,\sigma)$$
$$= \frac{\beta\sigma\gamma x^{\gamma-1}}{(\lambda+x^{\gamma})\left[1-\overline{\alpha}\lambda^{\sigma}(\lambda+x^{\gamma})^{-\sigma}\right]} \times \left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+x^{\gamma})^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}\right]\right)^{\beta-1}$$

$$\times \exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+x^{\gamma})^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}\right]\right)^{\beta}\right\} \tag{7}$$

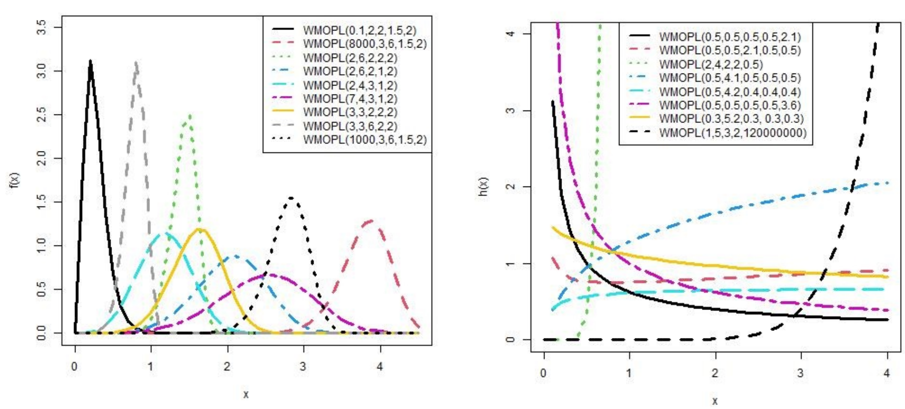
$\overline{\alpha}=1-\alpha, \sigma>0, \beta>0$, and $\gamma>0$ are shape parameters, whereas $\alpha>0$, and $\lambda>0$ are the scale parameters. Consequently $X \sim WMOPL\left(\alpha,\beta,\sigma,\gamma,\lambda\right)$is a random variable with density function from Equation (7).

The hazard rate and survival functions of the WMOPL distribution are respectively, given by

$$h\left(x\right)=\frac{\beta\sigma\gamma x^{\gamma-1}}{\left(\lambda+x^{\gamma}\right)^{-\sigma}\left[1-\overline{\alpha}\lambda^{\sigma}(\lambda+x^{\gamma})^{-\sigma}\right]}\times\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+x^{\gamma})^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}\right]\right)^{\beta-1},$$

$$S\left(x\right)=\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+x^{\gamma})^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}\right]\right)^{\beta}\right\},$$

some distributions are treated as special cases in the new model; for example, for $\alpha=1$, the WMOPL model simplifies the Weibull Power Lomax distribution. When $\beta=1$, we have Marshall-Olkin Power Lomax. It follows the Power Lomax distribution when $\alpha=\beta=1$. Figure 1 provides some plots of the WMOPL density and hazard rate functions for different values of the parameters.



(a) (b)

FIGURE 1. The possible plots of the density (a) and hazard rate (b) functions.

Figure 1 depicts the probability density and hazard rate function for the WMOPL distribution in various configurations. Figure 1(a) depicts The WMOPL probability density figure may exhibit variable behavior. It can take on right-and left-skewed, symmetrical, and asymmetrical forms. Figure 1(b) demonstrates that the WMOPL hazard rate curves may have a falling failure rate, a rising failure rate, a bathtub shape, an inverted bathtub shape, a reversed-J form, or a decreasing-increasing decreasing failure rate, indicating that the suggested model is a good lifetime model. The WMOPL distribution has a great deal of flexibility when it comes to modeling skewed data; hence, it is frequently employed in fields such as engineering, biological trials, and reliability research.

2.1. Expansion of the CDF function of WMOPL distribution. We propose two linear representations of the WMOPL density in this section. The CDF from Equation (6) can be represented as

$$F\left(x\right)=\sum_{\delta=1}^{\infty}\frac{\left(-1\right)^{\delta}}{\delta!}\left(-\ln\left[1-\left(1-\frac{\alpha\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}\right)\right]\right)^{\delta\beta}, \tag{8}$$

by utilizing the power series $e^{x}=\sum_{i=0}^{\infty}\frac{x^{i}}{i!}$. According to Flajolet and Sedgewick, (2009), and it's generalization by Ghazani, (2021), we have

$$\left[-\ln\left(1-y\right)\right]^{b}=y^{b}+\sum_{k=0}^{\infty}P_{k}\left(b\right)y^{k+b+1},$$

for any real parameter b and $0<y<1$, $P_k\left(b\right)$ are polynomials of the Stirling type for the nonnegative value of k. Using $P_{-1}\left(j\beta\right)=0$ the expansion that results from Equation (8) is

$$\left[-\ln(1-y)\right]^{\delta\beta}=\sum_{k=0}^{\infty}\mathcal{P}_{k-1}(\delta\beta)y^{k+\delta\beta} \tag{9}$$

then, using Equation (2), the CDF from Equation (7) may be represented as

$$F\left(x\right)=\sum_{\delta=1}^{\infty}\sum_{k=0}^{\infty}\frac{\left(-1\right)^{\delta}}{\delta!}P_{k-1}\left(\delta\beta\right)\left[1-\frac{\alpha\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}\right]^{k+\delta\beta},$$

for |y|<1 and real non-integer b, the generalized binomial expansion holds

$$\left(1-y\right)^{c}=\sum_{m=0}^{\infty}\left(-1\right)^{m}\binom{c}{m}y^{m},$$

then by substituting generalized binomial expansion, $F(x)$ can be expressed as

$$F\left(x\right)=\sum_{\delta=1}^{\infty}\sum_{k,m=0}^{\infty}\frac{\left(-1\right)^{\delta+m}\alpha^{m}\lambda^{\sigma m}\mathcal{P}_{k-1}\left(\delta\beta\right)}{\Gamma\left(\delta+1\right)}\binom{k+\delta\beta}{m}\left(\lambda+x^{\gamma}\right)^{-\sigma m}\left[1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}\right]^{-m}, \tag{10}$$

for $|y| < 1$ and integer positive m, by using a power series expansion in Equation (10), we obtain

$$F(x) = \sum_{\delta=1}^{\infty} \sum_{k,m=0}^{\infty} \frac{(-1)^{\delta+m} \alpha^m P_{k-1}(\delta\beta)}{\Gamma(\delta+1)} \binom{k+\delta\beta}{m} \left[\lambda^{\sigma m} (\lambda + x^\gamma)^{-\sigma m}\right] \times \left(\sum_{p=0}^{\infty} (-1)^p \binom{-m}{p} \left(\overline{\alpha} \lambda^\sigma (\lambda + x^\gamma)^{-\sigma}\right)^p \right),$$

then the WMOPL distribution is attained as

$$F(x) = \sum_{m,p=0}^{\infty} \mathcal{M}_{m,p} \overline{G}(x; \sigma, \lambda, \gamma)^{m+p}, \tag{11}$$

where $\overline{G}(x; \sigma, \lambda, \gamma) = 1 - G(x; \sigma, \lambda, \gamma)$, is the Power Lomax survival function and

$$M_{m,p} = \sum_{\delta=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{\delta+m+p} \alpha^m P_{k-1}(\delta\beta)}{\Gamma(\delta+1)(1-\alpha)^{-p}} \binom{k+\delta\beta}{m} \binom{-m}{p}.$$

Based on Lehman type-II (L-II) CDF $\mathcal{F}_e(x) = 1 - \{1 - G(x)\}^e$ where e is the power parameter that is derived from G(x) as the baseline distribution. Every LT-II Power Lomax can be expressed in exponentiated Power Lomax (EPL) densities. For every real e and by $\mathcal{F}_e(x) = 1 - \{1 - G(x)\}^e$the power series converges everywhere

$$\mathcal{F}_e(x) = \sum_{s=1}^{\infty} (-1)^{s+1} \binom{e}{s} G(x)^s,$$

differentiating the above equation yields

$$f_e(x) = \sum_{s=0}^{\infty} (-1)^s \binom{e}{s+1} h_{s+1}(x),$$

where $h_{s+1}(x) = (s+1) G(x)^s g(x)$ is the EPL density. If e is an integer that is positive, the index ends at e. Then, the density function of X follows as

$$f(x) = \sum_{(m,p)\epsilon J} M_{m,p} f_{m+p}(x; \sigma, \lambda, \gamma),$$

$$f(x) = \sum_{(m,p)\epsilon J} \sum_{s=0}^{\infty} (-1)^s \mathcal{M}_{m,p} \binom{m+e}{s+1} (s+1) G(x)^s g(x) \tag{12}$$

Equation (12) demonstrates that the WMOPL density function is a linear combination of the EL density function and the LTI Power Lomax densities.

3. Statistical properties

This section focuses on some statistical properties of the WMOPL distributions, such as the quantiles function, the moment generating function, moments, conditional moments, mean residual life, and the residual life function.

3.1. Quantiles function.

Theorem 3.1. *Let X be a random variable with WMOPL distribution. The quantile function Q_p is defined by F(Q(p))=p and for every $0 < p < 1, \lambda, \sigma, \gamma > 0$ expressed by the equation:*

$$Q_p = \lambda^{\frac{1}{\gamma}} \left[\left(\frac{\left[\alpha - \overline{\alpha} e^{-(-\ln(1-p))^{1/\beta}}\right]^{\frac{1}{\sigma}}}{e^{-(-\ln(1-p))^{1/\beta\sigma}}} \right) - 1 \right]^{\frac{1}{\gamma}}, \tag{13}$$

Proof. The CDF given in Equation (6) can be written as

$$F\left(Q\left(p\right)\right) = 1 - \exp\left\{ - \left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda + {Q_p}^{\gamma})^{-\sigma}}{1 - \overline{\alpha}\lambda^{\sigma}(\lambda + {Q_p}^{\gamma})^{-\sigma}} \right] \right)^{\beta} \right\} = p,$$

then

$$\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda + {Q_p}^{\gamma})^{-\sigma}}{1 - \overline{\alpha}\lambda^{\sigma}(\lambda + {Q_p}^{\gamma})^{-\sigma}} \right] = -(-\ln\left(1-p\right))^{\frac{1}{\beta}},$$

so

$$\frac{\alpha\lambda^{\sigma}(\lambda + {Q_p}^{\gamma})^{-\sigma}}{1 - \overline{\alpha}\lambda^{\sigma}(\lambda + {Q_p}^{\gamma})^{-\sigma}} = e^{-(-\ln(1-p))^{\frac{1}{\beta}}},$$

$$(\lambda + {Q_p}^{\gamma})^{-\sigma} = \frac{e^{-(-\ln(1-p))^{\frac{1}{\beta}}}}{\lambda^{\sigma}\left(\alpha - \overline{\alpha} e^{-(-\ln(1-p))^{\frac{1}{\beta}}}\right)},$$

so, we have

$$\lambda + {Q_p}^{\gamma} = \frac{\lambda\left[\alpha - \overline{\alpha} e^{-(-\ln(1-p))^{\frac{1}{\beta}}}\right]^{\frac{1}{\sigma}}}{e^{-(-\ln(1-p))^{\frac{1}{\beta\sigma}}}},$$

hence,

$$Q_p = \lambda^{\frac{1}{\gamma}} \left[\left(\frac{\left[\alpha - \overline{\alpha} e^{-(-\ln(1-p))^{1/\beta}}\right]^{\frac{1}{\sigma}}}{e^{-(-\ln(1-p))^{1/\beta\sigma}}} \right) - 1 \right]^{\frac{1}{\gamma}}.$$

□

Remark 3.1. The three quantiles of WMOPL may be derived by setting $p = 0.25$, $p = 0.5$, and $p = 0.75$ in Equation (13).

3.2. Moments and generating functions. Moment analysis can investigate a distribution's most significant properties and characteristics (e.g., mean, variance, skewness, and kurtosis). We extract the WMOPL distribution's ordinary moments, moment generating function (MGF), Conditional moments, mean residual life, and mean deviations. The following theorem illustrates the WMOPL's n^{th} raw moment.

Theorem 3.2. *The n^{th} moments about the origin of the WMOPL distribution are given by*

$$\mu'_n = \sum_{(m,p)\epsilon J} \sum_{s=0}^{m+p} \sum_{k=0}^{s} (-1)^{s+k} \sigma \lambda^{\frac{n}{\gamma}} (s+1) \binom{s}{k} \binom{m+p}{s+1} \mathcal{M}_{m,p} \times \frac{\Gamma\left(\frac{n}{\gamma}+1\right) \Gamma\left(k\sigma+\sigma-\frac{n}{\gamma}+1\right)}{\Gamma\left((k+1)\sigma\right)} \quad (14)$$

Proof. The raw moment of the WMOPL distribution is given by

$$\mu'_n = \int_0^\infty x^n f(x)\, dx = \sum_{(m,p)\epsilon J} \sum_{s=0}^{\infty} (-1)^{s+1} \binom{m+p}{s+1} M_{m,p} \sigma \gamma \lambda^{\sigma} (s+1) \times \int_0^\infty x^{n+\gamma-1} (\lambda + x^\gamma)^{-\sigma-1} \left[1 - \lambda^\sigma (\lambda + x^\gamma)^{-\sigma}\right]^s dx$$

let $\lambda(\lambda + x^\gamma)^{-1} = z$, we have

$$x^n = \left(\frac{\lambda}{z}\right)^{\frac{n}{\gamma}} (1-z)^{\frac{n}{\gamma}},$$

using the binomial expansion, we obtain

$$(1 - z^\sigma)^s = \sum_{k=0}^{\infty} (-1)^k \binom{s}{k} z^{k\sigma},$$

after some algebraic utilizations, we have

$$\mu'_n = \sum_{(m,p)\epsilon J} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{s+k} \binom{s}{k} \binom{m+p}{s+1} M_{m,p} \sigma \lambda^{\frac{n}{\gamma}} (s+1) \times \int_0^1 z^{k\sigma+\sigma-\frac{n}{\gamma}} (1-z)^{\frac{n}{\gamma}} dz$$

then

$$\mu'_n = E\left[X^n\right] = \sum_{(m,p)\epsilon J} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{s+k} \binom{s}{k} \binom{m+p}{s+1} M_{m,p} \sigma \lambda^{\frac{n}{\gamma}} (s+1) \times \frac{\Gamma\left(\frac{n}{\gamma}+1\right) \Gamma\left(k\sigma+\sigma-\frac{n}{\gamma}\right)}{\Gamma\left((k+1)\sigma\right)}.$$

□

Remark 3.2. n^{th} central Moments μ_n of X can be determined as

$$\mu_n = \sum_{j=0}^{n} \sum_{k=0}^{s} (-1)^{s+k+j} \binom{n}{j} \binom{s}{k} M_{m,p}$$

$$\times \frac{\Gamma\left(\frac{1}{\gamma}+1\right)\Gamma\left(k\sigma+\sigma-\frac{1}{\gamma}+1\right)+\Gamma\left(\frac{n-j}{\gamma}+1\right)\Gamma\left(k\sigma+\sigma-\frac{n-j}{\gamma}+1\right)}{\Gamma\left((k+1)\sigma\right)}$$

Proof. Based on the definition of the central moment as

$$\mu_n = E(X-\mu)^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\mu_1'\mu_{n-j}',$$

and by substituting Equation (14), the proof is completed. □

Theorem 3.3. *Considering the WMOPL distribution, the* n^{th} *conditional moment,* $E\left(X^n \mid X>t\right)$, *is given by*

$$E\left(X^n \mid X>t\right) = \frac{1}{S\left(x\right)} \times \sum_{(m,p)\epsilon J}\sum_{s=0}^{m+p}\sum_{k=0}^{s}(-1)^{s+k}\sigma\lambda^{\frac{n}{\gamma}}\left(s+1\right)\binom{s}{k}\binom{m+p}{s+1}M_{m,p}$$
$$\times B\left(\frac{n}{\gamma}+1, k\sigma+\sigma-\frac{n}{\gamma}+1, t\right).$$

Proof. $E\left(X^n \mid X>t\right)$, is given by

$$E\left(X^n \mid X>t\right) = \frac{1}{S\left(t\right)}J_n\left(t\right) = \frac{1}{S\left(t\right)}\int_t^\infty x^n f\left(x\right)dx,$$

then

$$J_n\left(t\right) = \sum_{(m,p)\epsilon J}\sum_{s=0}^{m+p}\sum_{k=0}^{s}(-1)^{s+k}\sigma\lambda^{\frac{n}{\gamma}}\left(s+1\right)\binom{s}{k}\binom{m+p}{s+1}M_{m,p}$$
$$\times B\left(\frac{n}{\gamma}+1, k\sigma+\sigma-\frac{n}{\gamma}+1, t\right), \tag{15}$$

where $B(\alpha,\beta,t)$ denotes the incomplete beta function. □

The following result demonstrates the Mean Residual Life (MRL) in applying conditional moments in the WMOPL distribution. The MRL function represents the item's projected remaining life , $X-t$, if it survives to time x.

Remark 3.3. The WMOPL MRL function can be written

$$m_X\left(t\right) = E\left(X-t \mid X>t\right) = \frac{1}{S\left(t\right)}J_1\left(t\right)-t$$
$$= \sum_{(m,p)\epsilon J}\sum_{s=0}^{m+p}\sum_{k=0}^{s}(-1)^{s+k}\sigma\lambda^{\frac{1}{\gamma}}\left(s+1\right)\binom{s}{k}\binom{m+p}{s+1}M_{m,p}$$
$$\times B\left(\frac{1}{\gamma}+1, k\sigma+\sigma-\frac{1}{\gamma}+1, t\right)-t$$

where $J_1\left(t\right)$ can be acquired from Equation (15).

Further use of conditional moments is to determine the mean deviations from the mean and the median, as illustrated in the following result.

Remark 3.4. If M denotes the median, then the mean deviation from the median of the WMOPL distribution can be calculated as

$$\begin{aligned}\delta_M =&2\sum_{(m,p)\epsilon J}\sum_{s=0}^{m+p}\sum_{k=0}^{b}\sum_{b=0}^{\infty}(-1)^{s+m}\binom{m+p}{s+1}\binom{b\gamma}{k}M_{m,p}\left(s+1\right)\\ &\times\lambda^{\gamma}\frac{sb(1+\lambda M^{\gamma})^{\gamma-k(b+1)}}{\Gamma\left(b+1\right)\left[1-\frac{1}{\gamma\sigma}+b+2\right]}\\ &-\sum_{(m,p)\epsilon J}\sum_{s=0}^{m+p}\sum_{k=0}^{s}(-1)^{s+k}\sigma\lambda^{\frac{1}{\gamma}}\left(s+1\right)\binom{s}{k}\binom{m+p}{s+1}M_{m,p}\\ &\times\frac{\Gamma\left(\frac{1}{\gamma}+1\right)\Gamma\left(k\sigma+\sigma-\frac{1}{\gamma}+1\right)}{\Gamma\left(\left(k+1\right)\sigma\right)}.\end{aligned}$$

Proof. The median deviation is defined as

$$\begin{aligned}\delta_M =&E\left|X-M\right|=\int_0^{\infty}\left|x-M\right|f\left(x\right)dx\\ =&\int_0^{M}\left(M-x\right)f\left(x\right)dx+\int_M^{\infty}\left(x-M\right)f\left(x\right)dx=2J_1\left(M\right)-\mu,\end{aligned}$$

where

$$\begin{aligned}J_1\left(M\right)=&\int_M^{\infty}xf\left(x\right)dx\\ =&\sum_{(m,p)\epsilon J}\sum_{s=0}^{m+p}\sum_{k=0}^{s}(-1)^{s+k}\sigma\lambda^{\frac{1}{\gamma}}\left(s+1\right)\binom{s}{k}\binom{m+p}{s+1}M_{m,p}\\ &\times B\left(\frac{1}{\gamma}+1,k\sigma+\sigma-\frac{1}{\gamma}+1,M\right).\end{aligned}$$

Using the R software, the mean, variance, skewness, and kurtosis of the WMOPL distribution are estimated numerically for various values of the parameters. The values in Table 1 demonstrate that the skewness of the WMOPL distribution can range between $(-3.1164, 10.7706)$, whilst The PL distribution's skewness can only rang between $(2.1572, 3.1908)$. The various for the WMOPL kurtosis ranges from 0.0162 to12.7148, whereas the PL distribution's kurtosis ranges from only 2.3541 to 6.9312. In addition, the WMOPL model can be negatively or positively biased. Consequently, the WMOPL distribution is a flexible distribution that can be utilized to simulate skewed data.

□

4. Reliability measures

Several notable WMOPL reliability measurements are derived in this section.

TABLE 1. Moments of the WMOPL distribution for selected values of the parameters.

parameters	mean	variance	Skewness	kurtosis
$\alpha=0.5,\beta=3,\lambda=1.5,\gamma=2,\sigma=1$	1.0468	0.0879	0.2709	0.0162
$\alpha=1.5,\beta=1.5,\lambda=0.5,\gamma=2,\sigma=3$	0.4026	0.0289	0.6042	0.5328
$\alpha=1.5,\beta=1.5,\lambda=2,\gamma=3,\sigma=0.5$	2.3316	2.2933	3.1344	12.7148
$\alpha=1.5,\beta=2,\lambda=0.5,\gamma=3,\sigma=1.5$	0.7231	0.0327	0.1732	0.0523
$\alpha=1.5,\beta=2,\lambda=1.5,\gamma=3,\sigma=0.5$	2.0125	0.7129	1.3463	3.5434
$\alpha=1.5,\beta=2,\lambda=3,\gamma=0.5,\sigma=1.5$	10.2410	3.8222	5.4017	5.4394
$\alpha=1.5,\beta=3,\lambda=1.5,\gamma=2,\sigma=0.5$	2.8716	1.3634	0.9146	1.3688
$\alpha=2,\beta=2,\lambda=0.5,\gamma=3,\sigma=1$	1.0113	0.0827	0.4612	0.4565
$\alpha=2,\beta=2,\lambda=3,\gamma=0.5,\sigma=1$	10.9778	8.5188	8.5550	2.8985
$\alpha=2,\beta=3,\lambda=1,\gamma=0.5,\sigma=2$	0.7597	0.4058	1.7520	4.8229
$\alpha=2.5,\beta=0.5,\lambda=1,\gamma=1.5,\sigma=3$	3.6754	8.4897	10.7706	8.5938
$\alpha=2.5,\beta=0.5,\lambda=3,\gamma=1.5,\sigma=1$	10.8325	3.3571	10.4399	9.0928
$\alpha=2.5,\beta=3,\lambda=1.5,\gamma=0.5,\sigma=1$	10.2484	9.7620	-3.1164	7.0293
$\alpha=3,\beta=1.5,\lambda=0.5,\gamma=1,\sigma=2$	0.7205	0.3615	-2.2727	10.2366

4.1. Residual life function. The residual lifetime is defined as the period remaining until the event of interest occurs at age $t>0$.

Theorem 4.1. *Let X has WMOPL distribution,*

- *the survival function of the residual lifetime $R(t)$ is given by*

$$S_{R(t)}\left(x\right)=\frac{S\left(x+t\right)}{S\left(t\right)}=\frac{\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+(x+t)^{\gamma})^{-\sigma}}{1-\overline{\alpha}(\lambda+(x+t)^{\gamma})^{-\sigma}}\right]\right)^{\beta}\right\}}{\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+t^{\gamma})^{-\sigma}}{1-\overline{\alpha}(\lambda+t^{\gamma})^{-\sigma}}\right]\right)^{\beta}\right\}},\quad x>0,$$

- *the associated CDF is given by*

$$F_{R(t)}\left(x\right)=\frac{\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+t^{\gamma})^{-\sigma}}{1-\overline{\alpha}(\lambda+t^{\gamma})^{-\sigma}}\right]\right)^{\beta}\right\}-\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+(x+t)^{\gamma})^{-\sigma}}{1-\overline{\alpha}(\lambda+(x+t)^{\gamma})^{-\sigma}}\right]\right)^{\beta}\right\}}{\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+t^{\gamma})^{-\sigma}}{1-\overline{\alpha}(\lambda+t^{\gamma})^{-\sigma}}\right]\right)^{\beta}\right\}},$$

- *the corresponding pdf is given by*

$$\begin{aligned}f_{R(t)}\left(x\right)=&\alpha\beta\gamma\sigma\lambda^{\sigma}\times\frac{\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+(x+t)^{\gamma})^{-\sigma}}{1-\overline{\alpha}(\lambda+(x+t)^{\gamma})^{-\sigma}}\right]\right)^{\beta}\right\}}{\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+t^{\gamma})^{-\sigma}}{1-\overline{\alpha}(\lambda+t^{\gamma})^{-\sigma}}\right]\right)^{\beta}\right\}}\\&\times\frac{\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+(x+t)^{\gamma})^{-\sigma}}{1-\overline{\alpha}(\lambda+(x+t)^{\gamma})^{-\sigma}}\right]\right)^{\beta-1}}{\left(x+t\right)^{1-\gamma}\left(\lambda+\left(x+t\right)^{\gamma}\right)\left[1-\overline{\alpha}(\lambda+t^{\gamma})^{-\sigma}\right]},\end{aligned}$$

- *the associated hazard rate function is given by*

$$h_{R(t)}(x) = \alpha\beta\gamma\sigma\lambda^{\sigma} \times \frac{\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+(x+t)^{\gamma})^{-\sigma}}{1-\overline{\alpha}(\lambda+(x+t)^{\gamma})^{-\theta}}\right]\right)^{\beta-1}}{(x+t)^{1-\gamma}\left(\lambda+(x+t)^{\gamma}\right)\left[1-\overline{\alpha}(\lambda+t^{\gamma})^{-\sigma}\right]}.$$

4.2. Stress-strength reliability. Stress-strength models describe the life of system components that have a random strength X_1 and are subject to an unpredictably high-stress X_2. The component performs appropriately as long as $X_1 > X_2$ but fails when $X_1 < X_2$. If the stress surpasses the system's strength, the system fails. $R = P(X_1 > X_2)$ denotes the component's reliability. Stress-strength models have many applications, notably in reliability engineering concepts such as structures, aircraft structural fatigue failure, and rocket motor deterioration.

Theorem 4.2. *Assume that X_1 and X_2 to be independently distributed, with $X_1 \sim WMOPL(\alpha_1, \beta_1, \sigma_1, \gamma_1, \lambda_1)$ and $X_2 \sim WMOPL(\alpha_2, \beta_2, \sigma_2, \gamma_2, \lambda_2)$. Then, the parameter representing the Stress-strength relationship is as follows:*

$$R = 1 - \sum_{(m,p)\epsilon J}\sum_{s=0}^{\infty}\sum_{i=0}^{m+p}\sum_{j=0}^{\frac{\gamma_{1i}}{\gamma_2}}(-1)^{i+j+s}\mathcal{M}_{m,p}^{(1)}\mathcal{M}_{m,p}^{(2)}(s+1)\lambda_1{}^{(\sigma_1+1-i)}(\lambda_2)^{i}$$
$$\times\binom{m+p}{s+1}\binom{m+p}{i}\binom{\frac{\gamma_{1i}}{\gamma_2}}{j}\frac{\Gamma\left(\frac{s+m+p}{m+p}\right)\Gamma\left(\frac{\sigma_2-j}{\sigma_2}\right)}{\Gamma\left(\frac{s}{m+p}-\frac{j}{\sigma_2}\right)}.$$

Proof. By definition of the stress-strength parameter, we have

$$R = Pr(X_1 > X_2) = \int_0^{\infty} f_2(y)\left[1 - F_1(y)\right]dy = 1 - \int_0^{\infty} f_2(y)\,F_1(y)\,dy,$$

where by Equations (11) and (12) we have

$$F_1(y) = \sum_{m,p=0}^{\infty}\mathcal{M}_{m,p}^{(1)}\left[\overline{G}_1(y;\sigma_1,\lambda_1,\gamma_1)\right]^{m+p},$$

$$f_2(y) = \sum_{(m,p)\epsilon J}\sum_{s=0}^{\infty}(-1)^{s}M_{m,p}^{(2)}\binom{m+p}{s+1}(s+1)\left[G_2(y;\sigma_1,\lambda_1,\gamma_1)\right]^{s}g_2(y).$$

Now,

$$\int_0^{\infty} f_2(y)\,F_1(y)\,dy$$

$$= \sum_{(m,p)\epsilon J}\sum_{s=0}^{\infty}\mathcal{M}_{m,p}^{(1)}\mathcal{M}_{m,p}^{(2)}(-1)^{s}(s+1)\binom{m+p}{s+1}\int_0^{\infty}\overline{G}_1^{m+p}(y)G_2^{s}(y)g_2(y)dy$$

where $\overline{G}_1(y) = {\lambda_1}^{\sigma_1}(\lambda_1 + y^{\gamma_1})^{-\sigma_1}$, and $\overline{G}_2(y) = {\lambda_2}^{\sigma_2}(\lambda_2 + y^{\gamma_2})^{-\sigma_2}$, after some algebraic simplification, we get

$$(\lambda_2 + y^{\gamma_2}) = \left(\frac{\overline{G}_2(y)}{{\lambda_2}^{\sigma_2}}\right)^{\frac{-1}{\sigma_2}},$$

then $y = \left[\left(\frac{\overline{G}_2(y)}{{\lambda_2}^{\sigma_2}}\right)^{\frac{-1}{\sigma_2}} - \lambda_2\right]^{\frac{1}{\gamma_2}}$, by substituting in $\overline{G}_1(y)$, we have

$$\overline{G}_1(y) = {\lambda_1}^{\sigma_1}\left\{\lambda_1 + \lambda_2\left[\left(\overline{G}_2(y)\right)^{\frac{-1}{\sigma_2}} - 1\right]^{\frac{\gamma_1}{\gamma_2}}\right\}^{-\sigma_1},$$

so, we can write

$$\begin{aligned} A &= \int_0^\infty \overline{G}_1^{m+p}(y)\, G_2^s(y)\, g_2(y)\, dy \\ &= \int_0^\infty \left[{\lambda_1}^{-1}\left\{\lambda_1 + \lambda_2\left[\left(\overline{G}_2(y)\right)^{\frac{-1}{\sigma_2}} - 1\right]^{\frac{\gamma_1}{\gamma_2}}\right\}\right]^{(-\sigma_1)(m+p)} G_2^s(y)\, g_2(y) dy. \end{aligned}$$

Consider $G_2(y) = Z$, so $g_2(y)\, dy = dz$ and A can be rewritten as

$$A = \int_0^1 z^{\frac{s}{m+p}}\left\{1 - \frac{\lambda_2}{\lambda_1}\left[1 - (1-z)^{\frac{-1}{\sigma_2}}\right]^{\frac{\gamma_1}{\gamma_2}}\right\}^{(m+p)} dz,$$

by performing binomial series expansion in two steps, we have

$$\begin{aligned} \left\{1 - \frac{\lambda_2}{\lambda_1}\left[1 - (1-z)^{\frac{-1}{\sigma_2}}\right]^{\frac{\gamma_1}{\gamma_2}}\right\}^{m+p} &= \sum_{i=0}^{m+p} (-1)^i \binom{m+p}{i}\left(\frac{\lambda_2}{\lambda_1}\right)^i \left[1 - (1-z)^{\frac{-1}{\sigma_2}}\right]^{\frac{\gamma_{1i}}{\gamma_2}} \\ &= \sum_{i=0}^{m+p} \sum_{j=0}^{\frac{\gamma_{1i}}{\gamma_2}} (-1)^{i+j} \binom{m+p}{i}\binom{\frac{\gamma_{1i}}{\gamma_2}}{j}\left(\frac{\lambda_2}{\lambda_1}\right)^i (1-z)^{\frac{-j}{\sigma_2}}, \end{aligned}$$

as a result of substitution and simplification, we have

$$A = \sum_{i=0}^{m+p} \sum_{j=0}^{\frac{\gamma_{1i}}{\gamma_2}} (-1)^{i+j} {\lambda_1}^{(\sigma_1+1)} \binom{m+p}{i}\binom{\frac{\gamma_{1i}}{\gamma_2}}{j}\left(\frac{\lambda_2}{\lambda_1}\right)^i \int_0^1 (1-z)^{\frac{-j}{\sigma_2}} z^{\frac{s}{m+p}}\, dz,$$

then

$$A = \frac{\sum_{i=0}^{m+p}\sum_{j=0}^{\frac{\gamma_{1i}}{\gamma_2}} (-1)^{i+j} {\lambda_1}^{(\sigma_1+1)} \left(\frac{\lambda_2}{\lambda_1}\right)^i \binom{m+p}{i}\binom{\frac{\gamma_{1i}}{\gamma_2}}{j} \left(\Gamma\left(\frac{s+m+p}{m+p}\right)\Gamma\left(\frac{\sigma_2 - j}{\sigma_2}\right)\right)}{\Gamma\left(\frac{s}{m+p} - \frac{j}{\sigma_2}\right)}.$$

Then by substitution, the stress-strength parameter is represented as

$$R = 1 - \sum_{(m,p)\epsilon J} \sum_{s=0}^{\infty} \sum_{i=0}^{m+p} \sum_{j=0}^{\frac{\gamma_{1i}}{\gamma_2}} (-1)^{i+j+s} \mathcal{M}^{(1)}_{m,p} \mathcal{M}^{(2)}_{m,p} (s+1)$$

$$\times\,\lambda_1^{(\sigma_1+1-i)}\,(\lambda_2)^i\binom{m+p}{s+1}\binom{m+p}{i}\binom{\frac{\gamma_{1i}}{\gamma_2}}{j}\frac{\Gamma\left(\frac{s+m+p}{m+p}\right)\Gamma\left(\frac{\sigma_2-j}{\sigma_2}\right)}{\Gamma\left(\frac{s}{m+p}-\frac{j}{\sigma_2}\right)}.$$

□

5. Estimstion

In this section, we drive six methods to estimate the parameters of the WMOPL distribution.

5.1. Maximum likelihood (ML). Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with the WMOPL distribution and the unknown parameter vector $\theta = (\alpha, \beta, \sigma, \gamma, \lambda)^T$. Maximizing the log-likelihood function yields parameters' maximum likelihood estimate (MLE). The WMOPL distribution's log-likelihood function is provided by

$$\begin{aligned}\ell(\theta)\,\alpha n\log(\beta)&+n\log(\sigma)+n\log(\gamma)+(\gamma-1)\sum\log x_i-\sum\log\left(\lambda+x_i^\gamma\right)\\ -\sum\log\left[1-\overline{\alpha}\lambda^\sigma(\lambda+x_i^\gamma)^{-\sigma}\right]&+(\beta-1)\sum\log\left\{-\ln\left[\frac{\alpha\lambda^\sigma(\lambda+x_i^\gamma)^{-\sigma}}{1-\overline{\alpha}\lambda^\sigma(\lambda+x_i^\gamma)^{-\sigma}}\right]\right\}\\ -\sum\left\{-\ln\left\{\frac{\alpha\lambda^\sigma(\lambda+x_i^\gamma)^{-\sigma}}{1-\overline{\alpha}\lambda^\sigma(\lambda+x_i^\gamma)^{-\sigma}}\right\}\right\}^\beta&\end{aligned}\tag{16}$$

Let $\eta_i = \frac{\alpha\lambda^\sigma\left(\lambda+x_i^\gamma\right)^{-\sigma}}{1-\overline{\alpha}\lambda^\sigma\left(\lambda+x_i^\gamma\right)^{-\sigma}}$. Then, we can write as

$$\begin{aligned}\ell(\theta)\,\alpha n\log(\beta)&+n\log(\sigma)+n\log(\gamma)+(\gamma-1)\sum\log x_i-\sum\log\left(\lambda+x_i^\gamma\right)\\ -\sum\log\left[1-\overline{\alpha}\lambda^\sigma(\lambda+x_i^\gamma)^{-\sigma}\right]&+(\beta-1)\sum\log\{-\ln[\eta_i]\}-\sum\{-\ln\{\eta_i\}\}^\beta.\end{aligned}$$

The maximum likelihood estimators for the parameters $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\sigma}$, and $\widehat{\lambda}$ are generated by simultaneously solving the following derivatives concerning the five log-likelihood parameters:

$$\frac{\partial\ell(\theta)}{\partial\alpha}=-\frac{1}{\alpha}\sum_{i=1}^n\eta_i+\frac{1}{\alpha}\sum_{i=1}^n(1-\eta_i)\left[\frac{\beta-1}{\ln\eta_i}+\beta\{-\ln\eta_i\}^{\beta-1}\right]=0,$$

$$\frac{\partial l(\theta)}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^n\log\{-\ln\eta_i\}\times\left[1-\{-\ln\eta_i\}^\beta\right]=0,$$

$$\begin{aligned}\frac{\partial l(\theta)}{\partial\gamma}&=\frac{n}{\gamma}+\sum_{i=1}^n\log x_i-\sum_{i=1}^n\frac{x_i^\gamma\log x_i}{\lambda+x_i^\gamma}-\sum_{i=1}^n\frac{\sigma x_i^\gamma\log x_i\times\eta_i}{\alpha\lambda^\sigma\left(\lambda+x_i^\gamma\right)}\\ &+\sum_{i=1}^n\frac{x_i^\gamma\log x_i}{\left(\lambda+x_i^\gamma\right)}\left[1-\frac{x_i^\gamma\eta_i\log x_i}{\alpha}\right]\left[\frac{(\beta-1)}{\log\eta_i}-\beta\{-\log\eta_i\}^{\beta-1}\right]=0,\end{aligned}$$

$$\frac{\partial \ell(\theta)}{\partial \lambda}=$$

$$-\sum_{i=1}^{n}\frac{1}{\lambda+x_i^{\gamma}}+\sum_{i=1}^{n}\overline{\alpha}\sigma\eta_i\left[\frac{\lambda^{-1}+(\lambda+x_i^{\gamma})^{-1}}{\alpha}\right]+(\beta-1)\sum_{i=1}^{n}\frac{\left(\frac{1}{\lambda}+\frac{1}{\lambda+x_i^{\gamma}}\right)[\sigma-\overline{\alpha}\eta_i+\overline{\alpha}\sigma]}{\log\eta_i}$$

$$-\sum_{i=1}^{n}\beta\{-\log\eta_i\}^{\beta-1}\left(\frac{1}{\lambda}+\frac{1}{\lambda+x_i^{\gamma}}\right)[\sigma-\overline{\alpha}\eta_i+\overline{\alpha}\sigma]=0,$$

$$\frac{\partial \ell(\theta)}{\partial \sigma}=\frac{n}{\sigma}+(\beta-1)\sum_{i=1}^{n}\frac{\alpha\lambda^{\sigma}(\lambda+{x_i}^{\gamma})\left[\log\lambda-\sigma\log(\lambda+{x_i}^{\gamma})\right]}{\eta_i\log\eta_i}$$

$$-\sum_{i=1}^{n}\beta\{-\log\eta_i\}^{\beta-1}\alpha\lambda^{\sigma}\,(\lambda+{x_i}^{\gamma})\left[\log\lambda-\sigma\log(\lambda+{x_i}^{\gamma})\right]=0.$$

5.2. Ordinary least squares (OLS). Assume that the random sample $X_1, X_2, \ldots, X_n$ of size, n has the WMOPL distribution and the unknown parameters vector $\theta=(\alpha,\beta,\gamma,\sigma,\lambda)^T$ and CDF (4). The ordinary least squares (OLS) estimate $\widehat{\alpha}_{OLS}, \widehat{\beta}_{OLS}, \widehat{\gamma}_{OLS}, \widehat{\sigma}_{OLS}$ and $\widehat{\lambda}_{OLS}$ of $\alpha,\beta,\gamma,\sigma$ and λ can be determined numerically by maximizing the function.

$$OLS(\theta)=\sum_{i=1}^{n}\left[\frac{n+1-i}{n+1}-\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}(\lambda+x^{\gamma})^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}(\lambda+x^{\gamma})^{-\sigma}}\right]\right)^{\beta}\right\}\right]^2.$$

Assume that $\zeta(i,n)=\frac{n+1-i}{n+1}$ and η_i is given in equation (16). Then, we can write as

$$OLS(\theta)=\sum_{i=1}^{n}\left[\zeta(i,n)-\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\right]^2.$$

By solving the following equations, estimates can be obtained:

$$\frac{\partial OLS(\theta)}{\partial \alpha}=\sum_{i=1}^{n}\left[\zeta(i,n)-\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\right]\times\psi_1(x_i;\alpha,\beta,\lambda,\gamma,\sigma)=0,$$

$$\frac{\partial OLS(\theta)}{\partial \beta}=\sum_{i=1}^{n}\left[\zeta(i,n)-\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\right]\times\psi_2(x_i;\alpha,\beta,\lambda,\gamma,\sigma)=0,$$

$$\frac{\partial OLS(\theta)}{\partial \lambda}=\sum_{i=1}^{n}\left[\zeta(i,n)-\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\right]\times\psi_3(x_i;\alpha,\beta,\lambda,\gamma,\sigma)=0,$$

$$\frac{\partial OLS(\theta)}{\partial \gamma}=\sum_{i=1}^{n}\left[\zeta(i,n)-\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\right]\times\psi_4(x_i;\alpha,\beta,\lambda,\gamma,\sigma)=0,$$

$$\frac{\partial OLS(\theta)}{\partial \sigma}=\sum_{i=1}^{n}\left[\zeta(i,n)-\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\right]\times\psi_5(x_i;\alpha,\beta,\lambda,\gamma,\sigma)=0,$$

where

$$\psi_1\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)=\beta(-\ln\eta_i)^{\beta-1}\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\times\left(\frac{\eta_i}{\alpha}-\alpha\eta_i^2\right), \tag{17}$$

$$\psi_2\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)=(-\ln\eta_i)^{\beta}\log\eta_i\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\times\left(\eta_i^{\beta}\ln\eta_i\right), \tag{18}$$

$$\psi_3\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)=\beta(-\ln\eta_i)^{\beta-1}\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\times\left[\eta_i-\frac{\overline{\alpha}}{\alpha}{\eta_i}^2\right], \tag{19}$$

$$\psi_4\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)=\eta_i\sigma\beta{x_i}^{\gamma}\ln x_i\left(-\ln\eta_i\right)^{\beta-1}\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\times\left(\frac{\overline{\alpha}}{\alpha}-1\right), \tag{20}$$

$$\psi_5\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)=\beta(-\ln\eta_i)^{\beta-1}\exp\left\{-(-\ln\eta_i)^{\beta}\right\}\times\left[\frac{\eta_i}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}\right]. \tag{21}$$

5.3. Weighted least squares (WLS). By minimizing the following function concerning the parameters, the weighted least squares (WLS) estimates of $\alpha,\beta,\gamma,\sigma$ and λ for the WMOPL distribution are $\widehat{\alpha}_{WLS},\widehat{\beta}_{WLS},\widehat{\gamma}_{WLS},\widehat{\sigma}_{WLS}$ and $\widehat{\lambda}_{WLS}$, respectively.

$$WLS\left(\theta\right)=\sum_{t=1}^{b}\frac{\left(b+1\right)^2\left(b+2\right)}{t\left(b-t+1\right)}\left(1-\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}\right]\right)^{\beta}\right\}-\frac{t}{b+1}\right)^2,$$

Let $\omega\left(t,b\right)=\frac{(b+1)^2(b+2)}{t(b-t+1)}$, and $\S\left(t,b\right)=\frac{t}{b+1}$, then

$$WLS\left(\theta\right)=\sum_{t=1}^{b}\omega\left(i,b\right)\left(\exp\left\{-(-\ln\eta_t)^{\beta}\right\}-\S\left(t,b\right)\right)^2,$$

in the next step, solve the nonlinear equations.

$$\frac{\partial WLS\left(\theta\right)}{\partial\alpha}=\sum_{t=1}^{b}\omega\left(t,b\right)\left[\exp\left\{-(-\ln\eta_t)^{\beta}\right\}-\S\left(t,b\right)\right]\times\psi_1\left(x_t;\alpha,\beta,\lambda,\gamma,\sigma\right)=0,$$

$$\frac{\partial WLS\left(\theta\right)}{\partial\beta}=\sum_{t=1}^{b}\omega\left(t,b\right)\left[\exp\left\{-(-\ln\eta_t)^{\beta}\right\}-\S\left(t,b\right)\right]\times\psi_2\left(x_t;\alpha,\beta,\lambda,\gamma,\sigma\right)=0,$$

$$\frac{\partial WLS\left(\theta\right)}{\partial\lambda}=\sum_{t=1}^{b}\omega\left(t,b\right)\left[\exp\left\{-(-\ln\eta_t)^{\beta}\right\}-\S\left(t,b\right)\right]\times\psi_3\left(x_t;\alpha,\beta,\lambda,\gamma,\sigma\right)=0,$$

$$\frac{\partial WLS\left(\theta\right)}{\partial\gamma}=\sum_{t=1}^{b}\omega\left(t,b\right)\left[\exp\left\{-(-\ln\eta_t)^{\beta}\right\}-\S\left(t,b\right)\right]\times\psi_4\left(x_t;\alpha,\beta,\lambda,\gamma,\sigma\right)=0,$$

and

$$\frac{\partial WLS\left(\theta\right)}{\partial\sigma}=\sum_{t=1}^{b}\omega\left(t,b\right)\left[\exp\left\{-(-\ln\eta_t)^{\beta}\right\}-\S\left(t,b\right)\right]\times\psi_5\left(x_t;\alpha,\beta,\lambda,\gamma,\sigma\right)=0,$$

where (16), (17), (18), (19), (20), and (21) denote $\eta_{\mathfrak{A}t}$ and $\psi_j\left(x_t;\alpha,\beta,\lambda,\gamma,\sigma\right)$ $\left(j=1,2,3,4,5\right)$.

5.4. Cramer-Von Mises estimators (CM). Minimizing the following function yields the Cramér-Von Mises estimators for the unknown parameters of the WMOPL distribution.

$$C\left(\alpha,\beta,\lambda,\gamma,\sigma\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2\left(i-n\right)-1}{2n}+\exp\left\{-\left(-\ln\left[\frac{\alpha\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x^{\gamma}\right)^{-\sigma}}\right]\right)^{\beta}\right\}\right]^{2},$$

with respect to $\alpha,\beta,\gamma,\sigma$ and λ. Let $\delta\left(i,n\right)=\frac{2(i-n)-1}{2n}$ and $\eta_i=\frac{\alpha\lambda^{\sigma}\left(\lambda+x_i^{\gamma}\right)^{-\sigma}}{1-\overline{\alpha}\lambda^{\sigma}\left(\lambda+x_i^{\gamma}\right)^{-\sigma}}$. Then, we can write

$$CM\left(\alpha,\beta,\lambda,\gamma,\sigma\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\delta\left(i,n\right)+\exp\left\{-\left(-\ln\eta_i\right)^{\beta}\right\}\right]^{2}.$$

The following equations can be solved numerically to obtain these parameters:

$$\frac{\partial CM\left(\theta\right)}{\partial\alpha}=\sum_{i=1}^{n}\left[\delta\left(i,n\right)-\exp\left\{-\left(-\ln\eta_i\right)^{\beta}\right\}\right]\times\psi_1\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)=0,$$

$$\frac{\partial CM\left(\theta\right)}{\partial\beta}=\sum_{i=1}^{n}\left[\delta\left(i,n\right)-\exp\left\{-\left(-\ln\eta_i\right)^{\beta}\right\}\right]\times\psi_2\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)=0,$$

$$\frac{\partial CM\left(\theta\right)}{\partial\lambda}=\sum_{i=1}^{n}\left[\delta\left(i,n\right)-\exp\left\{-\left(-\ln\eta_i\right)^{\beta}\right\}\right]\times\psi_3\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)=0,$$

$$\frac{\partial CM\left(\theta\right)}{\partial\gamma}=\sum_{i=1}^{n}\left[\delta\left(i,n\right)-\exp\left\{-\left(-\ln\eta_i\right)^{\beta}\right\}\right]\times\psi_4\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)=0,$$

$$\frac{\partial CM\left(\theta\right)}{\partial\sigma}=\sum_{i=1}^{n}\left[\delta\left(i,n\right)-\exp\left\{-\left(-\ln\eta_i\right)^{\beta}\right\}\right]\times\psi_5\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)=0,$$

where $\psi_i\left(x_i;\alpha,\beta,\lambda,\gamma,\sigma\right)\left(i=1,2,3,4,5\right)$ are given by equations (17)–(21), respectively.

5.5. Maximum product of spacing (MPS). If the independently random sample $X1,X2,\ldots,Xn$ of n distributed random variables have the WMOPL distribution, then the MPS estimates $\widehat{\alpha}_{MPS},\widehat{\beta}_{MPS},\widehat{\lambda}_{MPS},\widehat{\gamma}_{MPS}$ and $\widehat{\sigma}_{MPS}$ of $\alpha,\beta,\lambda,\gamma$, and σ, can be obtained by maximizing the following function.

$$MPS\left(\alpha,\beta,\lambda,\gamma,\sigma\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln\delta_i\left(\alpha,\beta,\lambda,\gamma,\sigma\right),$$

Where the discrepancies between the cumulative distribution function values assessed at consecutive ordered values are represented by

$$\delta_{\zeta}\left(\alpha,\beta,\lambda,\gamma,\sigma\right)=F\left(x_{(\zeta)}\mid\alpha,\beta,\lambda,\gamma,\sigma\right)-F\left(x_{(\zeta-1)}\mid\alpha,\beta,\lambda,\gamma,\sigma\right),\quad\zeta=1,\ldots,n,$$

Where $F\left(x_{(0)}\mid\alpha,\beta,\lambda,\gamma,\sigma\right)=0$ and $F\left(x_{(\zeta+1)}\mid\alpha,\beta,\lambda,\gamma,\sigma\right)=1$.

By calculating the following derivatives concerning the five parameters, the maximum product of spacing estimators for the parameters is derived:

$$\frac{\partial MPS(\theta)}{\partial\alpha}=\frac{\psi_1(x_i|\alpha,\beta,\lambda,\gamma,\sigma)-\psi_1(x_{i-1}|\alpha,\beta,\lambda,\gamma,\sigma)}{\delta_i(\alpha,\beta,\lambda,\gamma,\sigma)}=0,$$

$$\frac{\partial MPS(\theta)}{\partial\beta}=\frac{\psi_2(x_i|\alpha,\beta,\lambda,\gamma,\sigma)-\psi_2(x_{i-1}|\alpha,\beta,\lambda,\gamma,\sigma)}{\delta_i(\alpha,\beta,\lambda,\gamma,\sigma)}=0,$$

$$\frac{\partial MPS(\theta)}{\partial\lambda}=\frac{\psi_3(x_i|\alpha,\beta,\lambda,\gamma,\sigma)-\psi_3(x_{i-1}|\alpha,\beta,\lambda,\gamma,\sigma)}{\delta_i(\alpha,\beta,\lambda,\gamma,\sigma)}=0,$$

$$\frac{\partial MPS(\theta)}{\partial\gamma}=\frac{\psi_4(x_i|\alpha,\beta,\lambda,\gamma,\sigma)-\psi_4(x_{i-1}|\alpha,\beta,\lambda,\gamma,\sigma)}{\delta_i(\alpha,\beta,\lambda,\gamma,\sigma)}=0,$$

$$\frac{\partial MPS(\theta)}{\partial\sigma}=\frac{\psi_5(x_i\mid\alpha,\beta,\lambda,\gamma,\sigma)-\psi_5(x_{i-1}\mid\alpha,\beta,\lambda,\gamma,\sigma)}{\delta_i(\alpha,\beta,\lambda,\gamma,\sigma)}=0.$$

Where $\psi_j(x_i\mid\alpha,\beta,\lambda,\gamma,\sigma)$ $(j=1,2,3,4,5)$ are defined by Equations (17)-(21), respectively.

5.6. Anderson–Darling (AD). By minimizing the function $AD(\theta)$, we may obtain the Anderson–Darling estimates (ADs) of the parameters $\alpha,\beta,\lambda,\gamma$, and σ, and, indicated by $\widehat{\alpha}_{ADE},\widehat{\beta}_{ADE},\widehat{\lambda}_{ADE},\widehat{\gamma}_{ADE}$ and $\widehat{\sigma}_{ADE}$.

$$AD(\theta)=-n-\frac{1}{n}\sum_{\tau=1}^{n}(2\tau-1)\left\{\ln F(x_\tau\mid\alpha,\beta,\gamma,\lambda,\sigma)+\ln\overline{F}(x_{n-\tau+1}\mid\alpha,\beta,\gamma,\lambda,\sigma)\right\},$$

so

$$\frac{\partial AD(\theta)}{\partial\alpha}=\sum_{\tau=1}^{n}(2\tau-1)\left\{\frac{\psi_1(x_\tau|\alpha,\beta,\lambda,\gamma,\sigma)}{F(x_\tau\mid\alpha,\beta,\lambda,\gamma,\sigma)}-\frac{\psi_1(x_{n-\tau+1}|\alpha,\beta,\lambda,\gamma,\sigma)}{\overline{F}(x_{n-\tau+1}\mid\alpha,\beta,\lambda,\gamma,\sigma)}\right\},$$

$$\frac{\partial AD(\theta)}{\partial\beta}=\sum_{\tau=1}^{n}(2\tau-1)\left\{\frac{\psi_2(x_\tau|\alpha,\beta,\lambda,\gamma,\sigma)}{F(x_\tau\mid\alpha,\beta,\lambda,\gamma,\sigma)}-\frac{\psi_2(x_{n-\tau+1}|\alpha,\beta,\lambda,\gamma,\sigma)}{\overline{F}(x_{n-\tau+1}\mid\alpha,\beta,\lambda,\gamma,\sigma)}\right\},$$

$$\frac{\partial AD(\theta)}{\partial\lambda}=\sum_{\tau=1}^{n}(2\tau-1)\left\{\frac{\psi_3(x_\tau|\alpha,\beta,\lambda,\gamma,\sigma)}{F(x_\tau\mid\alpha,\beta,\lambda,\gamma,\sigma)}-\frac{\psi_3(x_{n-\tau+1}|\alpha,\beta,\lambda,\gamma,\sigma)}{\overline{F}(x_{n-\tau+1}\mid\alpha,\beta,\lambda,\gamma,\sigma)}\right\},$$

$$\frac{\partial AD(\theta)}{\partial\gamma}=\sum_{\tau=1}^{n}(2\tau-1)\left\{\frac{\psi_4(x_\tau|\alpha,\beta,\lambda,\gamma,\sigma)}{F(x_\tau\mid\alpha,\beta,\lambda,\gamma,\sigma)}-\frac{\psi_4(x_{n-\tau+1}|\alpha,\beta,\lambda,\gamma,\sigma)}{\overline{F}(x_{n-\tau+1}\mid\alpha,\beta,\lambda,\gamma,\sigma)}\right\},$$

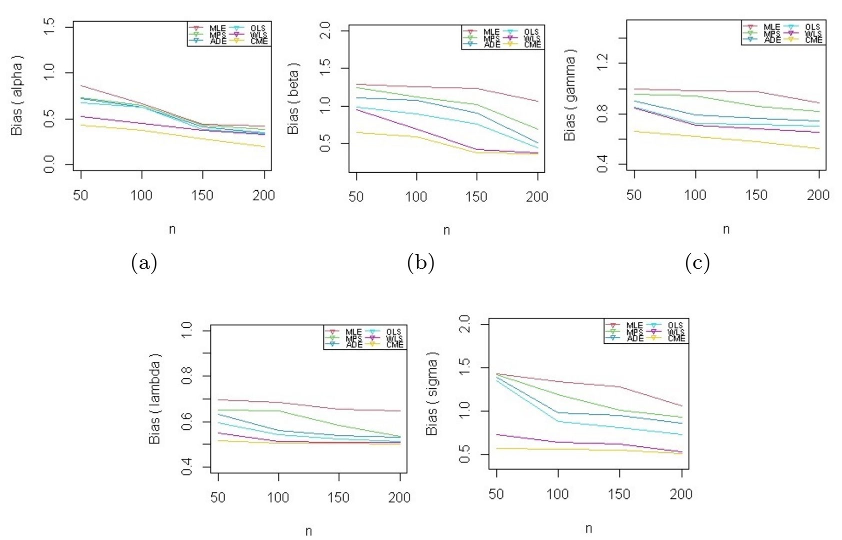
$$\frac{\partial AD(\theta)}{\partial\sigma}=\sum_{\tau=1}^{n}(2\tau-1)\left\{\frac{\psi_5(x_\tau|\alpha,\beta,\lambda,\gamma,\sigma)}{F(x_\tau\mid\alpha,\beta,\lambda,\gamma,\sigma)}-\frac{\psi_5(x_{n-\tau+1}|\alpha,\beta,\lambda,\gamma,\sigma)}{\overline{F}(x_{n-\tau+1}\mid\alpha,\beta,\lambda,\gamma,\sigma)}\right\},$$

6. Simulation results

This section shows the results of the six methods using simulations. The data are created using the WMOPL distribution with different values of n. We produce thousand random samples from the WMOPL distribution for each setting. We obtain the average values of the biases and root mean squared errors (RMSEs) associated with the estimations, so that

$$Bias_{\widehat{\theta}}=\frac{\sum_{i=1}^{1000}\left(\widehat{\theta}_i-\theta\right)}{1000} \quad \text{and} \quad RMSE_{\widehat{\theta}}=\sqrt{\frac{\sum_{i=1}^{1000}\left(\widehat{\theta}_i-\theta\right)^2}{1000}},$$

where $\theta=(\alpha,\beta,\gamma,\sigma,\lambda)^T$ and $\widehat{\theta}=\left(\widehat{\alpha},\widehat{\beta},\widehat{\gamma},\widehat{\sigma},\widehat{\lambda}\right)^T$. Figure 2 support the conclusion that all of the estimators are asymptotically unbiased since their biases tend to zero as n grows. The RMSEs of all estimators converge to zero as n grows large, demonstrating their consistency. Except in a few instances, the CME, WLS, and OLS respectively estimate outperform the other estimates in terms of minimal biases and RMSEs. Overall, MLEs tend to have the most significant biases and RMSEs compared to the other approaches. So estimating the WMOPL distribution's unknown parameters using the three techniques CME, WLS, and OLS respectively technique makes sense.



(a) (b) (c)

(d) (e)

FIGURE 2. Biases for the estimates of $(\alpha,\beta,\lambda,\gamma,\sigma) = (1,1,0.5,0.5,0.5)$ for different estimation methods.

6.1. Applications. In this section we demonstrate WMOPL distribution's flexibility using two real data sets. The WMOPL distribution is checked and compared with other competitive distributions such as Marshall-Olkin extended Power Lomax, Kumaraswamy Power Lomax, Marshall-Olkin Alpha Power Lomax, Beta Generalized Weibull (Singla et al., 2012), Beta Marshall Olkin Normal (Alizadeh et al., 2015), Marshall-Olkin Gumbel Lomax (Nwezza and Ugwuowo, 2020), Exponentiated Power Lomax, for its fit to the data sets. Using specific goodness-of-fit analytical measures, such as maximum log-likelihood, AIC (Akaike information criterion), BIC (Bayesian information criterion), and KS (Kolmogorov–Smirnov) statistics with their PV (p-value), we compared the competing distributions. The analyses are conducted in the R environment. The first set of data contains indomethacin plasma concentrations following intravenous administration. We used 66 observations from pooled data. These findings were published in (Kwan et al., 1976). Table 2 contains the data.

Table 2. Plasma Concentration data

1.5	0.94	0.78	0.48	0.37	0.19	0.12	0.11	0.08	0.07
0.05	2.03	1.63	0.71	0.70	0.64	0.36	0.32	0.20	0.25
0.12	0.08	2.72	1.49	1.16	0.80	0.80	0.39	0.22	0.12
0.11	0.08	0.08	1.85	1.39	1.02	0.89	0.59	0.40	0.16
0.11	0.10	0.07	0.07	2.05	1.04	0.81	0.39	0.30	0.23
0.13	0.11	0.08	0.10	0.06	2.31	1.44	1.03	0.84	0.64
0.42	0.24	0.17	0.13	0.10	0.09				

The second set of data is on aircraft windshield failure and repair times. These data were published by (Murthy et al., 2004) and analyzed in (Ramos et al., 2013). Table 3 contains the data.

Table 3. Aircraft windshield failure

0.046	1.436	2.592	0.140	1.492	2.600	0.150	1.580	2.670	0.248
1.719	2.717	0.280	1.794	2.819	0.313	1.915	2.820	0.389	1.920
2.878	0.487	1.963	2.950	0.622	1.978	3.003	0.900	2.053	3.102
0.952	2.065	3.304	0.996	2.117	3.483	1.003	2.137	3.500	1.010
2.141	3.622	1.085	2.163	3.665	1.092	2.183	3.695	1.152	2.240
4.015	1.183	2.341	4.628	1.244	2.435	4.806	1.249	2.464	4.881
1.262	2.543	5.140							

Table 4 shows The WMOPL distribution fits these data better than the Marshall-Olkin extended Power Lomax, Marshall-Olkin Alpha Power Lomax, Beta Generalized Weibull, Beta Marshall Olkin Normal.

Table 5 shows the novel distribution fits this data set better than the Type II Topp- Leone Power Lomax, Exponentiated Power Lomax, Marshall-Olkin Gumbel Lomax, and Kumaraswamy Lomax. The goodness-of-fit measures for

Table 4. The goodness-of-fit measures of considered models for data set 1.

Model	Parameters	Estimates	-log lik	AIC	BIC	K-S	p-value
WMOPL	α	1.1370	196.8414	403.6829	414.6311	0.0962	0.8185
	β	0.5475					
	λ	0.1882					
	γ	0.2860					
	σ	0.3842					
MOEPL	α	0.8742	216.7913	459.9423	496.2843	0.1088	0.7475
	β	0.6901					
	γ	0.1961					
	λ	0.2544					
MOAPL	α	0.9969	220.91524	550.8389	506.8354	0.1067	0.4406
	β	0.6778					
	θ	0.2016					
	λ	2.5753					
BGW	a	0.1533	230.0527	506.1053	527.1409	0.1275	0.2338
	b	0.1150					
	c	17.0159					
	α	1.0342					
	β	11.1016					
BMON	a	6.3730	243.0828	2065.363	2081.904	0.1875	0.2413
	b	0.5420					
	c	0.0076					
	μ	11.1698					
	σ	41.7503					

the fitted WMOPL model and another fitted distributions to both data sets are presented in Table 4 and Table 5, respectively.

Additionally, we estimate the unknown parameters from data set I and data set II using the estimating methods outlined. The six methods used to estimate the WMOPL parameters and the KS and PV values are reported in Tables 6 and 7 for both data sets. The Figures in these tables show that the WMOPL parameters can be estimated using the OLS technique for data set I and the WLS method for data set II. All estimating approaches, however, perform well on both sets of data. Figure 3 displays the histograms and estimated densities derived from the estimating methods for both data sets. Figure 3 supports the statistics in Table 6 and Table 7.

TABLE 5. The goodness-of-fit measures of considered models for data set 2.

Model	Parameters	Estimates	-log lik	AIC	BIC	K-S	p-value
WMOPL	α	1.8614	60.29747	130.5949	141.3106	0.0806	0.8349
	β	1.6281					
	λ	1.0810					
	γ	1.3884					
	σ	1.3782					
TIITLPL	θ	213.2225	68.9327	269.0398	269.5461	0.1530	0.6097
	α	3.6880					
	β	1.2282					
	λ	186.8420					
MOGL	p	23.6608	69.04595	206.1838	216.8995	0.09571	0.6911
	μ	1.0716					
	σ	1.6599					
	α	62.2514					
	λ	28.1483					
EPL	α	7.7389	71.9512	215.6657	224.2383	0.1439	0.7598
	β	1.9232					
	λ	0.0118					
	c	7.7389					
KWL	a	1.6991	60.4338	209.7353	218.3078	0.09332	0.4285
	b	60.5673					
	α	2.5649					
	β	65.0640					

TABLE 6. The parameter estimations for data set 1 using various approaches, KS statistics, and the accompanying p-values.

Method	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\gamma}$	$\widehat{\sigma}$	-log lik	AIC	BIC	K-S	p-value
OLS	1.1370	0.5475	0.1882	0.2860	0.3842	196.8414	403.6829	414.6311	0.0962	0.8185
WLS	1.2201	0.4964	0.2624	0.7325	0.6107	136.1586	282.3172	293.2655	0.1002	0.7801
CME	1.2743	47.2702	1.2246	0.0269	0.1306	270.3152	550.6304	561.5787	0.1057	0.7230
MPS	3.3130	3.5847	2.8739	3.5716	3.6499	258.6457	527.2914	538.2397	0.1116	0.6603
ADE	0.8542	2.0043	0.1169	0.9405	1.7210	307.2467	624.4933	635.4416	0.1230	0.5394
MLE	0.8542	2.0043	0.1169	0.9405	0.5394	346.1392	682.2784	713.2267	0.1453	0.3346

TABLE 7. The parameter estimations for data set 2 using various approaches, KS statistics, and the accompanying p-values.

Method	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\gamma}$	$\widehat{\sigma}$	-log lik	AIC	BIC	K-S	p-value
OLS	1.6583	1.4412	1.4287	1.1528	1.1073	74.40548	158.811	169.5266	0.101424	0.705342
WLS	1.8614	1.6281	1.0810	1.3884	1.3782	60.29747	130.5949	141.3106	0.080688	0.834944
CME	1.6805	1.1406	1.4513	1.2467	1.0084	103.8194	217.6387	228.3544	0.129268	0.123738
MPS	1.5961	1.3293	1.5068	1.2179	0.9481	87.51124	185.0225	195.7381	0.11326	0.568848
ADE	1.2454	1.4915	1.4011	1.6754	1.4499	85.17651	180.353	191.0687	0.104908	0.658069
ne MLE	0.8968	0.8042	0.6443	0.8680	0.4757	173.327	356.6541	367.3697	0.160465	0.145761

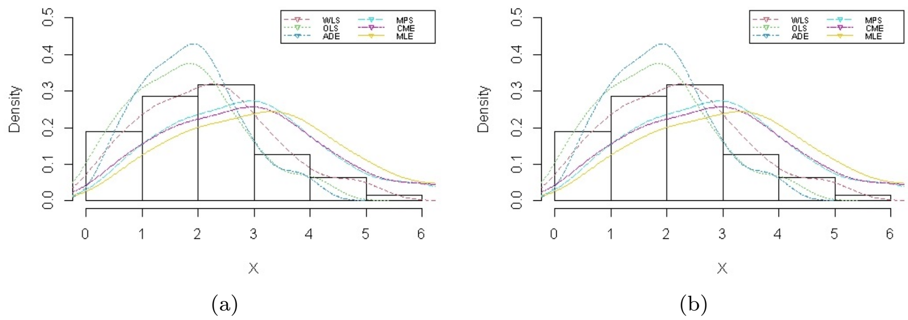


FIGURE 3. The fitted densities of the WMOPL distribution for different methods for data set 1 (a) and data set 2 (b).

7. Conclusions

We introduce the five-parameter Weibull Marshall–Olkin Power Lomax (WMOPL) distribution, which contains some well-known distributions as unique models. The WMOPL failure rate function can analyze lifetime data efficiently. The novel distribution can substitute for various generalized forms of the Power Lomax and Weibull distributions. Some statistical properties of the proposed model such as the quantiles function, moments, mean residual life and mean deviations, variance, skewness, kurtosis, and reliability measures like the residual life function, reversed residual life functions, and stress-strength reliability, are discussed. The model parameters are estimated by the six methods including Maximum likelihood, Ordinary least squares, Weighted least squares, Cramer-Von Mises, Maximum product of spacing, and Anderson–Darling methods, and a simulation study is conducted to assess the performance of the various estimators. We demonstrate that the proposed distribution can provide better fits and flexibility than other distributions through two applications to real data.

Conflicts of interest: The authors declare no conflict of interest.

Data availability: Not applicable

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