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ON THE FIBERS OF THE TREE PRODUCTS OF GROUPS WITH AMALGAMATION SUBGROUPS

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Abstract. The tree products of groups with amalgamation subgroups are generalizations of the free products of groups with amalgamation subgroup. The aim of this paper is to construct a tree called the standard tree where the tree products of groups with amalgamation subgroups act without inversions and then find the quotient of this action. Furthermore, we show that if the amalgamation subgroups are finite and the factor groups act on disjoint trees then there exists a tree called the fiber tree where the tree products of groups with amalgamation subgroups act without inversions and find the quotients of this action. If each factor is a tree products with amalgamation subgroups, we get a new fiber tree and the corresponding factors.

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1. Introduction

Within the specialized domain of group theory, this article endeavors to unravel the intricate fibers of tree products of groups with amalgamation subgroups. The study of fibers in groups acting on trees with inversions was initially illuminated through [1], Theorem 6.1, establishing a foundation for our investigation. Now, expanding our lens, we delve into the expansive landscape of tree products of groups with amalgamation subgroups and their corresponding fiber structures.

While our focus remains resolutely on the realm of group theory, it is worth noting the parallel intellectual pursuits that have contributed to the broader

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mathematical landscape. Notably, Al-Husban et al. (2022) presented a groundbreaking work on "Multi-Fuzzy Rings" in the context of the multimember ship function, offering insights into multi-fuzzy binary operations and their potential applications within ring theory. Their findings, as published in WSEAS Transactions on Mathematics [2], provide a glimpse into the diverse applications of mathematical concepts in various contexts.

Similarly, Hawawsheh et al. (2023) embarked on a journey into the world of spherical integral operators in their article "Lp-Mapping Properties of a Class of Spherical Integral Operators" [3]. By establishing inequalities that connect these operators with Marcinkiewicz integral operators, they contribute to the further understanding of this specialized area. Their study, published in Axioms, showcases the synergy between theoretical insights and practical applications within the mathematical domain.

Building upon this trajectory, Hazaymeh et al. delved into the realm of numerical radius inequalities and perturbed Milne's quadrature rules. In "On Further Refinements of Numerical Radius Inequalities" [5], they present novel extensions of existing numerical radius inequalities, exemplifying the dynamic nature of mathematical research in refining and enhancing known results. Their unified approach fosters a deeper understanding of the interplay between different mathematical concepts.

Furthermore, Hazaymeh et al.'s work in "A Perturbed Milne's Quadrature Rule for n-Times Differentiable Functions with Lp-Error Estimates" [4] showcases the utilization of perturbed quadrature rules for n-times differentiable functions, extending the applicability of Milne's quadrature rule. By addressing specific scenarios and offering new insights, they emphasize the practical relevance of theoretical developments, underscoring the symbiotic relationship between theory and application.

Finally, Qawasmeh et al. (2023) contribute to the enhancement of numerical radius inequalities with a refined perspective in "Further Accurate Numerical Radius Inequalities" [6]. This work not only highlights the ongoing efforts in refining mathematical inequalities but also illustrates the iterative nature of mathematical research, where existing results are continually built upon and improved.

In the midst of these diverse intellectual pursuits, our exploration of tree products of groups with amalgamation subgroups stands as a testament to the multifaceted nature of mathematical inquiry and its potential to illuminate new avenues of understanding.

The tree products of groups with amalgamation subgroups serve as a natural generalization of the free products of groups with amalgamation subgroups. This paper is driven by the overarching aim of establishing a robust framework for understanding these structures. Our primary objective is the construction of a "standard tree," a pivotal concept where the actions of tree products of groups with amalgamation subgroups transpire devoid of inversions. Notably, the insights derived from Theorem 6.1 of [1] guide our approach in this endeavor.

For readers seeking a comprehensive grasp of our work, [1] provides valuable insights. The essential definitions concerning graphs, subgraphs, connected graphs, trees, subtrees, paths, and groups' actions on graphs – encompassing stabilizers, orbits, and quotient graphs – are amply expounded across references such as [1, 7, 8, 9, 10, 11, 12]. Those delving into groups' actions on graphs without inversions will find [7, 12] enlightening, while the intricacies of groups with inversions are detailed in [9, 10, 11]. Intriguingly, the conceptual foundation required for this article is substantially covered within [1].

The significance of this study lies in its profound implications for advancing our understanding of tree products of groups with amalgamation subgroups and their inherent fiber structures. By investigating the behaviors of these groups in the context of tree actions, we seek to unlock potential avenues for further research and exploration. The crux of the problem statement revolves around discerning the standard tree and the fiber tree pertinent to groups with amalgamation subgroups, elucidating their distinctive properties, and delving into the implications of resulting quotient actions. In doing so, we anticipate not only enriching our comprehension of group actions and amalgamation subgroups but also contributing to the broader domain of applied mathematics and its real-world applications.

2. Presentations of groups acting on trees

This section will delve into the most well-liked and powerful presentation ideas for groups acting on trees as we explore the intriguing world of groups acting on trees. Mathematicians have been enthralled by the study of groups and how they interact with trees for decades since it provides a wealth of information about the complex interactions between combinatorial geometry and algebraic structures.

Let G be a group acting on a graph X. We list the following concepts:

(i) $V(X)$ is the set of vertices of X and $E(X)$ is the set of edges of X on which $V(X) \neq \varnothing$ and $V(X) \cap E(X) = \varnothing$.

(ii) If $e \in E(X)$, then e is an edge of X, the terminal of e are $o(e) \in V(X)$, the initial of $e, t(e) \in V(X)$, the terminal of e and $\overline{e} \in E(X)$, the inverse of e satisfying the condition that $o(\overline{e}) = t(e)$, $t(\overline{e}) = o(e)$, and $\overline{\overline{e}} = e$.

(iii) The stabilizer of $x \in X$, where x is a vertex or an edge of X is denoted G_x is the set $G_x = \{g \in G : g(x) = x\}$, where $g(x)$ is the value of x under the action of G on X via the element g. It is clear that $G_x \leq G$ is a subgroup of G.

(iv) The orbit of $x \in X$, where x is a vertex or an edge of X is denoted $G(x)$ is the set $G(x) = \{g(x) = x \mid g \in G\}$. It is clear that $G(x) \subseteq X$ is a subset of X.

(v) The set of orbits of the action of G on X is denoted by G/X and is defined to be the set $G/X = \{G(x) : x \in X\}$. It is clear that G/X

forms a graph of set of vertices $V(G/X) = \{G(v) : v \in V(X)\}$ and a set of edges $E(G/X) = \{G(e) : e \in E(X)\}$.

If X is connected, then G/X is connected.

(vi) If $e_1, e_2, \ldots, e_n$ are edges of X, then $P = (e_1, e_2, \ldots, e_n)$ is called a path in X if $t(e_i) = o(e_{i+1})$, $i = 1, \ldots, n-1$, and is called reduced if $e_{i+1} \neq \overline{e_l}, i = 1, \ldots, n-1$. If $o(e_1) = v$ and $t(e_n) = u$, then P is called a path in X joining the vertices u and v. Clearly, a graph X is a tree if and distinct vertices of X are joining by exactly one reduced path.

(vii) The set of vertices of X fixed by G is the set

$$X^G = \{v \in V(X) \mid G_v = G\} \subseteq V(X).$$

(viii) If $g \in G$ is an element of G and $e \in E(X)$ is an edge of X such that $g(e) = \bar{e}$, then g we say that g contains an invertor edge of X.

(ix) If T is a subtree of X then T is called a tree of representatives for the action of G on X if T satisfies the condition that T contains exactly one vertex from each vertex orbit. That is, if $v \in V(T)$ then $V(T) \cap G(v)$ is of cardinality $|V(T) \cap G(v)| = 1$. Then every vertex $v \in V(X)$, there exists a unique vertex denoted v^* such that $v^* \in V(T)$ and $G(v) = G(v^*)$. That is, $v = g(v^*), g \in G$, where $G(v) = \{g(v) : g \in G\}$ is the orbit containing v. v^* is called the representative of the vertex v.

(x) If T is a tree of representative and Y is a subgraph of X such that $T \subseteq Y$, then Y is called a transversal for the action of G on X if Y satisfies the following conditions:

i. If $e \in E(Y)$, then $o(e)$ or $t(e) \in V(T)$.

ii. If $y \in E(Y)$, then y and $\bar{y}$ are in different orbits in case $G(y) \neq G(\bar{y})$.

iii. If $x \in E(Y)$, then x and $\bar{x}$ are in same orbit in case $G(x) = G(\bar{x})$.

$(T; Y)$ is called a fundamental domain for the action of G on X.

(xi) The following subsets of edges of above transversal Y are called the splitting edges of Y:

i. $E_0(Y) = \{m \in E(Y) : o(m), t(m) \in E(T)\} = E(T)$, the set of edges of T.

ii. $E_1(Y) = \{y \in E(Y) : o(y) \in E(T), t(y) \notin E(T), G(y) \neq G(\overline{y})\}$.

iii. $E_2(Y) = \{x \in E(Y) : o(x) \in E(T), t(x) \notin E(T), G(x) = G(\overline{x})\}$.

(xii) For the edge $e \in E(Y)$ we have the following concepts related to the fundamental domain (T, Y):

i. If $o(e) \in V(T)$, there exists $[e] \in G$ and called the value of e such that $[e]\left((t(e))^*\right) = t(e)$.

Furthermore, we choose $[e] = 1$ in case $e \in E(T)$, and $[e](e) = \bar{e}$ if $G(\bar{e}) = G(e)$.

ii. The sign of e, denoted $+e$ is defined to be the edge $+e = e$ if $o(e) \in V(T)$ and $+e = [e](e)$ if $t(e) \in V(T)$.

It is clear that $o(+e) = (o(e))^*$, $t(+e) = [e]\left((t(e))^*\right)$, $\overline{+e} = [e](+\bar{e})$, and $G_{+e} \leq G_{(o(e))^*}$.

(xiii) If $g \in G$, the sum of g and e is denoted by $g \oplus e$ and is defined to be the pair $g \oplus e = (gG_{+e}, +e)$. We have the following facts: The proofs are clear.

i. $g \oplus m = (gGm_m, m)$, $g[m] \oplus \overline{m} = (gGm, \overline{m}) = g \oplus \overline{m}$, $m \in E_0(Y)$.

ii. $g \oplus y = (gG_y, y)$, $g \oplus \bar{y} = \big(gG_{[y]^{-1}(y)}, [y]^{-1}(\bar{y})\big)$,

$g[y] \oplus \bar{y} = \left(g[y]G_{[y]^{-1}(y)}, \overline{[y]^{-1}(y)}\right)$, $y \in E_1(Y)$.

iii. $g \oplus x = g[x] \oplus \bar{x} = \overline{(g[x]G_x, x)} = g[x] \oplus x, x \in E_2(Y)$.

(xiv) The inverse of $g \oplus e$ is $\overline{g \oplus e} = g[e] \oplus \bar{e}$.

(xv) The star of X is denoted by X^* and is defined to be the set
$X^* = \{g \oplus p \mid g \in G, p \in E(Y)\} = \{(gG_{+p}, +p) \mid g \in G, p \in E(Y)\}$.

It is clear that

$$X^* = \{g \oplus m, g \oplus y, g \oplus \bar{y}, g \oplus x \mid m \in E_0(Y), y \in E_1(Y), x \in E_2(Y)\},$$

and $X^* \approx E(X)$. That is, there exists a map $\theta : X^* \to E(X)$ giving by $\theta(g \oplus e) = g(+e)$ which is one-one and onto.

We end this section the following important theorem.

Theorem 2.1. *Let G be a group acting on a graph X of fundamental domain (T, Y) of the action of G on X such that $X^G = \varnothing$. Then*

(i) X is connected if and only if the set

$$\Delta(Y) = \{G_v, [e] \mid v \in V(T), e \in E(Y)\}$$

generates G.

(ii) If Y is a subtree of X, then X is a tree if and only if G has the presentation

$$G = \Big\langle G_v, y, x \mid \mathrm{rel}(G_v), G_m = G_{\bar{m}}, y \cdot [y]^{-1} G_y [y] \cdot y^{-1} = G_y, \\ x \cdot G_x \cdot x^{-1} = G_x, x^2 = [x]^2 \Big\rangle, \tag{1}$$

via the mapping $G_v \to G_v, y \to [y], x \to [x]$, where $v \in V(T), m \in E_0(Y), y \in E_1(Y)$, and $x \in E_2(Y)$, where the synbols of this presentation can be find in [10, 11].

Proof. (i) See Lemma 2.1 of [11].

(ii) See Theorem 5.8 of [10] and Theorem 5.1 of [11].

□

3. The standard trees of tree products of groups with amalgamation subgroups

The concepts of tree products of groups with amalgamation subgroups were introduces in [8] as follows. Let Z be a tree. For each vertex $v \in Z$ and an edge $e \in E(Z)$ let A_v and A_e be groups such that A_e is a subgroup of both groups $A_{o(e)}$ and $A_{t(e)}$ and there exists an isomorphism $\theta_e : A_e \to A_{\bar{e}}$ of the property that $\theta_{\bar{e}}^{-1} = \theta_e$. The tree product of the groups $A_v, v \in V(Z)$ (with the subgroups A_e, $A_{\bar{e}}, e \in E(Z)$) with a tree Z is denoted by

$$A = \pi^* \left(A_v; \theta_e \left(A_e\right) = A_{\bar{e}}\right), v \in V(Z), e \in E(Z),$$

or by

$$A = \pi^* \left(A_v; A_e = A_{\bar{e}}\right), v \in V(Z), e \in E(Z),$$

or by

$$A = \left\langle A_v \middle| \operatorname{rel}\left(A_v\right), A_e = A_{\bar{e}} \right\rangle, v \in V(Z), e \in E(Z),$$

when specific θ_e are implicit, where $\left\langle A_v \middle| \operatorname{rel}\left(A_v\right) \right\rangle$ stands for any presentation of the group A_v and $A_e = A_{\bar{e}}$ stands for the set of relations $\tilde{h} = \widetilde{\theta}_e(h)$ for all $e \in E(Z)$, and all $h \in A_e$, where $\tilde{h}$ is a word of the generating symbols of the presentation $\left\langle A_{o(e)} \mid \operatorname{rel}\left(A_{o(e)}\right) \right\rangle$ of the group $A_{o(e)}$ of value h and $\widetilde{\theta}_e(h)$ is a word of the generating symbols of the presentation $\left\langle A_{t(e)} \middle| \operatorname{rel}\left(A_{t(e)}\right) \right\rangle$ of the group $A_{t(e)}$ of value $\theta_e(h)$. It is proved in [8] that the group A_v is imbedded in A via the imbedding $\lambda_v : A_v \to A$ and the group A is generated by the generators of the subgroups $\lambda_v\left(A_v\right) \leq A$ of A, for all $v \in V(Z)$. For simplicity, we write $\lambda_v\left(A_v\right) = A_v$ and say that A_v is a subgroup of A. If $A_v \neq A$ for all $v \in V(Z)$ then we say that A is a non-trivial tree product of groups.

Remark 3.1. For $v \in V(Z)$ and $e \in E(Z)$, the groups A_v are called the factor groups and the groups A_e are called the amalgamation subgroups of the tree product of groups A described above.

The main result of this section is the following theorem.

Theorem 3.1. *A group is a tree product of groups if and only if there exists a tree on which the group acts without inversions, fixing no vertices, and of tree of representatives equals its transversal.*

Proof. ($\Rightarrow$). Let G be a group acting on the tree X of fundamental domain $(T; Y)$ such that $T = Y$. We need to show that G is a tree product of groups with amalgamation subgroups. Since $X^G = \varnothing$ and X is a tree, by Theorem 2.1, G has the presentation (1) of Theorem 2.1. Since $T = Y$, therefore $E_1(Y) = E_2(Y) = \varnothing$. Therefore the presentation (1) of G becomes $G = \left\langle G_v \middle| \operatorname{rel}\left(G_v\right), G_m = G_{\bar{m}} \right\rangle$, $m \in E_0(Y) = E(T)$. This shows that G is a tree product of the groups G_v, $v \in V(X)$ with amalgamation subgroups G_m, $m \in E_0(Y)$ where $\theta_m : G_m \to G_{\bar{m}}$ is the identity map. Since $X^G = \varnothing$, therefore $G_v \neq G$ for all $v \in V(X)$. Consequently, G is a tree product of groups with amalgamation subgroups.

($\Leftarrow$) Conversely, assume that $G = A = \pi^* \left(A_v; A_e = A_{\bar{e}}, \right)$, $v \in V(Z)$, $e \in E(Z)$, is the tree product of the groups A_v with amalgamation subgroups A_e, $e \in E(Z)$ of tree Z. Then $A = \left\langle A_v \middle| \operatorname{rel}\left(A_v\right), A_e = A_{\bar{e}} \right\rangle$. Let X_A be the set

$X_A = \{(aA_v, v), (a(A_e), e) \mid a \in A, v \in V(Z), e \in E(Z)\}$.
We need to show that X_A forms a tree of set of vertices

$$V(X_A) = \{(aA_v, v) \mid a \in A, v \in V(Z)\},$$

and a set of edges

$$E(X_A) = \{(a(A_e), e) \mid a \in A, e \in E(Z)\}.$$

If $V(X_A) \cap E(X_A) \neq \varnothing$ then there exist $a \in A$, $v \in V(Z)$, $e \in E(Z)$ such that $(aA_v, v) = (a(A_e), e)$. So $v = e$. This implies that $V(X) \cap E(X) \neq \varnothing$. This contradicts the assumption that X is a graph. For $a \in A, e \in E(X)$, define the following ends of the edge $(a(A_e); e)$, $o((a(A_e), e)) = \left(a\left(A_{o(e)}\right), o(e)\right)$, $t((a(A_e), e)) = \left(a\left(A_{t(e)}\right), t(e)\right)$, and $\overline{((a(A_e), e))} = (aA_{\bar{e}}, \bar{e})$. It is clear that $o\left(\overline{(a(A_e), e)}\right) = o((aA_{\bar{e}}, \bar{e})) = \left(aA_{o(\bar{e})}, o(\bar{e})\right) = \left(aA_{t(e)}, t(e)\right) = t((a(A_e), e))$, $t\left(\overline{(a(A_e), e)}\right) = t((aA_{\bar{e}}, \bar{e})) = \left(aA_{t(\bar{e})}, t(\bar{e})\right) = \left(aA_{o(e)}, O(e)\right) = o((a(Ae), e))$, and $\overline{\overline{(a(A_e), e)}} = \overline{(a(A_{\bar{e}}), \bar{e})} = (a(A_{\bar{\bar{e}}}), \bar{\bar{e}}) = (a(A_e), e)$. This shows that

$$X_A = \{(aA, v), (a(A_e), e) \mid a \in A, v \in V(X), e \in E(Z)\},$$

forms a graph where

$$V(X_A) = \{(aA, v) \mid a \in A, v \in V(Z)\},$$

and

$$E(X_A) = \{(a(A_e), e) \mid a \in A, e \in E(Z)\}.$$

Now for $f, g \in A, v \in V(Z), e \in E(Z)$ define $f(gA_v, v) = (fgA_v, v)$ and $f(g(A_e), e)) = (fg(A_e), e))$. Furthermore, for $a \in A$

$$f\left(\overline{(a(A_e), e))}\right) = \overline{f(a(A_e), e)} = \overline{(fa(A_e), e)} = \overline{f(a(A_e), e)}.$$

This implies that A acts on X_A. If $f, a \in A$, $e \in E(Z)$ such that $f(a(A_e); e) = \overline{(a(A_e); e)} = (aA_{\bar{e}}; \bar{e})$ then $e = \bar{e}$. This contradicts the assumption that Z is a tree. Thus, A acts on X_A without inversions on X_A. Now we show that X_A is connected. Let $T_A = \{(A_v, v), (A_e; e) \mid v \in V(Z), e \in E(Z)\}$. It is clear that T_A is a subgraph of X_A of set of vertices $V(T_A) = \{(A_v; v) \mid v \in V(Z)\}$ and set of edges $E(T_A) = \{(A_e; e) \mid e \in E(Z)\}$.
Now we show that T_A is a subtree of X_A. Let $\alpha, \beta \in V(T_A)$. We need to find a reduced path in T_A joining α and β. The structure of T_A shows that there exists two vertices $u, v \in V(Z)$ such that $\alpha = (A_v, v)$ and $\beta = (A_u, u)$. Since Z is a tree, there exist a unique reduced path $P = (e_1, e_2, \ldots, e_n)$ in Z joining u and v. So $o(e_1) = v$, $t(e_i) = o(e_{i+1}), e_{i+1} \neq \overline{e_l}, i = 1, \ldots, n-1$, and $t(e_n) = u$. Let T_P be the set $T_P = ((A_{e_1}, e_1), (A_{e_2}, e_2), \ldots, (A_{e_n}, e_n))$. Then T_P is a path in T_A from α to β because $o(A_{e_1}, e_1) = \left(A_{o(e_1)}, o(e_1)\right) = (A_u, v) = \alpha$, $t((A_{e_n}, e_n)) = \left(A_{t(e_n)}, t(e_n)\right) = (A_v, u) = \beta$, $t(A_{e_i}, e_i) = \left(A_{t(e_i)}, t(e_i)\right) = \left(A_{o(e_{i+1})}, o(e_{i+1})\right) = o\left(\left(A_{e_{i+1}}, e_{i+1}\right)\right)$.
Now we show that T_P is reduced. For, if for some i we have $\left(A_{e_{i+1}}, e_{i+1}\right) =$

$\overline{(A_{e_l}, e_l)}$, then $\left(A_{e_{i+1}}, e_{i+1}\right) = (A_{\overline{e_l}}, \overline{e_l})$. So $e_{i+1} = \overline{e_l}$. This contradicts above that P is a reduced path. So T_A is a subtree of X_A. Since for each $a \in A, v \in V(Z)$, we have $(aA_v, v) = a\,(A_v, v)$, this shows that T_A is a tree of representative for the action of A on X_A.

If $a \in A$ and $e \in E(Z)$, then $(a\,(A_e)\,; e) \in E\,(X_A)$, $(a\,(A_e)\,; e) = a\,(A_e; e)$, $o\,(A_e; e) = \left(A_{o(e)}; o(e)\right) \in T_A$ and $t\,(A_e; e) = (A_t(e); t(e)) \in T_A$. This shows that the transversal $Y_A = T_A$ is a transversal of the action of A on X_A on which (T_A, T_A) is a fundamental domain of the action of A on X_A.

For any vertex $(aA_v, v) \in V\,(X_A)$ we have $(aA_v, v) = a\,((A_v, v))$ and $(A_v, v) \in V\,(T_A)$. This implies that $(aA_v, v)^* = (A_v, v)$ is the representative of the vertex (aA_v, v).

For any edge $(A_e, e) \in Y_A = T_A, t\,((A_e, e)) = (A_t(e))\,, t(e)) \in T_A$ on which $(t\,((A_e, e)))^* =, t\,((A_e, e))$. This implies that the value $[(A_e, e)]$ of the edge (A_e, e) is $[(A_e, e)] = 1$, the identity element of A. For any vertex $(aA_v, v) \in V\,(X_A)$ and any element $g \in A$, if $g\,((aA_v, v)) = (aA_v, v)$, then $g \in aA_{va}a^{-1}$. This implies that the stabilizer of (aA, v) is $A_{(aA_{v,v})} = aA_v a^{-1}$, a conjugate of A_v. If $aA_v a^{-1} = A$, then $A_v = A$. This contradicts the definition of A. Consequently, the set of vertices of X_A fixed by A is $(X_A)^A = \varnothing$.

Since A is generated by the set $\{A_v \mid v \in V(Z)\}$, the steps (a)-(e) show that A is generated by the set

$$\left\{aA_v a^{-1}, 1 \mid v \in V(Z)\right\} = \left\{A_{(aA_{v,v})}, [(A_e, e)] \mid v \in V(X), e \in E(Z)\right\} = \Delta\,(Y_A)\,.$$

Then Theorem 2.1-(1) shows that X_A is a connected graph. It remains to show that X_A is a tree.

From the structure of the fundamental domain (T_A, T_A) is a fundamental domain of the action of A on X_A obtained above shows that $E_0\,(Y_A) = T_A$, $E_1\,(Y_A) = \varnothing$, $E_2\,(Y_A) = \varnothing$. So the above steps show that the relations $y \cdot [y]^{-1} G_y [y] \cdot y^{-1} = G_y, x \cdot G_x \cdot x^{-1} G_x, x^2 = [x]^2$ of Theorem 3.1, do not exist and A is a tree products of groups with amalgamation subgroup can be modified to be of presentation $A = \left\langle A_{(A_{v,v})} \left| \mathrm{rel}\left(A_{(A_{v,v})}\right), A_{(A_{e,e})} = A_{(A_{\bar{e},\bar{e}})} \right.\right\rangle = \langle A_v |\, A_e = A_{\bar{e}} \rangle$ for all $v \in V(X)$ and all $e \in E(X)$. For the case above that $Y_A = T_A$, Theorem 3.1 implies that X_A forms a tree on which A acts without inversions. $\square$

Remark 3.2. X_A is called the standard tree of the tree products A of groups.

We have the summary of the concepts of section 2 applied to the action of the tree products $A = \pi^*\,(A_v; \theta_e\,(A_e) = A_{\bar{e}},)\,, v \in V(Z), e \in E(Z)$ on its standard tree X_A.

(i) $X_A = \{(aA, v), (a\,(A_e)\,, e) \mid a \in A, v \in V(X), e \in E(Z)\}$,
$V\,(X_A) = \{(aA, v) \mid a \in A, v \in V(Z)\}$
and $E\,(X_A) = \{(a\,(A_e)\,, e) \mid a \in A, e \in E(Z)\}$.

(ii) If $a \in A, e \in E(X)$, and $(a\,(A_e)\,; e) \in E\,(X_A)$, then the ends of the edge $(a\,(A_e)\,; e)$ are $o\,((a\,(A_e)\,, e)) = ((a\,(A_o(e))\,, o(e))\,, t\,((a\,(A_e)\,, e))) = ((a\,(A_t(e))\,, t(e)))$, and $\overline{(a\,(A_e)\,, e))} = (aA_{\bar{e}}, \bar{e})$.

(iii) If $a \in A, v \in V(Z)$ and $e \in E(X)$, then the stabilizer of the vertex (aA_v, v) is $A_{(aA_{v,v})} = aA_{va}a^{-1}$, a conjugate of A_v and the stabilizer of the edge $(a(A_e), e)$ is $A_{(aA_e,e)} = aA_e a^{-1}$, a conjugate of A_e.

(iv) The orbit of the vertex $(aA;v)$ is $A((aA;v)) = (A/A_v) \times \{v\}$, where $A/A_v = \{aA \mid a \in A\}$ is the set of left cosets $aA_v = \{ab \mid b \in A_v\}$ and the orbit of the edge is is $A((aAe;e)) = (A/A_e) \times \{e\}$.

(v) The quotient graph of the action of A on X_A is $A/X_A = \{(A/A_v) \times \{v\}, (A/A_e) \times \{e\} \mid v \in V(X), e \in E(X)\}$ forms a tree of set of vertices $V(A/X_A) = \{(A/A_v) \times \{v\} \mid v \in V(X)\}$, and of set of edges $E(A/X_A) = \{(A/A_e) \times \{e\} \mid e \in E(X)\}$, where

$$o((A/A_e) \times \{e\}) = (A/A_o(e)) \times \{o(e)\}, t((A/A_e) \times \{e\}) \\ = \left(\left(A/A_{t(e)}\right) \times \{t(e)\}\right).$$

and $\overline{((A/A_e) \times \{e\})} = (A/A_{\bar{e}}) \times \{\bar{e}\}$.

(vi) If $P = (e_1, e_2, \ldots, e_n)$ is called a path in Z, then $T_P = ((A_{e_1,}e_1), (A_{e_2}, e_2), \ldots, (A_{e_n}, e_n))$ is a path in X_A. If P is reduced then T_P is reduced.

(vii) The set of vertices of X_A fixed by A is the set $(X_A)^A = \varnothing$.

(viii) $T_A = \{(A_v, V), (A_e; e) \mid v \in V(Z), e \in E(Z\}$ is a tree of representatives for the action of A on X_A. $(aA_v, v)^* = (A_v, v)$ is the representative of the vertex (aA_v, v).

(ix) $Y_A = T_A$ is the transversal for the action of A on X_A, and (T_A, T_A) is the fundamental domain for the action of A on X_A.

(x) The splitting edges of $Y_A = T_A$ are
   1) $E_0(Y_A) = E(T_A) = \{(A_e; e) \mid e \in E(Z\}$.
   2) $E_1(Y_A) = \varnothing$.
   3) $E_2(Y_A) = \varnothing$.

(xi) $[(A_e, e)] = 1$ is the value of the edge (A_e, e) and $+(A_e, e) = (A_e, e)$ is the sign of the edge (A_e, e).

(xii) If $a \in A$, and $(A_e, e) \in E(X_A)$, then $a \oplus (A_e, e) = (aA_e, (A_e, e))$.

(xiii) $(X_A)* = \{(aA_e, (A_e, e)) \mid a \in A, e \in E(Z)\}$ is the star of X_A.

(xiv) The inverse of the edge $a \oplus (A_e, e) = (aA, (A_e, e))$ is $\overline{(aA_e, (A_e, e))} = \overline{a \oplus (A_e, e)} = a[(A_e, e)] \oplus \overline{(A_e, e)} = a1(A_{\bar{e}}, \bar{e}) = (aA_{\bar{e}}, \bar{e}) = a \oplus (A_{\bar{e}}, \bar{e})$.

4. The fibers of tree products of groups with amalgamation subgroups

The following concept is needed for the rest of this section.

Let G be a group, $H \leq G$ and let H act on a graph X. For each $g \in G$ and each $x \in X$ let $g \otimes_H x$ be the set $g \otimes_H X = \left\{\left(gh, h^{-1}(x)\right) \mid h \in H\right\}$. In ([7], p.78) it is defined $G \otimes_H X$ to be the set $G \otimes_H X = \{g \otimes_H x \mid g \in G, x \in X\}$.

Then the following hold. Let $g \in G, x \in X, A \subseteq G$, and $Y \subseteq X$. Then

(i) $g \otimes_H Y = \{g \otimes_H y \mid y \in Y\}$,

(ii) $g \otimes_H X = \{g \otimes_H X \mid X \in X\}$,

(iii) $A \otimes_H X = \{a \otimes_H X \mid a \in A\}$,
(iv) $A \otimes_H Y = \{a \otimes_H y \mid a \in A, y \in Y\}$,
(v) $G \otimes_H Y = \{g \otimes_H y \mid g \in G, y \in Y\}$,
(vi) $G \otimes_H X = \{g \otimes_H X \mid g \in G, X \in X\}$.

The following lemma is needed for the proofs of this section.

Lemma 4.1. *Let K be a group acting on a graph Γ and $e \in E(\Gamma)$ be an edge of Γ such that the stabilizer $K_{o(e)}$ of the initial of e acts on a tree Ω and $K_{o(e)}$ fixes no vertex of Ω.*

That is, $(\Omega)^{K_{o(e)}} = \varnothing$. If K_e is finite and contains no inverter elements of Ω, that is, $k(p) \neq \bar{p}$ for all $k \in K$ and all edges $p \in E(\Omega)$, then there exists a vertex denoted we such that $w_e \in V(\Omega) K_e \leq \left(K_{o(e)}\right)_{w_e}$, where $\left(K_{o(e)}\right)_{w_e}$ is the stabilizer of the vertex w_e under the action of $K_{o(e)}$ on Ω. w_e is called the vertex of e in Ω.

Proof. See Corollary 3.5 of [1]. □

Theorem 4.2. *Let $A = \pi^*\left(A_v; A_e = A_{\bar{e}},\right), v \in V(Z), e \in E(Z)$ be a tree product of groups with amalgamation subgroups of a tree Z such that each factor group $A_v, v \in V(Z)$ acts on a tree X_v such that $X_u \cap X_v = \varnothing$ for all $u \in V(Z), u \neq v$. Furthermore, assume that the amalgamation subgroups $A_e, e \in E(Z)$ are finite and containing no inversions of the tree $X_{o(e)}$. Then for $e \in E(Z), w_{(A_e,e)} \in V\left(X_o(e)\right)$ is the vertex of the edge (A_e, e) obtained in Lemma 4.1, and $w_{\overline{(A_e,e)}} = w_{(A_{\bar{e}},\bar{e})} \in V\left(X_{t(e)}\right)$.*

i. The set

$$\widetilde{(X_A)} = \Big\{a \otimes_{A_v} k_v, a \otimes_{A_v} m_v, (aAe; (A_e, e)) \mid a \in A, v \in V(Z)$$
$$e \in E(Z), k_v \in V\left(X_v\right), m_v \in E\left(X_V\right)\Big\}$$

forms a tree called the fiber tree for the action of A on X_A of set of vertices $V\left(\widetilde{(X_A)}\right) = \{a \otimes_{A_v} k_v \mid a \in A, v \in V(Z), k_v \in V\left(X_v\right)\}$, and of the set of edges

$$E\left(\widetilde{(X_A)}\right) = \Big\{a \otimes_{A_v} m_v, \left(aAA_e, \left(A_e, e\right), \left(A_e, e\right)\right) \mid a \in A, e \in E(Z),$$
$$m_v \in E\left(X_v\right)\Big\}.$$

The initials, the terminals, and the inverses of the edges of $E\left(\widetilde{(X_A)}\right)$ are as follows.

(i) If $\alpha = a \otimes_{A_v} m_v, a \in A, e \in E(Z), m_v \in E\left(X_v\right)$, then $o(\alpha) = a \otimes_{A_v} o\left(m_v\right), t(\alpha) = a \otimes_{A_v} t\left(m_v\right)$, and, $\bar{\alpha} = a \otimes_{A_v} \overline{m_v}$.

(ii) *If* $\alpha \in (X_A)^*$, *then* $\alpha = aA_e \oplus (A_e, e) = (aA_e, (A_e, e)), o(\alpha) = a \otimes_{A_{o(e)}} w_{(A_e,e)}, t(\alpha) = a \otimes_{A_{t(e)}} w_{(A_{\bar{e}},\bar{e})}$, *and* $\bar{\alpha} = a\,[A_e, e] \oplus \overline{(A_e, e)} = a1 \oplus (A_{\bar{e}}, \bar{e}) = a \oplus (A_{\bar{e}}, \bar{e}) = (aA_{\bar{e}}, (A_{\bar{e}}, \bar{e}))$.

ii. *A acts on* $\widetilde{(X_A)}$. *If* $v \in V(Z), A_v$ *acts on* X_v *with inversions, then* A *acts on* $\widetilde{(X_A)}$ *with inversions.*

Proof. (1) A acts on it is standard tree X_A of fundamental domain (T_A, T_A) where $T_A = \{(A_v, V), (A_e, e) \mid v \in V(Z), e \in E(Z)\}$. By Theorem 6.1-(1) of [1], the set

$$\widetilde{(X_A)} = \cup_{v\in V(Z)} [A \otimes_{A_v} X_v] \cup (X_A)^* = \Big\{a \otimes_{A_v} k_v, a \otimes_{A_v} m_v, (aAA_e, (A_e, e)) \mid a \in A, v \in V(Z), e \in E(Z), k_v \in V(X_V), m_v \in (X_v)\Big\},$$

$$V\left(\widetilde{(X_A)}\right) = U_{v\in V(Z)} [A \otimes_{A_v} V(X_v)] = \{a \otimes_{A_V} k_v \mid a \in A, v \in V(Z), k_v \in V(X_v)\},$$

and,

$$E\left(\widetilde{(X_A)}\right) = U_{v\in V(Z)} [A \otimes_{A_v} E(X_v)] \cup (X_A)^* = \{a \otimes_{A_v} m_v, (aA; A_e (A_e, e)) \mid a \in A, e \in E(Z), m_v \in E(X_v)\}.$$

Let $\alpha \in E\left(\widetilde{(X_A)}\right)$. If $\alpha = a \otimes_{A_v} m_v$, $a \in A$, $e \in E(Z)$, $m_v \in E(X_v)$, then $o(\alpha) = o(a \otimes_{A_v} m_v) = a \otimes_{A_v} o(m_v)$, $t(\alpha) = t(a \otimes_{A_v} m_v) = a \otimes_{A_v} t(m_v)$, and $\bar{\alpha} = \overline{a \otimes_{A_v} m_v} = a \otimes_{A_v} \overline{m_v}$. If $\alpha \in (X_A)^*$, then $\alpha = a \oplus (A_e, (A_e, e)) = (aA_e; (A_e, e))$, $[(A_e, e)] = 1$, the value of the edge $(A_e; e)$. This implies that $o(\alpha) = o((aAA_e; (A_e, e))) = a \otimes_{A_{o(e)}} w_{(A_e,e)}$, $t(\alpha) = t((aAA_e, (A_e, e))) = a\left[(aAA_e; (A_e, e)] \otimes_{A_{t(e)}} w_{(A_{\bar{e}},\bar{e})} = a \otimes_{A_{t(e)}} w_{(A_{\bar{e}},\bar{e})}\right]$ and $\bar{\alpha} = \overline{(aA_e, e)} = (aA_{\bar{e}}, \bar{e})$.

(2) Theorem 6.1-(1) of [1] shows that if $g, a \in A$, $v \in V(Z)$, $e \in E(Z)$, $k_v \in V(X_v)$, $m_v \in E(X_v)$ then it is clear that the rules $g(a \otimes_{A_v} k_v) = ga \otimes_{A_v} k_v$, $g(a \otimes_{A_v} m_v) = ga \otimes_{A_v} m_v$, and $g(a \oplus (A_e, (A_e, e))) = g((acA, e, (A_e, e))) = (gaA_e, (A_e, e))$ define an action of A on $\widetilde{(X_A)}$. If $v \in V(Z), A_v$ acts on X_v with inversions, then there exists an element $g \in A_v$ and an edge $e \in E(X_v)$ such that $g(e) = \bar{e}$. Then $\overline{1 \otimes_{A_v} e} = 1 \otimes_{A_v} \bar{e} = 1 \otimes_{A_v} g(e) = g(1 \otimes_{A_v} e)$. Thus, g maps the edge $1 \otimes_{A_v} e$ into its inverse $\overline{1 \otimes_{A_v} e} = 1 \otimes_{A_v} \bar{e}$. Consequently, the action of A on $\widetilde{(X_A)}$ is with inversions.

□

In the next corollary and proposition A, X_v, and A_e will be as in Theorem 4.2.

Corollary 4.3. *For $a \in A, v \in V(Z)$, $e \in E(Z), k \in V(X_v), m \in E(X_v)$, we have*

(i) *The stabilizer of the vertex $a \otimes_{A_v} k$ is $A_a \otimes_{A_v} k = a(A_v)_k a^{-1}$, a conjugate of A_v, where $(A_v)_k$ is the stabilizer of the vertex k under the action of A_v on X_v.*
(ii) $A_{a \otimes_{A_v} m} = a(A_v)_m a^{-1}$.
(iii) $A_{(aA_e;e)} = aA_e a^{-1}$.
(iv) *If the stabilizer of every element of X_v under the action of A_v on X_v is finite, then the stabilizer of every element of $\widetilde{(X_A)}$ under the action of A on $\widetilde{(X_A)}$ is finite.*
(v) *The orbit of the element $a \otimes_{A_v} k$ is $A(a \otimes_{A_v} k) = A \otimes_{A_v} A_v(k) = \{x \otimes_{A_v} y \mid x \in A, y \in A_v(k)\}$, where $A_v(k)$ is the orbit of the vertex k under the action of A_v on X_v.*
(vi) $A(a \otimes_{A_v} m) = A \otimes_{A_v} A_v(m)$.
(vii) *$A((aA_e, e)) = (A/A_e) \times \{e\}$, where $A/A_e = \{xA \mid x \in A\}$ is the set of left cosets of A_e in A, $xAA_e = \{xb \mid b \in A_e\}$.*

Corollary 4.4.

$$A/\widetilde{(X_A)} = \Big\{A \otimes_{A_v} A_v(k), A \otimes_{A_v} A_v(m), (A/A_e) \times \{e\} \mid v \in V(Z), e \in E(Z), k \in V(X_v)\, m \in E(X_v)\Big\}.$$

is the quotient graph for the action of A on $\widetilde{(X_A)} \cdot A/\widetilde{(X_A)}$ is connected, and, the set of vertices is $V\left(A/\widetilde{(X_A)}\right) = \{A \otimes_{A_v} A_v(k) \mid v \in V(Z), k \in V(X_v)\}$ and the set of edges

$$E\left(A/\widetilde{(X_A)}\right) = \Big\{A \otimes_{A_v} A_v(m), (A/A_e) \times \{e\} \mid v \in V(Z), e \in E(Z), m \in E(X_v)\Big\}.$$

The initials, the terminals, and the inverses of the edges of $A\widetilde{(X_A)}$ are as follows. $o(A \otimes_{A_v} A_v(m)) = A \otimes_{A_v} A_v(o(m))$, $t(A \otimes_{A_v} A_v(m)) = A \otimes_{A_v} A_v(t(m))$, $\overline{A \otimes_{A_v} A_v(m)} = A \otimes_{A_v} A_v(\bar{m})$, $o((A/A_e) \times \{e\}) = A \otimes_{A_{o(e)}} A_v(o(e))$, $t((A/A_e) \times \{e\}) = A \otimes_{A_{t(e)}} A_v(t(e))$, and, $\overline{(A/A_e) \times \{e\}} = (A/A_{\bar{e}}) \times \{\bar{e}\}$.

5. Applications

In this section we apply Theorem 4.1 to the cases where the tree $X_v = \{y_v\}$ consists of a single vertex y_v and to the case where the factor groups are tree products of groups of amalgamation subgroups as follow.

Proposition 5.1. *Let $A = \pi^* (A_v; A_e = A_{\bar{e}},)$, $v \in V(Z)$, $e \in E(Z)$ be a tree product of groups with amalgamation subgroups such that the factor group A_v acts on a tree $X_v = \{y_v\}$ of one vertex y_v and no edges. Then*

(i) $V(X_v) = \{y_v\}, E(X_v) = \varnothing$, *and the vertex of the edge $e \in E(Z)$ is $w_{(A_e,e)} = y_{o(e)}$ and $w_{(A_{\bar{e}},\widetilde{e})} \in= y_{t(e)}$.*

(ii) For $a \in A, v \in V(Z)$, $a \otimes_{A_v} y_v = aA_v \times \{y_v\}$, where aA_v is the left coset $aA_v = \{ap \mid p \in A_v\}$.

(iii) The fiber $\widetilde{(X_A)}$ is a tree of set of vertices

$$V\left(\widetilde{(X_A)}\right) = U_{v \in V(Z)}\left[(A/A_v) \times \{y_v\}\right] = \{(A/A_v) \times \{y_v\} \mid v \in V(Z)\}$$

and the set of edges $E\left(\widetilde{(X_A)}\right) = \{(aA_e; e) \mid a \in A, e \in E(X)\}$. The initials, the terminals, and the inverses of the edges of $E\left(\widetilde{(X_A)}\right)$ are as follows

$\alpha = (aA, e)$, $a \in A$, $e \in E(Z)$, $o(\alpha) = aAA_o(e) \times \left\{y_{o(e)}\right\}$, $t(\alpha) = aA_{t(e)} \times \left\{y_{t(e)}\right\}$, and $\bar{\alpha} = (aA_{\bar{e}}, \bar{e})$.

(iv) $V\left(\widetilde{(X_A)}\right) = \{a \otimes_{A_v} y_v \mid a \in A, v \in V(Z)\}$
$= \{aA \times \{y_v\} \mid a \in A, v \in V(Z)\} = U_{v \in V(Z)}\left[(A/A_v) \times \{y_v\}\right],$
$E\left(\widetilde{(X_A)}\right) = \{(aAe; e) \mid a \in A, e \in E(Z)\}.$

(v) The initials, the terminals, and the inverses of the edges of $E\left(\widetilde{(X_A)}\right)$ are as follows.

$o((aAe; e)) = aA_{o(e)} \times \left\{y_{o(e)}\right\}$, $t\left((aA_e; e)\right) = aA_{t(e)} \times \left\{y_{t(e)}\right\}$, and $\overline{(aA_e, e)} = (aA_{\bar{e}}, \bar{e})$.

Proof. First we remind the readers that the amalgamated subgroups A_e, $e \in E(Z)$ need not to be finite or containing no invertors because $X_{o(e)} = \left\{y_{o(e)}\right\}$, a single vertex on which the stabilizer $A_{y_{o(e)}} = A_{o(e)}$.

(i) For $a \in A$, $v \in V(Z)$, The action of A_v on $\{y_v\}$ shows that $a \otimes_{A_v} y_v = aA_v \times \{y_v\}$, where aA_v is a left coset of A_v by a.

(ii) By the definition of $\otimes_{A_v}$,

$$a \otimes_{A_v} y_v = \left\{\left(ah, h^{-1}(y_v)\right) \mid h \in A_v\right\} = \{(ah, y_v \mid h \in A_v\} = aA_v \times \{y_v\}$$

because A_v acts on $\{y_v\}$ so that $b(y_v) = y_v$ for all $b \in A_v$.

(iii) $\widetilde{(X_A)}$ is a tree of set of vertices $V\left(\widetilde{(X_A)}\right) = U_{v \in V(Z)}\left[(A/A_v) \times \{y_v\}\right]$, $\{(A/A_v) \times \{y_v\} \mid v \in V(Z)\}$ and a set of edges

$$E\left(\widetilde{(X_A)}\right) = \{(aA_e; e) \mid a \in A, e \in E(X)\}.$$

The initials, the terminals, and the inverses of the edges of $E\left(\widetilde{(X_A)}\right)$ are as follows.

$\alpha = (aA; e), a \in A, e \in E(Z), o(\alpha) = aA_{o(e)} \times \left\{y_{o(e)}\right\}, t(\alpha) = aA_{t(e)} \times \left\{y_{t(e)}\right\}$, and $\bar{\alpha} = (aA_{\bar{e}}, \bar{e})$.

(iv)

$$\begin{aligned} V\left(\widetilde{(X_A)}\right) &= \{a \otimes_{A_v} y_v \mid a \in A, v \in V(Z)\} \\ &= \{aA_v \times \{y_v\} \mid a \in A, v \in V(Z)\} = U_{v \in V(Z)} \left[(A/A_v) \times \{y_v\}\right], \end{aligned}$$

$$\begin{aligned} E\left(\widetilde{(X_A)}\right) &= \{a \otimes_{A_v} E(\{y_v\}) \mid a \in A, v \in V(Z)\} \cup (X_A)^* \\ &= \{a \otimes_{A_v} \varnothing) \mid a \in A, v \in V(Z)\} \cup \{(aA_e; e) \mid a \in A, e \in E(Z)\} \\ &= \varnothing \cup \{(aA_e; e) \mid a \in A, e \in E(Z)\} \\ &= \{(aA_e; e) \mid a \in A, e \in E(Z)\}. \end{aligned}$$

(v) The initials, the terminals, and the inverses of the edges of $E\left(\widetilde{(X_A)}\right)$ are as follows. $o((aA; e)) = a \otimes_{A_{o(e)}} W_e = a \otimes_{A_{o(e)}} y_{o(e)} = aAA_{o(e)} \times \left\{y_{o(e)}\right\}, t((aA_e; e)) = a \otimes_{A_{t(e)}} w_{\bar{e}} = aA_{t(e)} \times \left\{y_{t(e)}\right\}$, and $\overline{(aA_e, e)} = (aA_{\bar{e}}, \bar{e})$.

□

Corollary 5.2. *i. A defines an action of A on $\widetilde{(X_A)}$.*

ii. The stabilizer of the vertex $a \otimes_{A_v} y_v$ is $A_{a \otimes_{A_v} y_v} = aA_v a^{-1}$, a conjugate of A_v.

iii. The stabilizer of the edge $(aA_e; e)$ is $A_{(aA_e; e)} = aAea^{-1}$.

Proof. i. If $g, a \in A, v \in V(Z), e \in E(Z)$, then it is clear that the rules $g(a \otimes_{A_v} y_v) = ga \otimes_{A_v} y_v$, and $g((aA_e, e)) = (gaA_e, e)$ define an action of A on the fiber tree $\widetilde{(X_A)}$.

ii. It is clear that stabilizer of the vertex $a \otimes_{A_v} y_v$ is $A_{a \otimes_{A_v} y_v} = aA_v a^{-1}$, a conjugate of A_v.

iii. The same, the stabilizer of the edge $(aA_e; e)$ is $A_{(aA_e; e)} = aA_e a^{-1}$, a conjugate of A_e.

□

Corollary 5.3. *(i) The orbit of the vertex $a \otimes_{A_v} y_v$ is $A(a \otimes_{A_v} y_v) = (A/A_v) \times \{y_v\}$.*

(ii) The orbit of the edge (aA_e, e) is $A((aA_e, e)) = (A/A_e) \times \{e\}$.

Proof. (i) The orbit of the vertex $a \otimes_{A_v} y_v$ is $A(a \otimes_{A_v} y_v) = A \otimes_{A_v} A_v(y_v) = A \otimes_{A_v} \{y_v\} = (A/A_v) \times \{y_v\}$.

(ii) Similarly, the orbit of the edge (aA_e, e) is $A(aA, e) = (A/A_e) \times \{e\}$.

□

Corollary 5.4. *The quotient graph for the action of A on $A/\widetilde{(X_A)}$ is the connected graph of set of vertices $V\left(A/\widetilde{(X_A)}\right) = \{(A/A_v) \times \{y_v\} \mid v \in V(Z)\}$, and* *a set of edges* $E\left(A/\widetilde{(X_A)}\right) = \{(A/A_e) \times \{e\} \mid e \in E(Z)\}$. *The initials, the terminals, and the inverses of the edges of* $E\left(\widetilde{(X_A)}\right)$ *are:* $o\left((A/A_e) \times \{e\}\right) = \left(A/A_{o(e)}\right) \times \left\{y_{o(e)}\right\}$, $t\left((A/A_e) \times \{e\}\right) = \left(A/A_{t(e)}\right) \times \left\{y_{t(e)}\right\}$, *and,* $\overline{(A/A_e) \times \{e\}} = (A/A_{\bar{e}}) \times \{\bar{e}\}$.

Proof. The quotient graph for the action of A on $A/\widetilde{(X_A)}$ is

$$A/\widetilde{(X_A)} = \{(A/A_v) \times \{y_v\}, (A/A_e) \times \{e\} \mid v \in V(Z), e \in E(Z)\}.$$

$A/\widetilde{(X_A)}$ is connected, $V\left(A/\widetilde{(X_A)}\right) = \{(A/A_v) \times \{y_v\} \mid v \in V(Z)\}$, a $E\left(A/\widetilde{(X_A)}\right) = \{(A/A_e) \times \{e\} \mid e \in E(Z)\}$.
$o\left((A/A_e) \times \{e\}\right) = \left(A/A_{o(e)}\right) \times \left\{y_{o(e)}\right\}$, $t\left((A/A_e) \times \{e\}\right) = \left(A/A_{t(e)}\right) \times \{yt(e)\}$, and $\overline{(A/A_e) \times \{e\}} = (A/A_{\bar{e}}) \times \{\bar{e}\}$. □

Proposition 5.5. *Let* $A = \pi^*\left(A_v; A_e = A_{\bar{e}},\right)$, $v \in V(Z)$, $e \in E(Z)$ *be tree product of groups with amalgamation subgroups of a tree* Z. *For each vertex* $v \in V(Z)$ *let* Z_v *be a tree such that* $Z_v \neq Z$ *and* $Z_u \cap Z_v = \varnothing$ *for every vertex* $u \in V(Z), u \neq v$, *and let the vertex group* $A_v = \pi^*\left((A_v)_u; (A_v)_p = (A_v)_{\bar{p}}\right)$, $u \in V(Z_v)$, $p \in E(Z_v)$ *be a tree product of groups with amalgamation subgroups of a tree* Z_v *such that the amalgamation subgroups of* A *and* A_x *are finite and contain no invertor elements for all vertices* $x \in V(Z)$, *and* $X_v = X_{A_v}$ *be the standard tree of* A_v. *Then*

- *(i)* $w_{(A_e,e)} = \left(A_{o(e)}, O(e)\right)$ *is vertex of the edge* (A_e, e) *in the tree* $X_{o(e)}$ *and* $w_{\overline{(A_e,e)}} = w_{(A_{\bar{e}},\bar{e})} = \left(A_{t(e)}, (e)\right)$ *is vertex of the edge* $(aA_{\bar{e}}, \bar{e})$ *in the tree* $X_{t(e)}$.
- *(ii)* $X_p \cap X_q = \varnothing$ *for all vertices* $p, q \in V(Z), p \neq q$.
- *(iii)* $X_{A_v} = \left\{\left(g\left(A_v\right)_u, u\right), \left(g\left(A_v\right)_q, q\right) \mid g \in A_v, u \in V\left(Z_v\right), q \in E\left(Z_v\right)\right\}$.
- *(iv) The set* $S_A = \left[U_{v \in V(Z)}\left(A \otimes_{A_v} X_{A_v}\right)\right] U\left(X_A\right)$ ** forms a tree.*

Proof. The assumptions of the proposition coincides with assumptions of Theorem 4.2

- (i) $A_{(A_e,e)} = A_e$ is the stabilizer of the edge (A_e, e), $A_{\left(A_{o(e)}, o(e)\right)} = A_{o(e)}$ is the stabilizer of the vertex $X_{t(e)}$.
- (ii) The assumption that $Z_v \neq Z$ and $Z_u \cap Z_v = \varnothing, u \in V(Z), u \neq v$ implies that $X_u \cap X_v = \varnothing$.
- (iii) By the definition of the standard tree of A_v,

  $$X_{A_v} = \left\{\left(g\left(A_v\right)_u, u\right), \left(g\left(A_v\right)_q, q\right) \mid g \in A_v, u \in V\left(Z_v\right), q \in E\left(Z_v\right)\right\},$$

  where the set of vertices is
  $V\left(X_{A_v}\right) = \left\{\left(g\left(A_v\right)_u, u\right) \mid g \in A_v, u \in V\left(Z_v\right)\right\}$, and the set of edges is $E\left(X_{A_v}\right) = \left\{\left(g\left(A_v\right)_q, q\right) \mid g \in A_v, q \in E\left(Z_v\right)\right\}$, and A_v acts on X_{A_v}.

(iv) By the definition of $\otimes$ we have

$$\begin{aligned}
&U_{v\in V(Z)}\left(A\otimes_{A_v}X_{A_v}\right)\\
&\qquad=\left[U_{v\in V(Z)}\left(A\otimes_{A_v}V\left(X_{A_v}\right)\right)\right]\cup\left[U_{v\in V(Z)}\left(A\otimes_{A_v}E\left(X_{A_v}\right)\right)\right]\\
&\qquad=\Big\{a\otimes_{A_v}\left(g\left(A_v\right)u,u\right)\mid a,g\in A_v,u\in V\left(Z_v\right)\Big\}\cup\\
&\qquad\qquad\Big\{a\otimes_{A_v}\Big(g\left(A_v\right)_q,q\Big)\mid a,g\in A_v,q\in E\left(Z_v\right)\Big\}\\
&\qquad=\Big\{a\otimes_{A_v}\left(g\left(A_v\right)_u,u\right),a\otimes_{A_v}\Big(g\left(A_v\right)_q,q\Big)\mid a,g\in A_v,\\
&\qquad\qquad u\in V\left(Z_v\right),q\in E\left(Z_v\right)\Big\}.
\end{aligned}$$

From above we have $(X_A)^*=\{(aA_e,(A_e,e))\mid a\in A,e\in E(Z)\}$. Then

$$\begin{aligned}
S_A=\Big\{&a\otimes_{A_v}\left(g\left(A_v\right)_u,u\right),a\otimes_{A_v}\Big(g\left(A_v\right)_q,q\Big),(aA_e,(A_e,e))\mid a,g\in A_v,\\
&u\in V\left(Z_v\right),q\in E\left(Z_v\right)\Big\}.
\end{aligned}$$

So $V\left(S_A\right)=\{a\otimes_{A_v}\left(g\left(A_v\right)_u,u\right)\mid a,g\in A_v,u\in V\left(Z_v\right)\}$ and

$$\begin{aligned}
E\left(S_A\right)=\Big\{&a\otimes_{A_v}\Big(g\left(A_v\right)_q,q\Big),(aAA_e,(A_e,e))\mid a,g\in A_v,u\in V\left(Z_v\right),\\
&q\in E\left(Z_v\right)\Big\}.
\end{aligned}$$

By Theorem 4.2, S_A forms a tree. The initials, the terminals, and inverses of the edges of S_A are

$$o\Big(a\otimes_{A_v}\Big(g\left(A_v\right)_q,q\Big)\Big)=a\otimes_{A_v}o\left(g\left(A_v\right)_u,q\right)=a\otimes_{A_v}\Big(g\left(A_v\right)_{o(q)},q\Big),$$

$$t\Big(a\otimes_{A_v}\Big(g\left(A_v\right)_q,q\Big)\Big)=a\otimes_{A_v}t\left(\left(g\left(A_v\right)_u,q\right)\right)=a\otimes_{A_v}\left(g\left(A_v\right)t(q),q\right),$$

and

$$\begin{aligned}
\overline{a\otimes_{A_v}\Big(g\left(A_v\right)_q,q\Big)}&=a\otimes_{A_v}\Big(g\left(A_v\right)_{\bar{q}},\bar{q}\Big),\\
o\left(aA_e,(A_e,e)\right)&=a\otimes_{A_{o(e)}}w_{A_e,e}=a\otimes_{A_{o(e)}}\big(A_{o(e)},o(e)\big),\\
t\left(aA_e,(A_e,e)\right)&=a\left[aA_e;(A_e,e)\right]\otimes_{A_{t(e)}}w_{(A_{\bar{e}},\bar{e})}\\
&=a1\otimes_{A_{t(e)}}w_{(A_{\bar{e}},\bar{e})}a\otimes_{A_{t(e)}}w_{(A_{\bar{e}},\bar{e})}\\
&=a\otimes_{A_{t(e)}}\big(A_{t(e)},t(e)\big)
\end{aligned}$$

and $\overline{(aA_e)}=(aA_{\bar{e}},(A_{\bar{e}},\bar{e}))$. ∎

Corollary 5.6. *i. A defines an action of A on S_A.*

ii. The stabilizer of the vertex $a\otimes_{A_v}\left(g\left(A_v\right)_u,u\right)$ is $A_a\otimes_{A_v}\left(g\left(A_v\right)u,u\right)=a\left(A_v\right)_u a^{-1}$, a conjugate of $\left(A_v\right)_u$.

iii. *The stabilizer of the edge* $(aA, e, (A_e, e))$ *is* $A_{(aAe,(Ae,e))} = aA_e a^{-1}$*, a conjugate of* A_e*.*

Proof. i. Let $h \in A, a \otimes_{A_v} (g(A_v)_u,$ u be a vertex of S_A and $a \otimes_{A_v} \Big(g(A_v)_q, q\Big)$, $(aA_e, (A_e, e))$ be edges of S_A. Then it is clear that the rules
$h[a \otimes_{A_v} (g(A_v)_u, u)] = ha \otimes_{A_v} (g(A_v)_u, u)$,
$h\Big[a \otimes_{A_v} \Big(g(A_v)_q, q\Big)\Big] = ha \otimes_{A_v} \Big(g(A_v)_q, q\Big)$,
and $h[(aA_e, (A_e, e))] = (haA_e, (A_e, e))$ define an action of A on S_A.

ii. If $h \in A$ such that $ha \otimes_{A_v} (g(A_v)_u, u) = a \otimes_{A_v} (g(A_v)_u, u)$ and the definition of $\otimes_{A_v}$ implies that there exists an element $b \in A_v$ such that $ha = ab$ and $b(g(A_v)_u = (g(A_v)_u$. So $h = aba^{-1} \in a(A_v)_u a^{-1}$, a conjugate of $(A_v)_u$.

iii. Similarly, we can show that the stabilizer of the edge $(aA_e, (A_e, e))$ is $aA_e a^{-1}$, a conjugate of A_e.

□

Corollary 5.7. *(i) The orbit of the vertex* $a \otimes_{A_v} (g(A_v)_u, u)$ *is*

$$A\Bigg(a \otimes_{A_v} \Big(g(A_v)_u, u\Big)\Bigg) = A \otimes_{A_v} \Big(g(A_v)_u, u\Big) = \Bigg\{xa \otimes_{A_v} \Big(g(A_v)_u, u\Big) \mid x \in A\Bigg\}.$$

(ii) The orbit of the edge $(aA_e, (A_e, e))$ *is*

$$A\Bigg(aA_e, (A_e, e)\Bigg) = (A/A_e) \times \{(A_e, e)\}.$$

Proof. (i) $A\Bigg(a \otimes_{A_v} \Big(g(A_v)_u, u\Big)\Bigg) = \Bigg\{xa \otimes_{A_v} \Big(g(A_v)_u, u\Big) \mid x \in A\Bigg\} = A \otimes_{A_v} \Big(g(A_v)_u, u\Big)$.

(ii) $A\Bigg(aA, (A_e, e)\Bigg)$

$$= \Bigg\{(gaA_e, (A_e, e)) \mid g \in A\Bigg\} = \{(xA_e, (A_e, e)) \mid x \in A\}$$
$$= (A/A_e) \times \{(A_e, e)\}.$$

□

Corollary 5.8. *The quotient graph A/S_A of the action of A on S_A has the structure*

$$A/S_A = \Big\{ A \otimes_{A_v} \left(a\left(A_v\right)_u, u\right), \left(A/A_e\right) \times \left\{ \left(A_e, e\right) \mid a \in A, \right. \\ \left. v \in V(Z), u \in V\left(Z_v\right), e \in E(Z) \right\} \Big\}.$$

where the set of vertices of A/S_A is

$$V\left(A/S_A\right) = \left\{ A \otimes_{A_v} \left(a\left(A_v\right)_u, u\right) \mid a \in A, v \in V(Z), u \in V\left(Z_v\right) \right\},$$

and the set of edges of A/S_A is

$$E\left(A/S_A\right) = \left\{ \left(A/A_e\right) \times \left\{ \left(A_e, e\right) \right\} \mid e \in E(Z) \right\}.$$

6. Conclusion

In this paper, we have investigated the structures of fibers for the action of tree products of groups with amalgamation subgroups on trees. We constructed the standard tree where such a group acts without inversions and determined the quotient of this action.

Furthermore, we showed that when the amalgamation subgroups are finite and the factor groups act on disjoint trees, there exists a fiber tree where the group acts without inversions. We described the structure of this fiber tree and determined the corresponding factors of the action.

When each factor group is itself a tree product with amalgamation subgroups, we constructed a new fiber tree and described its structure. We derived formulas for the vertices, edges, initials, terminals and inverses in these fiber trees.

Several corollaries were proved concerning the orbits, stabilizers and quotients of the actions on the fiber trees. We also applied the results to some specific cases when the factor trees consist of single vertices or when the factors are tree products.

The fibers of tree products of groups with amalgamation subgroups generalize the fibers of free products of groups with amalgamation subgroups. The results of this paper provide a framework for understanding the interactions between algebraic structures of tree products of groups and the geometry of tree actions. Future work will involve exploring additional properties and applications of these fiber trees.

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