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CONVOLUTION SUM OF DIVISOR FUNCTIONS GIVEN THE CONDITIONS OF COPRIME†

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Abstract. The study of convolution sums for divisor functions is an area that has been extensively researched by many mathematicians including Ramanujan. The aim of this paper is to find the formula for convolution sum of divisor functions with coprime conditions.

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1. Introduction

Throughout this paper, $\mathbb{N}$ will be denoted by the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and p be a positive prime integer. For $d, l \in \mathbb{N}$, $s \in \mathbb{N}_0$ and $n \in \mathbb{N} - \{1\}$, we define some necessary divisor functions and their convolution sums for later use:

$$\sigma_s(l) = \sum_{d|l} d^s, \quad \sigma(l) := \sigma_1(l) = \sum_{d|l} d, \quad \zeta_s(l) := l^s, \quad \zeta(l) := \zeta_0(l) = l^0,$$

$$K(n) := \sum_{m=1}^{n-1} m\sigma(m)\sigma(n-m), \quad U(n) := \sum_{\substack{m=1 \\ \gcd(m,n-m)=1}}^{n-1} m\sigma(m)\sigma(n-m).$$

Glaisher ([3], [9, Theorem 12.7]) proved

$$\sum_{m=1}^{n-1} \sigma(m)\sigma(n-m) = \frac{1}{12}\left(5\sigma_3(n) - (6n-1)\sigma(n)\right) \tag{1}$$

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and

$$\sum_{k=1}^{n-1} k\sigma(k)\sigma(n-k) = \frac{n}{24}\left(5\sigma_3(n) - (6n-1)\sigma(n)\right). \tag{2}$$

Later Ramanujan [7] obtained (1) again, and appears also in [2, p. 300]. Recently, in [4], S. Kong, Y. Li and D. Kim were considered convolution sums of the form

$$\sum_{\substack{m=1\\ \gcd(m,n-m)=1}}^{n-1} \sigma(m)\sigma(n-m).$$

In this paper, we prove the following theorem, which is slightly different form of the convolution sums considered in [3, 4, 7].

Theorem 1.1. *Let* $\alpha \in \mathbb{N} - \{1\}$. *Then*

$$\frac{24U(p^\alpha)}{p^{2\alpha-1}(p^2-1)} - (-1)^\alpha = \begin{cases} 5(p-1)\big(\sigma_4(p^n) - p^2\sigma_4(p^{n-1})\big), & \text{if } \alpha = 2n+1, \\ 5(p-1)^2(p+1)\sigma_4(p^{n-1}), & \text{if } \alpha = 2n. \end{cases}$$

Corollary 1.2. *If* $\alpha \in \mathbb{N} - \{1\}$ *then*

$$24U(p^\alpha) \equiv 0 \pmod{p^{2\alpha-1}} \quad \text{and} \quad \frac{24U(p^\alpha)}{p^{2\alpha-1}(p^2-1)} \equiv (-1)^\alpha \pmod{5}.$$

2. Preliminaries

It is well known that arithmetical functions are rings for addition and Dirichlet convolution sums [5, 6, 8]. Let h_1 and h_2 be arithmetical functions. The Dirichlet convolution sum $h_1 * h_2$ of h_1 and h_2 is defined by

$$(h_1 * h_2)(m) = \sum_{d|m} h_1(d)h_2\left(\frac{m}{d}\right) \tag{3}$$

for all $m \in \mathbb{N}$. Here, d are positive divisors of m. Define the function δ by $\delta(1) = 1$ and $\delta(m) = 0$ for $m > 1$. Then $h_1 * \delta = \delta * h_1 = h_1$. An arithmetical function h_1^{-1} is called an inverse of h_1 if $h_1 * h_1^{-1} = h_1^{-1} * h_1 = \delta$. h_1 is a unit if and only if $h_1(1) \neq 0$. It is well known that

$$h_1^{-1}(1) = \frac{1}{h_1(1)} \tag{4}$$

and

$$h_1^{-1}(m) = -\frac{1}{h_1(1)} \sum_{\substack{d|m\\ d>1}} h_1(d)h_1^{-1}\left(\frac{m}{d}\right) \tag{5}$$

for all $m > 1$.

Let p be a prime and $1 \le k \le p^n - 1$ with $k, n \in \mathbb{N}$. Let $gcd(k, p^n - k) = d$. Then there exist $a, b \in \mathbb{N}$ such that $k = ad$, $p^n - k = db$ and $p^n = d(a+b)$. Hence we obtain

$$\gcd(k, p^n - k) = 1 \text{ or } p \text{ or } \ldots \text{ or } p^{n-1}. \tag{6}$$

If $1 \le i \le n-1$ and

$$p^i | k \text{ then } p^i | (p^n - k). \tag{7}$$

So, if $\gcd(k, p^n - k) = p^i$ then $\gcd(\frac{k}{p^i}, \frac{p^n-k}{p^i}) = 1$, $p \nmid \frac{k}{p^i}$ and $p \nmid \frac{p^n-k}{p^i}$ and

$$k\sigma(k)\sigma(p^n - k) = \left(p^i \sigma(p^i)^2\right) \left(\frac{k}{p^i}\sigma\left(\frac{k}{p^i}\right)\sigma\left(\frac{p^n - k}{p^i}\right)\right). \tag{8}$$

By (7) and (8), we deduce that

$$\sum_{k=1}^{p^n-1} k\sigma(k)\sigma(p^n - k) = \sum_{i=0}^{n-1} \sum_{\substack{k=1 \\ \gcd(k,p^n-k)=p^i}}^{p^n-1} k\sigma(k)\sigma(p^n - k). \tag{9}$$

By (8) and (9), we have

$$K(p^n) = \sum_{k=1}^{p^n-1} k\sigma(k)\sigma(p^n - k) = \sum_{i=0}^{n} p^i \sigma(p^i)^2 U(p^{n-i}) = [(\zeta_1\sigma^2) * U](p^n). \tag{10}$$

Here, $K(1) := U(1) := 0$. By (2) and (10), we have

$$U(p^n) = \left((\zeta_1\sigma^2)^{-1} * K\right)(p^n) = \left((\zeta_1\sigma^2)^{-1} * \left(\frac{5}{24}\zeta_1\sigma_3 - \frac{1}{4}\zeta_1^2\sigma + \frac{1}{24}\zeta_1\sigma\right)\right)(p^n). \tag{11}$$

3. Convolution sums of divisor functions

Using (11), to obtain the result of Theorem 1.1, it is necessary to find the inverse divisor function. In order to find the inverse divisor functions, we introduce some properties of multiplicative functions that are mainly used in this article.

Proposition 3.1. [5, Chapter 1]
(1) If f and g are multiplicative functions then $f * g$ is also a multiplicative function.
(2) If f is a multiplicative function then f^{-1} is also a multiplicative function.
(3) If f and g are multiplicative functions then fg is a multiplicative function.
(4) A multiplicative function f is completely multiplicative if and only if $f(g * h) = fg * fh$ for all arithmetical functions g and h.

Remark. It is easily checked that

$$\zeta_s(mn) = (mn)^s = m^s n^s = \zeta_s(m)\zeta_s(n)$$

with $s \in \mathbb{N}_0$. So, ζ_s is a completely multiplicative function. On the other hand, we note that

$$\sigma_s(m) = \sum_{d|m} d^s = \sum_{d|m} d^s \left(\frac{m}{d}\right)^0 = (\zeta_s * \zeta)(m). \tag{12}$$

Thus,

$$\sigma_s \text{ is a multiplicative function} \tag{13}$$

by Proposition 3.1(1).

The Möbius function, which is well known to us and plays an important role in the inverse arithmetical function, is introduced below. Möbius function $\mu(n)$ ([1, Definition 6.2]), $n \in \mathbb{N}$, is defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \text{ is not square-free}, \\ (-1)^k, & \text{if } n = p_1 p_2 \cdots p_k, \end{cases} \tag{14}$$

where p_i are distinct prime integers.

It is easily check that

$$\zeta^{-1} = \mu \quad \text{and} \quad \zeta_1^{-1} = \zeta_1 \mu. \tag{15}$$

So, we get

Lemma 3.2. *Let $p_1, \ldots, p_r$ be distinct primes. Then*

$$\zeta_1^{-1}(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^r \cdot n, & \text{if } n = p_1 \cdots p_r, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.3. *Let $\alpha \in \mathbb{N}_0$. Then*

$$\sigma^{-1}(p^\alpha) = \begin{cases} 1, & \text{if } \alpha = 0, \\ -p-1, & \text{if } \alpha = 1, \\ p, & \text{if } \alpha = 2, \\ 0, & \text{if } \alpha \geq 3. \end{cases} \tag{16}$$

Proof. By (13) and Proposition 3.1, σ^{-1} is a multiplicative function. Thus, the proof can be easily completed by examining only the case of $\sigma^{-1}(p^m)$ with $m \in \mathbb{N}_0$. By (4) and (5), $\sigma^{-1}(1) = \frac{1}{\sigma(1)} = 1$,

$$\sigma^{-1}(p) = -\frac{1}{\sigma(1)} \sum_{\substack{d|p \\ d>1}} \sigma(d) \sigma^{-1}\left(\frac{p}{d}\right) = -\sigma(p)\sigma^{-1}(1) = -(p+1)$$

and

$$\sigma^{-1}(p^2) = -\big(\sigma(p)\sigma^{-1}(p) + \sigma(p^2)\sigma^{-1}(1)\big) = p.$$

By (5), we obtain

$$\sigma^{-1}(p^3) = -\big(\sigma(p)\sigma^{-1}(p^2) + \sigma(p^2)\sigma^{-1}(p) + \sigma(p^3)\sigma^{-1}(1)\big) = 0.$$

Using the mathematical induction, we want to show that

$$\sigma^{-1}(p^m) = 0$$

with $m > 2$. Assume that

$$\sigma^{-1}(p^i) = 0 \tag{17}$$

with $3 \le i \le m-1$. By (5) and (17),

$$\begin{aligned}\sigma^{-1}(p^m) &= -\frac{1}{\sigma(1)}\sum_{\substack{d|p^m\\ d>1}}\sigma(d)\sigma^{-1}(\frac{p^m}{d})\\ &= -\big(\sigma(p^{m-2})\sigma^{-1}(p^2)+\sigma(p^{m-1})\sigma^{-1}(p)+\sigma(p^m)\sigma^{-1}(1)\big)\\ &= 0\end{aligned}$$

with $m \ge 3$. Therefore, Theorem 3.3 is obtained. □

Corollary 3.4. *Let $p_1, p_2, \ldots, p_t$ be distinct primes. If n is an integer then*

$$\sigma^{-1}(n) = \begin{cases} 1, & \text{if } n=1,\\ (-1)^t\sigma(n), & \text{if } n=p_1\cdots p_t,\\ \sqrt{n}, & \text{if } n=p_1^2\cdots p_t^2,\\ (-1)^{t-r}(\prod_{i=1}^r p_i)(\prod_{i=1}^{r+1}(p_i+1)), & \text{if } n=p_1^2\cdots p_r^2 p_{r+1}\cdots p_t,\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 3.5. *Let $\alpha, k \in \mathbb{N}_0$. Then*

$$(\zeta_k\sigma)^{-1}(p^\alpha) = \begin{cases} 1, & \text{if } \alpha=0,\\ -p^k(p+1), & \text{if } \alpha=1,\\ p^{2k+1}, & \text{if } \alpha=2,\\ 0, & \text{if } \alpha\ge 3.\end{cases}$$

Proof. By Proposition 3.1, it is easily checked that

$$\delta = \zeta_k(\sigma * \sigma^{-1}) = (\zeta_k\sigma) * (\zeta_k\sigma^{-1}).$$

So,

$$(\zeta_k\sigma)^{-1}(p^\alpha) = \zeta_k(p^\alpha)\sigma^{-1}(p^\alpha).$$

The proof of Theorem 3.5 is completed by Theorem 3.3. □

Corollary 3.6. *Let $p_1, p_2, \ldots, p_t$ be distinct primes and $k \in \mathbb{N}_0$. Then $(\zeta_k\sigma)^{-1}(n) =$*

$$\begin{cases} 1, & \text{if } n=1,\\ (-1)^t n^k\sigma(n), & \text{if } n=\prod_{i=1}^t p_i,\\ n^k\sqrt{n}, & \text{if } n=\prod_{i=1}^t p_i^2,\\ (-1)^s(\prod_{i=1}^s p_i^k(p_i+1))(\prod_{j=s+1}^t p_j^{2k+1}), & \text{if } n=(\prod_{i=1}^s p_i)(\prod_{j=s+1}^t p_j^2),\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 3.7. *Let $\alpha \in \mathbb{N}_0$. Then*

$$(\sigma^2)^{-1}(p^\alpha) = \begin{cases} 1, & \text{if } \ \alpha=0,\\ -(p+1)^2, & \text{if } \ \alpha=1,\\ 2p^3+3p^2+2p, & \text{if } \ \alpha=2,\\ 2\cdot(-1)^\alpha p^{\alpha-1}(p+1)^2, & \text{if } \ \alpha\ge 3.\end{cases} \tag{18}$$

Proof. Since σ is a multiplicative function, $(\sigma^2)^{-1}$ is also a multiplicative function by Proposition 3.1. It is easily checked that $(\sigma^2)^{-1}(1) = 1$,

$$(\sigma^2)^{-1}(p) = -\sum_{\substack{d|p \\ d>1}} \sigma^2(d)(\sigma^2)^{-1}\left(\frac{p}{d}\right) = -(p+1)^2 \quad \text{and}$$

$$(\sigma^2)^{-1}(p^2) = -\sum_{\substack{d|p^2 \\ d>1}} \sigma^2(d)(\sigma^2)^{-1}\left(\frac{p^2}{d}\right) = 2p^3 + 3p^2 + 2p.$$

$$(\sigma^2)^{-1}(p^3) = -\sum_{\substack{d|p^3 \\ d>1}} \sigma^2(d)(\sigma^2)^{-1}\left(\frac{p^3}{d}\right) = 2\cdot(-1)^3 p^2(p+1)^2.$$

$$(\sigma^2)^{-1}(p^4) = -\sum_{\substack{d|p^4 \\ d>1}} \sigma^2(d)(\sigma^2)^{-1}\left(\frac{p^4}{d}\right) = 2\cdot(-1)^4 p^3(p+1)^2.$$

We use the same method as in Theorem 3.3, i.e., mathematical induction. Assume that for $3 \le i \le \alpha - 1$,

$$(\sigma^2)^{-1}(p^i) = -\sum_{\substack{d|p^i \\ d>1}} \sigma^2(d)(\sigma^2)^{-1}\left(\frac{p^i}{d}\right) = 2\cdot(-1)^i p^{i-1}(p+1)^2.$$

Then we obtain

$$\begin{aligned}
&(\sigma^2)^{-1}(p^\alpha) = -\sum_{\substack{d|p^\alpha \\ d>1}} \sigma^2(d)(\sigma^2)^{-1}\left(\frac{p^\alpha}{d}\right) \\
&= -\sum_{i=1}^{\alpha} \sigma^2(p^i)(\sigma^2)^{-1}(p^{\alpha-i}) \\
&= -\sigma^2(p^\alpha)(\sigma^2)^{-1}(1) - \sigma^2(p^{\alpha-1})(\sigma^2)^{-1}(p) - \sigma^2(p^{\alpha-2})(\sigma^2)^{-1}(p^2) \\
&- \sigma^2(p^{\alpha-3})(\sigma^2)^{-1}(p^3) - \sum_{i=1}^{\alpha-4} \sigma^2(p^i)(\sigma^2)^{-1}(p^{\alpha-i}) \\
&= -p\left(\sigma^2(p^{\alpha-1})(\sigma^2)^{-1}(1) + \sigma^2(p^{\alpha-2})(\sigma^2)^{-1}(p) + \sigma^2(p^{\alpha-3})(\sigma^2)^{-1}(p^2)\right) \\
&- \sigma^2(p^\alpha)(\sigma^2)^{-1}(1) - \sigma^2(p^{\alpha-1})(\sigma^2)^{-1}(p) \\
&- \sigma^2(p^{\alpha-2})(\sigma^2)^{-1}(p^2) - \sigma^2(p^{\alpha-3})(\sigma^2)^{-1}(p^3) - p(\sigma^2)^{-1}(p^{\alpha-1}) \\
&= -p(\sigma^2)^{-1}(p^{\alpha-1}) = 2\cdot(-1)^\alpha p^{\alpha-1}(p+1)^2.
\end{aligned}$$

This completes the proof of Theorem 3.7. $\square$

Corollary 3.8. *Let* $p_1, \ldots, p_t$ *be distinct primes and* $\alpha_k (s+1 \le k \le t) \in \mathbb{N} - \{1, 2\}$. *Then* $(\sigma^2)^{-1}(1) = 1$ *and*

$$(\sigma^2)^{-1}(p_1 \cdots p_r p_{r+1}^2 \cdots p_s^2 p_{s+1}^{\alpha_{s+1}} \cdots p_t^{\alpha_t})$$

$$= (-1)^r 2^{t-s} \prod_{i=1}^{r} (p_i+1)^2 \prod_{j=r+1}^{s} (2p_j^3+3p_j^2+2p_j) \cdot \prod_{k=s+1}^{t} (-1)^{\alpha_k} p_k^{\alpha_k-1}(p_k+1)^2.$$

Theorem 3.9. *Let $\alpha, k \in \mathbb{N}_0$. Then*

$$(\zeta_k\sigma^2)^{-1}(p^\alpha) = \begin{cases} 1, & \text{if } \alpha = 0, \\ -p^k(p+1)^2, & \text{if } \alpha = 1, \\ p^{2k+1}(2p^2+3p+2), & \text{if } \alpha = 2, \\ (-1)^\alpha 2p^{(k+1)\alpha-1}(p+1)^2, & \text{if } \alpha \geq 3. \end{cases}$$

Proof. It is trivial by Proposition 3.1 and Theorem 3.7. □

Corollary 3.10. *Let $k, r, s, t \in \mathbb{N}_0$, $\alpha_1, \ldots, \alpha_r \in \mathbb{N} - \{1, 2\}$, $p_1, \ldots, p_r, p_{r+1}, \ldots, p_s, p_{s+1}, \ldots, p_t$ be distinct primes and $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} p_{r+1} \cdots p_s p_{s+1}^2 \cdots p_t^2$. Then*

$$(\zeta_k\sigma^2)^{-1}(1) = 1 \quad \text{and}$$

$$(\zeta_k\sigma^2)^{-1}(n) = (-1)^{\alpha_1+\cdots+\alpha_r+s-r}\, 2^r \sigma(p_1p_2\cdots p_s)^2 \cdot \prod_{i=1}^{r} p_i^{(k+1)\alpha_i-1} \prod_{j=r+1}^{s} p_j^k \prod_{l=s+1}^{t} p_l^{2k+1}(2p_l^2+3p_l+2).$$

Using these results, we will prove Theorem 1.1.

Proof of Theorem 1.1

Proof. Let $f(p^i) := \big(\zeta_1\sigma^2\big)^{-1}(p^i)$ and $g(p^i) := \big(5\zeta_1\sigma_3 - 6\zeta_1^2\sigma + \zeta_1\sigma\big)(p^i)$ with $i \in \mathbb{N}_0$. By (11), it is easily checked that

$$24U(p^\alpha) = (g * f)(p^\alpha) = \sum_{i=1}^{\alpha} g(p^i)f(p^{\alpha-i}).$$

Here, $\alpha \in \mathbb{N}$. Assume that

$$\begin{aligned} 24U(p^\alpha) &= \sum_{i=1}^{\alpha} g(p^i)f(p^{\alpha-i}) \\ &= p^{2\alpha-1}(p^2-1)((-1)^\alpha + 5(p-1)\sum_{i=1}^{\alpha}(-1)^{\alpha+i}p^{2(i-1)}). \end{aligned} \tag{19}$$

Suppose that (19) holds up to $\alpha - 1$. Now we obtain

$$\begin{aligned} 24U(p^\alpha) &= \sum_{i=0}^{\alpha} g(p^i)f(p^{\alpha-i}) = g(p^\alpha)f(1) + g(p^{\alpha-1})f(p) \\ &+ g(p^{\alpha-2})f(p^2) + g(p^{\alpha-3})f(p^3) + \sum_{i=0}^{\alpha-4} g(p^i)f(p^{\alpha-i}) \\ &= g(p^\alpha)f(1) + g(p^{\alpha-1})f(p) + g(p^{\alpha-2})f(p^2) + g(p^{\alpha-3})f(p^3) \\ &+ p^2\big(g(p^{\alpha-1})f(1) + g(p^{\alpha-2})f(p) + g(p^{\alpha-3})f(p^2)\big) - p^2 24U(p^{\alpha-1}) \end{aligned} \tag{20}$$

and

$$5(p-1)(p^2-1)p^{4\alpha-3} = g(p^\alpha)f(1) + g(p^{\alpha-1})f(p) + g(p^{\alpha-2})f(p^2) \\ + g(p^{\alpha-3})f(p^3) + p^2\big(g(p^{\alpha-1})f(1) + g(p^{\alpha-2})f(p) + g(p^{\alpha-3})f(p^2)\big). \qquad (21)$$

A comparison of (20) and (21) yields the identity

$$\begin{aligned} 24U(p^\alpha) &= 5(p-1)(p^2-1)p^{4\alpha-3} - 24p^2U(p^{\alpha-1}) \\ &= 5(p-1)(p^2-1)p^{4\alpha-3} - p^2(p^2-1)p^{2\alpha-3} \\ &\times \left((-1)^{\alpha-1} + 5(p-1)\sum_{i=1}^{\alpha-1}(-1)^{\alpha-1+i}p^{2(i-1)}\right) \\ &= p^{2\alpha-1}(p^2-1)\left((-1)^\alpha + 5(p-1)\sum_{i=1}^{\alpha}(-1)^{\alpha+i}p^{2(i-1)}\right). \end{aligned}$$

Expanding and rearranging, we get

$$\frac{24U(p^\alpha)}{p^{2\alpha-1}(p^2-1)} - (-1)^\alpha = \begin{cases} 5(p-1)\big(\sigma_4(p^n) - p^2\sigma_4(p^{n-1})\big), & \text{if } \alpha = 2n+1, \\ 5(p-1)^2(p+1)\sigma_4(p^{n-1}), & \text{if } \alpha = 2n. \end{cases}$$

□

Conflicts of interest : The authors declare no conflict of interest.

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References

1. A. Dujella, *Number Theory*, Školska knjiga, Zagreb, 2021.
2. H.M. Farkas, *On an arithmetical function*, Contemp. Math. **382** (2005), 121-130.
3. J.W.L. Glaisher, *On certain sums of products of quantities depending upon the divisors of a number*, Mess. Math. **15** (1885), 1-20.
4. S. Kong, Y. Li and D. Kim, *Convolution sums of restricted divisor functions derived from Dirichlet convolution*, preprint.
5. P.J. McCarthy, *Introduction to arithmetical functions*, Springer Science and Business Media, 2012.
6. M.B. Nathanson, *Additive Number Theory*, The Classical Bases, Graduate Texts in Mathematics 164, Springer, 1996.
7. S. Ramanujan, *Collect papers*, AMS Chelsea Publishing, Province, RI, USA, 2000.
8. R. Vaidyanathaswamy, *The theory of multiplicative arithmetic functions*, Transactions of the American Mathematical Society **33** (1931), 579-662.
9. K.S. Williams, *Number Theory in the Spirit of Liouville*, London Mathematical Society, Student Texts 76, Cambridge, 2011.

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