J. Appl. Math. & Informatics Vol. **41***(2023), No. 6, pp. 1193 - 1207*
https://doi.org/10.14317/jami.2023.1193

AN iP_2 EXTENDED STAR GRAPH AND ITS HARMONIOUS CHROMATIC NUMBER

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ABSTRACT. In this paper, we introduce an iP_2 extension of a star graph S_n for $n \geq 2$ and $1 \leq i \leq n-1$. Certain general properties satisfied by order, size, domination (or Roman) numbers γ (or γ_R) of an iP_2 extended star graph are studied. Finally, we study how the parameters such as chromatic number and harmonious chromatic number are affected when an iP_2 extension process acts on the star graphs.

AMS Mathematics Subject Classification : 05C15, 05C69.
Key words and phrases : Extended star graph, domination number, roman domination number, chromatic number, harmonious chromatic number.

1. Introduction

A graph G consists of a non-empty set V of points or vertices, called nodes, and a set E of two point subsets of V, called edges. The cardinality of the set of nodes and edges are called order and size of G respectively. The number of edges incident with a node is known as the degree of the node. All the graphs considered here are undirected, connected and simple. For basic graph theoretic notation and terminology refer to [2].

A walk in a graph G is an alternating sequence of distinct nodes and edges which starts and terminates with a node. If no vertices are repeating in a walk, then it is called a path. The number of edges in a path is known as the length of the path. A graph G is said to be connected if there exists a path between every pair of nodes of G. A node of degree 1 is called a pendant node.

A graph G having $n-1$ pendant nodes and exactly one node of degree $n-1$ is said to be a star graph which is denoted by S_n, where $n > 0$.

For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in v : u \text{is adjacent to} V\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{u\}$. A subset $S \subset V$ of a graph $G = (V, E)$ is said to be a dominating set of G if

Received September 1, 2022. Revised May 2, 2023. Accepted August 14, 2023. *Corresponding author.

$N[S] = V$ or equivalently, every node not in S is adjacent to at least one node of S. The domination number $\gamma(G)$ of a graph G is the number of nodes in a minimal dominating set for G. That is,

$$\gamma(G) = \min\{|S| : S \text{is a minimal dominating set of G}\}$$

A node v in a graph G dominates itself and each of its adjacent nodes.

A vertex coloring or simply coloring of a graph G is a function $f : V(G) \to \mathbb{N}$ satisfying $f(u) = f(v) \Rightarrow \{u, v\}$ not belongs to $\phi(E(G))$ for all $u, v \in V(G)$; that means u and v are not adjacent in G. For $k \in \mathbb{N}$, a k-vertex coloring or a proper k-vertex coloring of G is a proper vertex coloring $c : V(G) \to \{1, 2, ..., k\}$. G is said to be k-colorable if G has a proper k- vertex coloring. The least $k \in \mathbb{N}$ such that G is k-vertex colorable is called the chromatic number of G and is denoted by $\chi(G)$. If $\chi(G) = k$, then G is called k-chromatic graph. Equivalently, state that, a proper coloring of a graph G is a labeling of nodes of G such that no two adjacent nodes receive the same color. A coloring of a graph G using at most k colors is called a k- coloring. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors required for coloring G properly.

An assignment of colors for the nodes of a graph G with the colors 0, 1 and 2 in such a way that each node u, say, colored 0 is adjacent to a minimum of one node v, say, colored 2 is called a Roman dominating function ϕ. The sum of the colors assigned for the nodes is known as the weight of the corresponding Roman dominating function and the minimum weight of all possible Roman dominating function of the graph G is known as the Roman domination number which is denoted by $\gamma_R(G)$.

A harmonious coloring of a graph G is a special type of proper coloring of G in which no color pairs repeats. The least number of colors required for a harmonious coloring of a graph G is called the harmonious chromatic number of G, which is denoted by $h(G)$.

Equivalently, say that a harmonious k-coloring of a finite graph G is a proper vertex k-coloring such that every pair of colors appears on at most one edge. The harmonious chromatic number h(G) of G is the minimum number k such that G has a harmonious k-coloring.The harmonious chromatic number introduced by Miller and Pritikin [7]. Further reference [8].

2. An iP_2 extension of star graphs

Definition 2.1. Let S_n be a non-trivial star graph on $n \geq 2$ nodes. The one of the node u_0 is adjacent to the remaining $n - 1$ pendant nodes, namely

$$\{u_1, u_2, \ldots ..., u_{n-1}\}.$$

Let P_2 be a path of length 1, which has two pendant nodes, say v_1 and w_1. If we superimpose a pendant node, say u_1 in S_n with a pendant node, say v_1 in P_2, then we obtain a new graph, called an $1P_2$ extended star graph and is denoted by $S^1_{n,2}$.

By the definition 2.1, an $1P_2$ extended star graph $S^1_{n,2}$ has $n+1$ nodes and n edges, since the superimposition of the path P_2 adds one more node and an edge into the existing star graph S_n.

Again we superimpose a pendant node, say u_2 in the $1P_2$ extended star graph $S^1_{n,2}$, which was a second pendant node in S_n, with a pendant node v_2 in a second path P_2. The resulting new graph is named as a $2P_2$ extended star graph, which is denoted by $S^2_{n,2}$. It has $n+2$ nodes and $n+1$ edges. Continue this procedure for each of the remaining pendant nodes of S_n.

For $1 \leq i, j \leq n-1$, let iP_2 denote i distinct paths of length 1, let (v_i, w_i) be the nodes of the i^{th} path P_2, both are pendant nodes and let u_j be the j^{th} pendant node in a star graph $S_n; n \geq 2$.

For all $i = j$, if we superimpose u_j with v_i, then the graph obtained is called *an iP_2 extended star graph* which is denoted by $S^i_{n,2}$.

For a particular value of n, where $n \geq 2$, there exists at most $n-1$ such P_2 extensions of a star graph $S_n; n \geq 2$.

It is clear that the superimposition of a pendant node in the star graph S_n and a pendant node in the path P_2 contributes a new node and a new edge to the existing graph.

Thus, an iP_2 extended star graph $S^i_{n,2}$ contains $n+i$ nodes and $n-1+i$ edges, where $n \geq 2, 1 \leq i \leq n-1$.

For $i = n-1$, we obtain an $(n-1)P_2$ extended star graph $S^{n-1}_{n,2}$, which is the highest P_2 extension of S_n. The number of nodes and edges in $S^{n-1}_{n,2}$ are $2n-1$ and $2n-2$ respectively.

Example 2.2. Consider a star graph S_5 and path graph P_2 which are given in the Figures 1 and 2. An $1P_2$ extended star graph $S^2_{5,2}$ obtained from S_5 and a path $1P_2$ by superimposing pendant nodes u_1 and v_1, which is given in the Figure 3.

Example 2.3. Consider a star graph S_5 and two path graphs $2P_2$ which are given in the Figures 4 and 5. The $2P_2$ extended star graph $S^2_{5,2}$ obtained from S_5 and two paths $2P_2$ by superimposing pendant nodes u_1 with v_1 and u_2 with v_2 which is given in the Figure 6.

Example 2.4. The $3P_2$ extended star graph $S^3_{5,2}$ obtained from S_5 and three paths $3P_2$ is shown in the Figure 9. The three path graphs $3P_2$ are in 7, 8

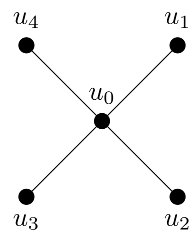

FIGURE 1. Star Graph S_5

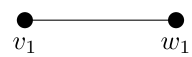

FIGURE 2. Path P_2

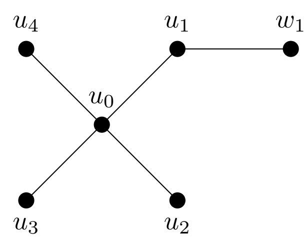

FIGURE 3. $1P_2$ Extended Star Graph $S^1_{5,2}$

Theorem 2.5. *For an iP_2 extended star graph $S^i_{n,2}; 1 \leq i \leq n-1$,*

$$\Delta(S^i_{n,2}) = \begin{cases} n-1 & \text{if } n \geq 3 \\ 2 & \text{if } n = 2, \end{cases} \tag{1}$$

where Δ denotes the maximum degree.

Proof. For $n = 2$, the star graph S_2 has two pendant nodes only. The superimposition of a pendant node in S_2 with a pendant node in the path P_2 results an $1P_2$ extended star graph $S^1_{2,2}$. But it will be a path P_3 on three nodes which has a node of maximum degree two.

For $n \geq 3$, after superimposing each pendant node $u_i, 1 \leq i \leq n-1$ in S_n with a pendant node in the path $iP_2, 1 \leq i \leq n-1$, it can be seen that all the pendant nodes in S_n will become nodes of degree 2 in the newly formed iP_2 extended star graph $S^i_{n,2}$. But there is no change in the degree of the node u_0, still it remains as the maximum degree node. □

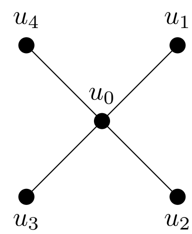

FIGURE 4. Star Graph S_5

FIGURE 5. Two Paths $2P_2$

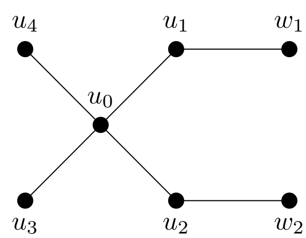

FIGURE 6. $2P_2$ Extended Star Graph $S^2_{5,2}$

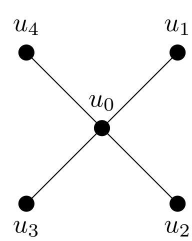

FIGURE 7. Star Graph S_5

FIGURE 8. Three Paths $3P_2$

Corollary 2.6. *Let $S^i_{n,2}; 1 \leq i \leq n-1$ be an iP_2 extended star graph obtained from a star graph S_n, then*

$$\Delta(S^i_{n,2}) = \begin{cases} \Delta(S_n) & \text{if } n \geq 3 \\ \Delta(S_n)+1 & \text{if } n = 2, \end{cases} \tag{2}$$

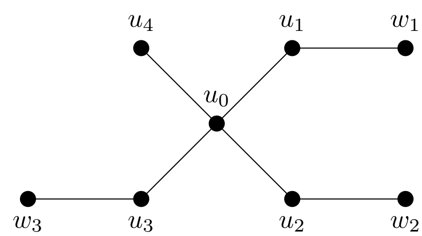

FIGURE 9. $3P_2$ Extended Star Graph $S^3_{5,2}$

where Δ denotes the maximum degree.

Proof. The maximum degree in the star graph $S_n; n \geq 2$ is $\Delta(S_n) = n - 1$. Therefore from Theorem 2.5, we will obtain the result. ☐

Theorem 2.7. *Let $S^i_{n,2}; 1 \leq i \leq n-1$ be an iP_2 extended star graph obtained from a star graph S_n, then*

$$\delta(S^i_{n,2}) = 1, \tag{3}$$

where δ denotes the minimum degree.

Proof. In each P_2 extension, we superimpose only one pendent node in P_2 with a pendent node in S_n at a time, for which the degree will become two. The other pendent node in P_2 still remains as a node of degree one. Therefore the minimum degree of an iP_2 extended star graph $S^i_{n,2}$ is one. ☐

Example 2.8. In the $4P_2$ extended star graph $S^4_{5,2}$ (Figure 10), we have the following facts :

- The number of nodes $|V| = 25 - 1 = 9$.
- The number of edges $|E| = 25 - 2 = 8$.
- The maximum degree $\Delta = 5 - 1 = 4$.
- The minimum degree $\delta = 1$

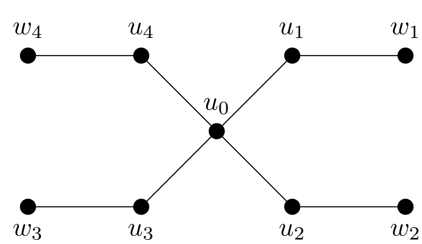

FIGURE 10. $4P_2$ Extended Star Graph $S^4_{5,2}$

3. The Domination Number of $S^i_{n,2}$

Theorem 3.1. *For an iP_2 extended star graph $S^i_{n,2}$,*

$$\gamma(S^i_{n,2}) = \begin{cases} i+1 & \text{if } n \geq 3, 1 \leq i \leq n-2 \\ 1 & \text{if } n = 2 \text{ and } i = 1, \end{cases} \tag{4}$$

where γ denotes the domination number.

Proof. For $n = 2$, we obtain an $1P_2$ extended star graph $S^1_{2,2}$, that is a path on three nodes, for which the domination number is

$$\gamma(S^1_{2,2}) = 1.$$

Now, consider the case $n \geq 3$. In an iP_2 extended star graph $S^i_{n,2}$, $1 \leq i \leq n-2$, all the nodes except the nodes in newly introduced P_2's which are not included in the superimposition are dominated by the node u_0 which was the maximum degree node in S_n. As P_2 has two pendent nodes, an iP_2 has $2i$-pendent nodes in total. Among these, i nodes are superimposed and which are dominated by the node u_0. Therefore, the remaining i nodes together with the node u_0 constitute a minimal dominating set for $S^i_{n,2}, 1 \leq i \leq n-2$ and hence the domination number for an iP_2 extended star graph $S^i_{n,2}, 1 \leq i \leq n-2$ is

$$\gamma(S^i_{n,2}) = i + 1.$$

□

Theorem 3.2. *Let S_n be a star graph on $n \geq 3$ nodes. If $S^{n-1}_{n,2}$ is an extended star graphs obtained from S_n by an $(n-1)P_2$ extension, then*

$$\gamma(S^{n-1}_{n,2}) = n - 1,$$

where γ denotes the domination number.

Proof. In an $(n-1)P_2$ extended star graph $S^{n-1}_{n,2}$, all the nodes adjacent to u_0 which was the maximum degree node in S_n are of degree two and these nodes are adjacent to the pendent nodes of the newly introduced P_2's. Therefore, these $n-1$ nodes altogether constitute a minimal dominating set for the graph $S^{n-1}_{n,2}$ and hence the domination number of the $(n-1)P_2$ extended star graph $S^{n-1}_{n,2}$ is

$$\gamma(S^{n-1}_{n,2}) = n - 1.$$

□

Corollary 3.3. *For $(n-1)P_2$ and $(n-2)P_2$ extensions,*

$$\gamma(S^{n-1}_{n,2}) = \gamma(S^{n-2}_{n,2}) = n - 1.$$

Proof. The result follows from Theorem 3.1 and Theorem 3.2. □

Corollary 3.4. *For $n \geq 3$,*

$$\gamma(S^{n-2}_{n,2}) = \gamma(S^{n-1}_{n,2}) = \Delta(S_n).$$

Proof. Since the maximum degree in the star graph S_n is $\Delta(S_n) = n-1$, from the Corollary 3.3, we have

$$\gamma(S_{n,2}^{n-2}) = \gamma(S_{n,2}^{n-1}) = n-1 = \Delta(S_n)$$

$\square$

Corollary 3.5. *For $n \geq 3$,*

$$\gamma(S_{n,2}^{n-2}) = \gamma(S_{n,2}^{n-1}) = \Delta(S_{n,2}^{n-1|})$$

Proof. The maximum degree in an $(n-1)P_2$ extended star graph $S_{n,2}^{n-1}$ is $n-1$, where $n \geq 3$, from the Corollary 3.3, we have

$$\gamma(S_{n,2}^{n-2}) = \gamma(S_{n,2}^{n-1}) = \Delta(S_{n,2}^{n-1})$$

$\square$

4. The Roman Domination Number of $S_{n,2}^{i}$

In [3], there are two characterizations on Roman domination number $\gamma_R(G)$ of a graph G given by the following propositions:

Proposition 4.1 ([3])**.** *Let $G = (V, E)$ be a connected graph on $|V|$ nodes. Then*

$$\gamma_R(G) = \gamma(G) + 1$$

if and only if G contains a node of degree $|V| - \gamma(G)$.

Proposition 4.2 ([3])**.** *Let $G = (V, E)$ be a connected graph on $|V|$ nodes. Then*

$$\gamma_R(G) = \gamma(G) + 2$$

if and only if the following two conditions hold:

(1) G does not contain a node of degree $|V| - \gamma(G)$.
(2) Either G contains a node of degree $|V| - \gamma(G) - 1$ or G contains two nodes u_1 and u_2 for which

$$|N[u_1] \cup N[u_2]| = |V| - \gamma(G) + 2$$

Based on proposition 4.1 and proposition 4.2, we have the following characterizations:

Theorem 4.3. *Let $S_{n,2}^{i}$ be an iP_2 extended star graph on $n+i$ nodes, then*

$$\gamma_R(S_{n,2}^{i}) = \begin{cases} n & \text{if} \quad n \geq 4, i = 1, 2, \ldots, m \text{ and } m < n, \\ 3 & \text{if} \quad n = 2, 3 \text{ and } i = 1, \\ 4 & \text{if} \quad n = 3, \text{ and } i = 2, \end{cases} \tag{5}$$

where γ_R denotes the Roman domination number.

Proof. For an iP_2 extended star graph $S^i_{n,2}$, where $n \geq 2, 1 \leq i \leq n-1$ we consider the following three cases:

Case(i): For $n \geq 3$ and $1 \leq i \leq n-2$.

For an extended star graph $S^i_{n,2}$, where $n \geq 3, 1 \leq i \leq n-2$, the number of nodes $|V(S^i_{n,2})| = n+i$ and $\gamma(S^i_{n,2}) = i+1$.Therefore,

$$\begin{aligned} |V(S^i_{n,2})| - \gamma(S^i_{n,2}) &= n + i - (i+1) \\ &= n-1 \end{aligned}$$

Also, the extended star graph $S^i_{n,2}$ contains a node of degree $n-1$, which will be the maximum degree node. That is, an extended star graph $S^i_{n,2}$, where $n \geq 3, 1 \leq i \leq n-2$ contains a node of degree $|V(S^i_{n,2})| - \gamma(S^i_{n,2})$. Therefore, from the Proposition 4.1, we have

$$\begin{aligned} \gamma_R(S^i_{n,2}) &= \gamma(S^i_{n,2}) + 1 \\ &= i+1+1 \\ &= i+2 \end{aligned}$$

Case (ii): $n=2$ and $i=1$. For $n=2$, we have $|V(S^1_{2,2})| = 3$ and $\gamma(S^1_{2,2}) = 1$. Therefore, $|V(S^1_{2,2})| - \gamma(S^1_{2,2}) = 3-1 = 2$ and $\gamma(S^1_{2,2})+1 = 2$. Also, the extended star graph $S^1_{2,2}$ contains a node of degree 2, which is the maximum degree node. That is, the extended star graph $S^1_{2,2}$ contains a node of degree $|V(S^1_{2,2})| - \gamma(S^1_{2,2})$ and therefore, from the Proposition 4.1, we have

$$\begin{aligned} \gamma_R(S^1_{2,2}) &= \gamma(S^1_{2,2}) + 1 \\ &= 1+1 = 2 \end{aligned}$$

Case (iii): $n \geq 3$ and $i = n-1$. For $n \geq 3$ and $i = n-1$, an extended star graph $S^{n-1}_{n,2}$ has $|V(S^{n-1}_{n,2})| = 2n-1$ nodes and it has the domination number $\gamma(S^{n-1}_{n,2}) = n-1$. Therefore,

$$\begin{aligned} |V(S^{n-1}_{n,2})| - \gamma(S^{n-1}_{n,2}) &= 2n-1-(n-1) \\ &= n, \end{aligned}$$

but $S^{n-1}_{n,2}$ does not contain a node of degree n.

Now, $|V(S^{n-1}_{n,2})| - \gamma(S^{n-1}_{n,2}) - 1 = n-1$. Furthermore, an extended star graph $S^{n-1}_{n,2}$ contains a node of degree

$$|V(S^{n-1}_{n,2})| - \gamma(S^{n-1}_{n,2}) - 1 = n-1$$

Therefore, from the Proposition 4.2, we have $\gamma_R(S^i_{n,2}) = \gamma(S^{n-1}_{n,2}) + 2 = n-1+2 = n+1$. Hence the proof. $\square$

5. The Chromatic Number of $S^i_{n,2}$

Theorem 5.1. *The chromatic number of an iP_2 extended star graph $S^i_{n,2}; n \geq 2, 1 \leq i \leq n-1$ is $\chi(S^i_{n,2}) = 2$.*

Proof. A star graph S_n can be considered as a union of paths of length at most 2, intersecting at a node, which is the maximum degree node in S_n. The superimposition of paths of length two at each pendant node of S_n increases the path length by a maximum of two. Therefore, an iP_2 extended star graph $S^i_{n,2}$, where $1 \leq i \leq n-1$ can be considered as a union of paths of length at most 4, intersecting at the same maximum degree node as in S_n.

Since all paths can be properly colored with minimum two colors, it can be concluded that the minimum number of colors required for a proper coloring of an iP_2 extended star graph $S^i_{n,2}$;$n \geq 2, 1 \leq i \leq n-1$ is two.

For the assurance of proper coloring, it is better to note that the coloring is to be started from the maximum degree node of $S^i_{n,2}$. □

5.1. The Chromatic Properties of $S^i_{n,2}$.

In [4], the authors proved the following theorem which states a relation connecting domination number, chromatic number and number of nodes of G.

Theorem 5.2. *Let $G = (V, E)$ be a graph with n number of nodes. Then,*

$$\gamma + \chi \leq n + 1,$$

where γ denotes the domination number and χ denotes the chromatic number of the graph G.

The above inequality can be verified for an iP_2 extended star graph $S^i_{n,2}$. For an iP_2 extended star graph $S^i_{n,2}$, we have

$$|V(S^i_{n,2})| = n + i, \quad \gamma(S^i_{n,2}) = i + 1, \quad \chi(S^i_{n,2}) = 2.$$

Therefore,

$$\begin{aligned}
\gamma(S^i_{n,2}) + \chi(S^i_{n,2}) &= i + 1 + 2 = i + 3 \\
&\leq i + n + 1 \quad \text{Since} \quad n \geq 2 \implies n + 1 \geq 3 \\
&= (n + i) + 1 \\
&= |V(S^i_{n,2})| + 1.
\end{aligned}$$

That is, $\gamma(S^i_{n,2}) + \chi(S^i_{n,2}) \leq |V(S^i_{n,2})| + 1$.

6. The Harmonious Colouring of $S^i_{n,2}$

Theorem 6.1. *For an iP_2 extended star graph $S^i_{n,2}$,*

$$h(S^i_{n,2}) = \begin{cases} n & \text{if } n \geq 4, i = 1, 2, \ldots, m \text{ and } m < n, \\ 3 & \text{if } n = 2, 3 \text{ and } i = 1, \\ 4 & \text{if } n = 3 \text{ and } i = 2, \end{cases} \tag{6}$$

where h denotes the harmonious chromatic number.

Proof. For the extension, we consider the non-trivial star graphs S_n only. If $n = 2$, from the star graph S_2, we obtain the $1P_2$ extended star graph $S^1_{2,2}$, which is a path of length three. Clearly, three colors are needed to color it harmoniously.

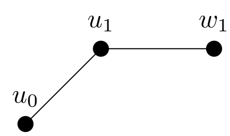

FIGURE 11. $1P_2$ Extended Star Graph $S^1_{2,2}$

Let c_1, c_2 and c_3 be the three colors. The possible distinct pairs are $(c_1, c_2), (c_1, c_3)$ and (c_2, c_3).

Colour assignments of the edges are as follows:
$(c_1, c_2) \to (u_0, u_1)$ and $(c_2, c_3) \to (u_1, v_2)$.
Here 3 colors are used.

If $n = 3$, from the star graph S_3, we obtain an $1P_2$ extended star graph $S^1_{3,2}$ and a $2P_2$ extended star graph $S^2_{3,2}$. Note that the graph $S^1_{3,2}$ has three edges. Since the maximum degree of $S^1_{3,2}$ is two, a minimum of three colors are needed for its harmonious coloring. Using these three colors, we can form distinct color pairs in $\binom{3}{2}$=3 ways. These color pairs can be assigned one by one to the edges of the $1P_2$ extended star graph $S^1_{3,2}$ in such a way that all the edges receive a distinct color pair. That is, the $1P_2$ extended star graph $S^1_{3,2}$ can be colored harmoniously using three colors.

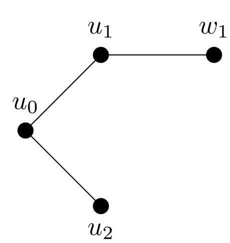

FIGURE 12. $1P_2$ Extended Star Graph $S^1_{3,2}$

Let c_1, c_2 and c_3 be the three colors. The possible distinct pairs are $(c_1, c_2), (c_1, c_3)$ and (c_2, c_3).

Colour assignments of the edges are as follows:

$(c_1, c_2) \to (u_0, u_1), (c_2, c_3) \to (u_1, w_1)$ and $(c_1, c_3) \to (u_0, u_2)$. Here, 3 colors are used.

But, for the $2P_2$ extended star graph $S^2_{3,2}$, since it has four edges, three color pairs are not enough to color it harmoniously. Therefore, we have to use a fourth color for its harmonious coloring. Using these four colors, $\binom{4}{2}$=6 distinct color pairs can be formed and can be assigned one by one to the edges of the $2P_2$ extended star graph $S^2_{3,2}$ in such a way that all the edges will receive a distinct color pair.

That is, the $2P_2$ extended star graph $S^2_{3,2}$ can be colored harmoniously using four colors.

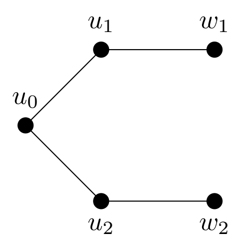

FIGURE 13. $2P_2$ Extended Star Graph $S^2_{3,2}$

Let c_1, c_2, c_3 and c_4 be four colors. The possible distinct pairs are $(c_1, c_2), (c_1, c_3), (c_1, c_4), (c_2, c_3), (c_2, c_4)$ and (c_3, c_4).

Colour assignments of the edges are as follows:
$(c_1, c_2) \to (u_0, u_1)$, $(c_2, c_3) \to (u_1, w_1)$, $(c_1, c_3) \to (u_0, u_2)$ and $(c_3, c_4) \to (u_2, w_2)$. Here 4 colors are used.

Thus, we have

$$h(S^1_{2,2}) = 3, h(S^1_{3,2}) = 3, h(S^2_{3,2}) = 4.$$

Consider the case $n \geq 4$. In an iP_2 extended star graph $S^i_{n,2}$, where $1 \leq i \leq n-1$, maximum edges occur when $i = n-1$. There are $2n-1$ nodes and $2n-2$ edges in the $(n-1)P_2$ extended star graph $S^{n-1}_{n,2}$, so that $2n-2$ distinct color pairs are needed for coloring it harmoniously. Since the maximum degree in $S^{n-1}_{n,2}$ is

$$\Delta(S^{n-1}_{n,2}) = n - 1,$$

at least n colors are needed for the harmonious coloring of $S^{n-1}_{n,2}$. Let $c_1, c_2, \ldots, c_n$ be the n distinct colors to be used for the harmonious coloring of $S^{n-1}_{n,2}$. Using these n colors, we can form distinct color pairs in $\binom{n}{2}$ ways. Now, we have,

$$\begin{aligned}\binom{n}{2} &= \frac{n(n-1)}{2} \\ &\geq \frac{4(n-1)}{2} \qquad \text{since } n \geq 4 \\ &= 2(n-1) = 2n-2\end{aligned}$$

$$\text{i.e.,} \quad \binom{n}{2} \geq 2n - 2.$$

That is, the number of distinct color pairs using n colors is greater than or equal to the maximum number of edges in the $(n-1)P_2$ extended star graph $S_{n,2}^{n-1}$. Therefore, each edge of the $(n-1)P_2$ extended star graph $S_{n,2}^{n-1}$ can be assigned the color pairs one by one from those $\binom{n}{2}$ distinct color pairs till each of the edge receives a distinct color pair.

The scheme of coloring is as follows: Suppose we color the maximum degree node with a color c_1. There are $n-1$ color pairs $(c_1, c_2), (c_1, c_3), ..., (c_1, c_n)$ are available with the color c_1. These color pairs can be assigned to the edges incident with the maximum degree node. There are $n-2$ color pairs

$$(c_2, c_3), (c_2, c_4), ..., (c_2, c_n)$$

are available with the color c_2. One of these color pairs can be assigned to the edge incident with the node colored with c_2. There are $n-3$ color pairs

$$(c_3, c_4), (c_3, c_5), ..., (c_3, c_n)$$

are available with the color c_3. One of these color pairs can be assigned to the edge incident with the node colored with c_3 and so on.

For the color c_{n-1}, there is only one color pair (c_{n-1}, c_n) is available that can be assigned to the edge incident with the second last node, i.e., the $(n-1)^{th}$ node which is colored with c_{n-1}.

Now, only one edge which is incident with the node colored with the color c_n is remained to color. Any color pair $(c_n, c_k); 1 \leq k \leq n-2$, which is not previously assigned for any edge can be used to assign for this last edge.

Thus, using n colors, an iP_2 extended star graph $S_{n,2}^{i}; n \geq 4$, where $1 \leq i \leq n-1$ can be colored harmoniously. □

Example 6.2. Consider $2P_2$ extension $S_{4,2}^2$ and $3P_2$ extension $S_{4,2}^3$ of the star graph S_4.

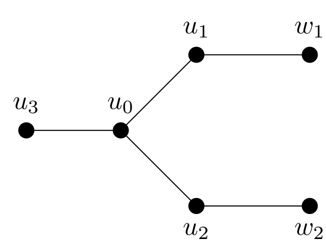

FIGURE 14. $2P_2$ Extended Star Graph $S_{4,2}^2$

Let c_1, c_2, c_3, and c_4 be the four colors to be used for the harmonious coloring. The possible distinct color pairs using these 4 colors are the following:

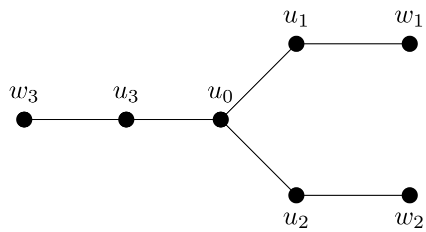

FIGURE 15. $3P_2$ Extended Star Graph $S^3_{4,2}$

$(c_1, c_2), (c_1, c_3), (c_1, c_4), (c_2, c_3), (c_2, c_4)$, and (c_3, c_4).

The assignments of color pairs are as follows:

For $S^2_{4,2}$: $(c_1, c_2) \rightarrow (u_0, u_1), (c_2, c_3) \rightarrow (u_1, w_1), (c_1, c_3) \rightarrow (u_0, u_2)$, $(c_1, c_4) \rightarrow (u_0, u_3)$ and $(c_3, c_4) \rightarrow (u_2, w_2)$.

For $S^3_{4,2}$: $(c_1, c_2) \rightarrow (u_0, u_1), (c_2, c_3) \rightarrow (u_1, w_1), (c_1, c_3) \rightarrow (u_0, u_2)$, $(c_1, c_4) \rightarrow (u_0, u_3)(c_3, c_4) \rightarrow (u_2, w_2)$ and $(c_2, c_4) \rightarrow (u_3, w_3)$.

7. Conclusion

A star graph can be extended to another graph structure in various ways such as edge removal and edge addition process. These types of extensions reflect on the graph parameters. In this paper, we have attempted to extend a star graph by superimposing paths P_2 of length 2 at their pendant nodes and study the associated graph parameters in terms of order, size, minimum degree δ, maximum degree Δ, domination number γ, Roman domination number γ_R, chromatic number χ and harmonious chromatic number h of the graph. In future, this work can be extended to develop new graph structures and initiate a study through algorithmic approach.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

Acknowledgments : The authors are thankful to the referee for the useful suggestions

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