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STUDY OF YOUNG INEQUALITIES FOR MATRICES

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Abstract. This paper investigates Young inequalities for matrices, a problem closely linked to operator theory, mathematical physics, and the arithmetic-geometric mean inequality. By obtaining new inequalities for unitarily invariant norms, we aim to derive a fresh Young inequality specifically designed for matrices.To lay the foundation for our study, we provide an overview of basic notation related to matrices. Additionally, we review previous advancements made by researchers in the field, focusing on Young improvements.Building upon this existing knowledge, we present several new enhancements of the classical Young inequality for nonnegative real numbers. Furthermore, we establish a matrix version of these improvements, tailored to the specific characteristics of matrices. Through our research, we contribute to a deeper understanding of Young inequalities in the context of matrices.

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1. Introduction

Let $M_n(C)$ be the space of n × n complex matrices and let $|||\cdot|||$ denote any unitarily invariant (or symmetric) norm on $M_n(C)$. So, $|||UAV||| = |||A|||$ for all $A \in M_n(C)$ and for all unitary matrices $U, V \in M_n(C)$. For $A \in M_n(C)$, the Hilbert-Schmidt norm, the trace norm, and the spectral norm of A are defined by $\|A\|_2 = \sum\limits_{j=1}^{n} s_j^2(A))^{1/2}$, $\|A\|_1 = \sum\limits_{j=1}^{n} s_j(A))$, and $\|A\| = s_1(A)$, respectively where $s_1(A) \geq ... \geq s_n(A)$are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix$|A| = (A^*A)^{1/2}$.

Note that $\|A\|_2 = (tr|A|^2)^{1/2}$ and $A_1 = tr|A|$, where tr is the usual trace functional. It is evident that these norms are unitarily invariant, and it is known

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that each unitarily invariant norm is a symmetric guage function of singular values [[8], p. 91].

The well-known Young inequality is a classic outcome attributed to William Henry Young (1863–1942), an English mathematician. For the first time, it officially appeared in a search W. H.Young [14].

Even though Young inequality looks very simple, it is very important in matrix theory. The classical Young inequality for two scalars is the v-weighted arithmetic–geometric mean inequality, which is a fundamental relation between two non-negative real numbers. This inequality says that if a, $d \geq 0$ and $0 \leq$ v ≤ 1, then

$$a^v d^{1-v} \leq va + (1-v)d,$$

with equality if and only if a = d, this inequality can be written as $ad \leq \frac{a^p}{p} + \frac{d^q}{q}$, if p,q > 1 such that $\frac{1}{p} + \frac{1}{q} = 1$. If v $= \frac{1}{2}$, we take out the arithmetic–geometric mean inequality $\sqrt{ad} \leq \frac{a+d}{2}$.

The matrix version of classical Young inequality proved by T. Ando [1], if $A, M, X \in M_n(\mathbb{C})$, then

$$s_i(AM) \leq (\frac{A^p}{p} + \frac{M^q}{q}),$$

for i = 1,...,n. so the trace version hold

$$tr\left|A^v M^{1-v}\right| \leq tr(vA + (1-v)M),$$

And the determinant version [[10], p.467]

$$\det(A^v M^{1-v}) \leq \det(vA + (1-v)M).$$

First refinement for Young inequality was finding by Hirzallah and Kittaneh [9] as the follows:

If $a, d \geq 0$ and $0 \leq v \leq 1$, then

$$(a^v d^{1-v})^2 + r^2(a-d)^2 \leq (va + (1-v)d)^2. \tag{1}$$

Next improvement was defined by Kittaneh and Manasrah [12] If $a, d \geq 0$ and $0 \leq v \leq 1$, then

$$a^v d^{1-v} + r(\sqrt{a} - \sqrt{d})^2 \leq va + (1-v)d, \tag{2}$$

where $r = min\{v, 1-v\}$.

Kittaneh and Manasrah give a generalization [6] that if $a, d > 0$ and $0 \leq v \leq 1$, then for $m = 1, 2, 3, ...$, we have

$$(a^v d^{1-v})^m + r^m(a^{\frac{m}{2}} - d^{\frac{m}{2}})^2 \leq (va + (1-v)d)^m \tag{3}$$

where $r = min\{v, 1-v\}$. And for matrix they proved generalization as following

$$\begin{aligned}&\left(\left|\left|\left|A^v X M^{1-v}\right|\right|\right|\right)^m + r^m\left(|||AX|||^{\frac{m}{2}} - |||XM|||^{\frac{m}{2}}\right)^2 \\ &\leq (v\,|||AX||| + (1-v)\,|||XM|||)^m\end{aligned} \tag{4}$$

Here some important Corollary, Theorem, and Lemma we will need it.

Lemma 1.1. [7] *Let Φ be a strictly increasing convex function defined on an interval I. Let $x, y, z,$ and w are points in I such that*

$$z - w \leq x - y,$$

where $w \leq z \leq x$ and $y \leq x$. Then

$$(0 \leq)\Phi(z) - \Phi(w) \leq \Phi(x) - \Phi(y).$$

Corollary 1.2. [7] *If $a, d > 0$ and $0 \leq v \leq 1$, $A, M, X \in M_n(\mathbb{C})$ such that A and M are positive semidefinite, then for $p \in \mathbb{R}$, $p \geq 1$ we have*

$$r^p((a+d)^p - (2\sqrt{ad})^p) \leq (va + (1-v)d)^p - (a^v d^{1-v})^p. \tag{5}$$

The matrix version of 5, [7]

$$r^p \left(\|AX + XM\|_2^p - 2^p \left\| A^{\frac{1}{2}} X M^{\frac{1}{2}} \right\|_2^p \right) \leq \|vA + (1-v)M\|_2^p - \left\| A^v X M^{1-v} \right\|_2^p. \tag{6}$$

Theorem 1.3. [11] *Let m be positive integer and v a positive real number such that $0 \leq v \leq 1$, then we have*

$$(a^v d^{1-v})^m + r^m \left(\frac{d^{m+1} - a^{m+1}}{d-a} - (m+1)(ad)^{\frac{m}{2}} \right) \leq (va + (1-v)d)^m,$$

where $r = \min\{v, 1-v\}$.

Theorem 1.4. [11] *Let $M, E, Y \in M_n(\mathbb{C})$ be positive definite matrices and $0 \leq v \leq 1$. Then for $m = 1, 2, 3, ...,$ we have:*

$$\left|\left|\left| A^v X M^{1-v} \right|\right|\right|^m + r^m \left(\frac{|||XM|||^{m+1} - |||AX|||^{m+1}}{|||XM||| - |||AX|||} - (m+1)(|||XM|||\, |||AX|||)^{\frac{m}{2}} \right)$$
$$\leq (v\, |||AX||| + (1-v)\, |||XM|||)^m.$$

2. Main results

At first we examined some inequalities for real number, after that we discuss the matrix version of them . Now by assuming $m = 1$ in the inequality of Theorem 1.3, then we have

$$r\left(\frac{d^2 - a^2}{d-a} - 2(ad)^{\frac{1}{2}}\right) \leq va + (1-v)d - a^v d^{1-v}. \tag{7}$$

simplifying this inequality we get again inequality 2

This inequality will help us prove theorem 2.1 which shows another way to prove corollary 2 in [7].

Theorem 2.1. *If $a, d > 0$, $0 \leq v \leq 1$, and $p \in \mathbb{R}$ ($p \in R$, $p \geq 1$), then*

$$r^p \left(\left(\frac{d^2 - a^2}{d-a} \right)^p - (2)^p (ad)^{\frac{p}{2}} \right) \leq (va + ((1-v)d)^p - (a^v d^{1-v})^p. \tag{8}$$

Proof. If $\Psi : [0, \infty) \to \mathbb{R}$ is a strictly increasing convex function, then

$$\Psi\left(r \frac{d^2 - a^2}{d-a}\right) - \Psi\left(2r(ad)^{\frac{1}{2}}\right) \leq \Psi\left(va + (1-v)d\right) - \Psi\left(a^v d^{1-v}\right).$$

Let $z = r\frac{d^2-a^2}{d-a}, w = 2r(ad)^{\frac{1}{2}}, x = va + (1-v)d$, and $y = a^v d^{1-v}$. Then based on the inequality 7 and the arithmetic-geometric mean inequality , we have

$$z - w \leq x - y.$$

Also $w \leq z \leq x, y \leq x$, then by lemma 1.1

$$\Psi(z) - \Psi(w) \leq \Psi(x) - \Psi(y).$$

□

Taking $\Psi(x) = x^p$, $p \in \mathbb{R}$ *(*$p \in R$, $p \geq 1$*), then we have*

$$r^p\left(\left(\frac{d^2-a^2}{d-a}\right)^p - (2)^p(ad)^{\frac{p}{2}}\right) \leq (va + ((1-v)d)^p - (a^v d^{1-v})^p.$$

Which is the same result in corollary 2 in [7].

In the next theorem we present prove for new result in same manner based on the following inequality

$$(a^v d^{1-v})^2 + r^2(a-d)^2 \leq (va + (1-v)d)^2$$

which we can write as

$$r^2\left(\frac{d^3-a^3}{d-a}\right) - r^2(3ad) \leq (va + (1-v)d)^2 - (a^v d^{1-v})^2 \tag{9}$$

Theorem 2.2. *If* $a, d > 0$ *and* $0 \leq v \leq 1$, *then for* $p \in \mathbb{R}$, $p \geq 1$ *we have*

$$r^{2p}\left[\left(d^2 + a^2 + ad\right)^p - (3ad)^p\right] \leq (va + (1-v)d)^{2p} - (a^v d^{1-v})^{2p}. \tag{10}$$

Proof. Let $z = r^2\left(\frac{d^3-a^3}{d-a}\right), w = r^2(3ad), x = (va + (1-v)d)^2$, and $y = (a^v d^{1-v})^2$, and $\Psi : [0, \infty) \to \mathbb{R}$ is a strictly increasing convex function. Then based on the inequality 9 and the arithmetic-geometric mean inequality, we have

$$z - w \leq x - y$$

Also $w \leq z \leq x$, $y \leq x$, then by Lemma 1.1

$$\Psi(w) - \Psi(z) \leq \Psi(x) - \Psi(y).$$

Now, taking $\Psi(x) = x^p$, where $p \in \mathbb{R}$, $p \geq 1$, we have

$$\Psi\left(r^2\frac{d^3-a^3}{d-a}\right) - \Psi\left(3r^2ad\right) \leq \Psi\left((va + (1-v)d)^2\right) - \Psi\left((a^v d^{1-v})^2\right),$$

then

$$\left(r^2\frac{d^3-a^3}{d-a}\right)^p - \left(3r^2ad\right)^p \leq (va + (1-v)d)^{2p} - (a^v d^{1-v})^{2p},$$

equivalently

$$r^{2p}\left[\left(d^2 + a^2 + ad\right)^p - (3ad)^p\right] \leq (va + (1-v)d)^{2p} - (a^v d^{1-v})^{2p}.$$

□

Depending on the result in the previous theory, we can conclude that if $2p = q$, so that we have following corollary

Corollary 2.3.

$$r^q\left[\left(d^2+a^2+ad\right)^{\frac{q}{2}}-(3ad)^{\frac{q}{2}}\right]\leq (va+(1-v)d)^q-(a^vd^{1-v})^q.$$

take $v=\frac{1}{2}$, *so* $r=\frac{1}{2}$, *now ew can get some inequalities by taking* $p=1,2,3,...,n$

I) $q=1$

$$\begin{aligned}
\frac{1}{2}\left[\left(d^2+a^2+ad\right)^{\frac{1}{2}}-(3ad)^{\frac{1}{2}}\right] &\leq \frac{1}{2}(a+d)-(ad)^{\frac{1}{2}}\\
\frac{1}{2}\left(d^2+a^2+ad\right)^{\frac{1}{2}}-\frac{(3ad)^{\frac{1}{2}}}{2}+(ad)^{\frac{1}{2}} &\leq \frac{1}{2}(a+d)\\
\frac{1}{2}\left(d^2+a^2+ad\right)^{\frac{1}{2}}+\left(1-\frac{\sqrt{3}}{2}\right)(ad)^{\frac{1}{2}} &\leq \frac{1}{2}(a+d)
\end{aligned}$$

II) $q=2$

$$\begin{aligned}
(\frac{1}{2})^2(a-d)^2 &\leq (\frac{1}{2})^2(a+d)^2-(ad)\\
(\frac{1}{2})^2\left(d^2+a^2+ad\right)-(\frac{1}{2})^2(3ad)+(ad) &\leq (\frac{1}{2})^2(a+d)^2\\
(\frac{1}{2})^2\left(d^2+a^2+ad\right)+(1-\frac{3}{4})(ad) &\leq (\frac{1}{2})^2(a+d)^2\\
(\frac{1}{2})^2\left(d^2+a^2+ad\right)+\frac{1}{4}(ad) &\leq (\frac{1}{2})^2(a+d)^2
\end{aligned}$$

III) $q=3$

$$\begin{aligned}
\frac{1}{2^3}\left[\left(d^2+a^2+ad\right)^{\frac{3}{2}}-(3ad)^{\frac{3}{2}}\right] &\leq \frac{1}{2^3}(a+d)^3-(ad)^{\frac{3}{2}}\\
\frac{1}{2^3}\left(a^2+ad+d^2\right)^{\frac{3}{2}}+\left(1-\frac{(3)^{\frac{3}{2}}}{2^3}\right)(ad)^{\frac{3}{2}} &\leq \frac{1}{2^3}(a+d)^3
\end{aligned}$$

IV) $q=4$

$$\begin{aligned}
\frac{1}{2^4}\left[\left(d^2+a^2+ad\right)^2-(3ad)^2\right] &\leq \frac{1}{2^4}(a+d)^4-(ad)^2\\
\frac{1}{2^4}\left(d^2+a^2+ad\right)^2-\frac{1}{2^4}(3ad)^2+(ad)^2 &\leq \frac{1}{2^4}(a+d)^4\\
\frac{1}{2^4}\left(d^2+a^2+ad\right)^2-\frac{1}{2^4}(3ad)^2+(ad)^2 &\leq \frac{1}{2^4}(a+d)^4\\
\frac{1}{2^4}\left(d^2+a^2+ad\right)^2+(1-\frac{3^2}{2^4})(ad)^2 &\leq \frac{1}{2^4}(a+d)^4
\end{aligned}$$

V) $q=n$, $n\in\mathbb{N}$

$$\frac{1}{2^n}\left(d^2+a^2+ad\right)^{\frac{n}{2}}+(1-\frac{3^{\frac{n}{2}}}{2^n})(ad)^{\frac{n}{2}}\leq\frac{1}{2^n}(a+d)^n.$$

Since $r^m(a^{\frac{m}{2}} - d^{\frac{m}{2}})^2 = r^m(a^m + d^m) - 2r^m\,(ad)^{\frac{m}{2}}$*, the inequality 3 becomes*
$r^m(a^m + d^m) - 2r^m\,(ad)^{\frac{m}{2}} \le (va + ((1-v)d)^m - (a^v d^{1-v})^m.$
Now, replace a *by* a^2*, and* d *by* d^2*, we have*

$$r^m(a^{2m} + d^{2m}) - 2r^m\,(ad)^m \le (va^2 + (1-v)d^2)^m - (a^v d^{1-v})^{2m}. \tag{11}$$

We obtain next theory by using previous result and same method of proof in theorem 2.1

Theorem 2.4. *If* $a, d > 0$ *and* $0 \le v \le 1$*, then for* $p \in \mathbb{R}$*,* $p \ge 1$ *we have*

$$r^p(a^2 + d^2)^p - (2rad)^p \le (va^2 + ((1-v)d^2)^p - (a^v d^{1-v})^{2p} \tag{12}$$

Proof. In 11 let $m = 1$.Then

$$r(a^2 + d^2) - 2r\,(ad) \le va^2 + ((1-v)d^2 - (a^v d^{1-v})^2.$$

Applying $\Psi : [0, \infty) \to \mathbb{R}$ is a strictly increasing convex function

$$\Phi\left(r(a^2 + d^2)\right) - \Phi\left(2r\,(ad)\right) \le \Phi\left(ta^2 + ((1-v)d^2\right) - \Phi\left((a^v d^{1-v})^2\right).$$

And let $\Psi(x) = x^p, p \in R$. Then

$$\begin{aligned} r^p(a^2 + d^2)^p - (2rad)^p &\le (ta^2 + ((1-v)d^2)^p - (a^v d^{1-v})^{2p} \\ &\le (ta^2)^p + ((1-v)d^2)^p - (a^v d^{1-v})^{2p}. \end{aligned}$$

□

Now we start to prove matrix version. In theorem 1.4 , let $m = 2$, and $|||.||| = \|.\|_2$, then we have

$$\begin{aligned} & r^2\left(\frac{\|XM\|_2^3 - \|AX\|_2^3}{\|XM\|_2 - \|AX\|_2} - 3\,\|AX\|_2\,\|XM\|_2\right) \\ & \le \left(v\,\|AX\|_2 + (1-v)\,\|XM\|_2\right)^2 - \left\|A^v X M^{1-v}\right\|_2^2. \end{aligned}$$

Applying lemma 1.1 on the above inequality, we are able to begin to prove the following result.

Theorem 2.5. *Let* $A, M, X \in M_n(\mathbb{C})$ *such that* A *and* M *are positive semi-definite. If* $\Phi : [0, \infty) \to \mathbb{R}$ *is a strictly increasing convex function, then we have*

$$\begin{aligned} & \Phi(r^2\frac{\|XM\|_2^3 - \|AX\|_2^3}{\|XM\|_2 - \|AX\|_2}) - \Phi\left(3r^2\,\|AX\|_2\,\|XM\|_2\right) \\ \le\ & \Phi\left(\left(v\,\|AX\|_2 + (1-v)\,\|XM\|_2\right)^2\right) - \Phi\left(\left\|A^v X M^{1-v}\right\|_2^2\right). \end{aligned}$$

equivalently

$$\begin{aligned} & \Phi(r^2\,\|AX\|_2^2 + \|XM\|_2^2 + \|AX\|_2\,\|XM\|_2) - \Phi\left(3r^2\,\|AX\|_2\,\|XM\|_2\right) \\ \le\ & \Phi\left(\left(v\,\|AX\|_2^2 + (1-v)\,\|XM\|_2^2\right)^2\right) - \Phi\left(\left\|A^v X M^{1-v}\right\|_2^2\right). \end{aligned}$$

In particular, when $\Phi(x) = x^p$ ($p \in R$, $p \geq 1$), we have

$$\begin{aligned} & r^{2p}\left(\|AX\|_2^2 + \|XM\|_2^2 + \|AX\|_2\|XM\|_2\right)^p - r^{2p}\left(3\|AX\|_2\|XM\|_2\right)^p \\ \leq\ & \left(v\|AX\|_2 + (1-v)\|XM\|_2\right)^{2p} + \left\|A^vXM^{1-v}\right\|_2^{2p}. \end{aligned}$$

If $p = 1$, we have

$$\begin{aligned} & r^2\left(\|AX\|_2^2 + \|XM\|_2^2 + \|AX\|_2\|XM\|_2\right) - r^2\left(3\|AX\|_2\|XM\|_2\right) \\ \leq\ & \left(v\|AX\|_2 + (1-v)\|XM\|_2\right)^2 - \left\|A^vXM^{1-v}\right\|_2^2. \end{aligned}$$

equivalently

$$r^2\|AX - XM\|_2^2 \leq \left(v\|AX\|_2 + (1-v)\|XM\|_2\right)^2 - \left\|A^vXM^{1-v}\right\|_2^2.$$

If $p = 2$, we have

$$\begin{aligned} & r^4\left(\|AX\|_2^2 + \|XM\|_2^2 + \|AX\|_2\|XM\|_2\right)^2 - r^4\left(3\|AX\|_2\|XM\|_2\right)^2 \\ \leq\ & \left(v\|AX\|_2 + (1-v)\|XM\|_2\right)^4 - \left\|A^vXM^{1-v}\right\|_2^4. \end{aligned}$$

Which are some forms of inequalities that may be useful in future.

Lemma 2.6. [1]
$A, M, X \in M_n(\mathbb{C})$ *such that* A *and* M *are positive semidefinite.*

$$\left\|A^{\frac{1}{2}}XM^{\frac{1}{2}}\right\|_2 \leq \left\|A^{\frac{1}{2}}X\right\|_2\left\|XM^{\frac{1}{2}}\right\|_2 \leq \|AX\|_2^{\frac{1}{2}}\|XM\|_2^{\frac{1}{2}}$$

.

Out of theory 2.5 take $2p = q$, we have the next Corollary.

Corollary 2.7. *Let* $A, M, X \in M_n(\mathbb{C})$ *such that* A *and* M *are positive semidefinite,and* $0 \leq v \leq 1$

$$\begin{aligned} & r^q\left(\|AX\|_2^2 + \|XM\|_2^2 + \|AX\|_2\|XM\|_2\right)^{\frac{q}{2}} - r^q\left(3\|AX\|_2\|XM\|_2\right)^{\frac{q}{2}} \\ \leq\ & \left(v\|AX\|_2 + (1-v)\|XM\|_2\right)^q - \left\|M^vXM^{1-v}\right\|_2^q. \end{aligned}$$

where $r = min\{v, 1-v\}$.

Let $v = \frac{1}{2} \Rightarrow r = \frac{1}{2}$
I) $q = 1$

$$\begin{aligned} & \frac{1}{2}\left(\|AX\|_2^2 + \|XM\|_2^2 + \|AX\|_2\|XM\|_2\right)^{\frac{1}{2}} - \frac{1}{2}\left(3\|AX\|_2\|XM\|_2\right)^{\frac{1}{2}} \\ \leq\ & \frac{1}{2}\left(\|AX\|_2 + (\|XM\|_2\right) - \left\|A^{\frac{1}{2}}XM^{\frac{1}{2}}\right\|_2. \end{aligned}$$

$$\begin{aligned} & \frac{1}{2}\left(\|AX\|_2^2 + \|XM\|_2^2 + \|AX\|_2\|XM\|_2\right)^{\frac{1}{2}} - \frac{1}{2}\left(3\|AX\|_2\|XM\|_2\right)^{\frac{1}{2}} \\ \leq\ & \frac{1}{2}\left(\|AX\|_2 + (\|XM\|_2\right) - \|AX\|_2^{\frac{1}{2}}\|XM\|_2^{\frac{1}{2}}\ .\text{(by lemma 2.6)} \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}\left(\|AX\|_2^2+\|XM\|_2^2+\|AX\|_2\|XM\|_2\right)^{\frac{1}{2}}-\frac{1}{2}\left(3\|AX\|_2\|XM\|_2\right)^{\frac{1}{2}} \\ &\quad+\|AX\|_2^{\frac{1}{2}}\|XM\|_2^{\frac{1}{2}} \\ \leq\;& \frac{1}{2}\left(\|AX\|_2+(\|XM\|_2\right). \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}\left(\|AX\|_2^2+\|XM\|_2^2+\|AX\|_2\|XM\|_2\right)^{\frac{1}{2}}-\left(1-\frac{\sqrt{3}}{2}\right)\|AX\|_2^{\frac{1}{2}}\|XM\|_2^{\frac{1}{2}} \\ \leq\;& \frac{1}{2}\left(\|AX\|_2+(\|XM\|_2\right). \end{aligned}$$

II) $q=2$:

$$\begin{aligned} &\frac{1}{2}^2\left[\|AX\|_2^2+\|XM\|_2^2-2\|AX\|_2\|XM\|_2\right] \\ \leq\;& \frac{1}{2}^2\left(\|AX\|_2+(\|XM\|_2\right)^2-\left\|A^{\frac{1}{2}}XM^{\frac{1}{2}}\right\|_2^2 (\text{lemma } 2.6) \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}^2\left[\|AX\|_2^2+\|XM\|_2^2-2\|AX\|_2\|XM\|_2\right] \\ \leq\;& \frac{1}{2}^2\left(\|AX\|_2+(\|XM\|_2\right)^2-\|AX\|_2\|XM\|_2 \end{aligned}$$

$$(\frac{1}{2})^2(\|AX\|_2-\|XM\|_2)^2 \leq (\frac{1}{2})^2(\|AX\|_2+\|XM\|_2)^2-\|AX\|_2\|XM\|_2$$

$$\begin{aligned} &(\frac{1}{2})^2\left(\|AX\|_2^2+\|XM\|_2^2+\|AX\|_2\|XM\|_2\right)-(\frac{1}{2})^2(3\|AX\|_2\|XM\|_2) \\ &+(\|AX\|_2\|XM\|_2) \\ &\leq (\frac{1}{2})^2(\|AX\|_2+\|XM\|_2)^2 \end{aligned}$$

$$\begin{aligned} &(\frac{1}{2})^2\left(\|AX\|_2^2+\|XM\|_2^2+\|AX\|_2\|XM\|_2\right)+(1-\frac{3}{4})(\|AX\|_2\|XM\|_2) \\ \leq\;& (\frac{1}{2})^2(\|AX\|_2+\|XM\|_2)^2 \end{aligned}$$

$$\begin{aligned} &(\frac{1}{2})^2\left(\|AX\|_2^2+\|XM\|_2^2+\|AX\|_2\|XM\|_2\right)+\frac{1}{4}(\|AX\|_2\|XM\|_2) \\ &\leq (\frac{1}{2})^2(\|AX\|_2+\|XM\|_2)^2 \end{aligned}$$

III) $q=3$

$$\frac{1}{2}^3\left[\left(\|AX\|_2^2+\|XM\|_2^2+\|AX\|_2\|XM\|_2\right)^{\frac{3}{2}}-(3\|AX\|_2\|XM\|_2)^{\frac{3}{2}}\right]$$

$$\leq \quad \frac{1}{2^3}(\|AX\|_2 + \|XM\|_2)^3 - (\|AX\|_2\|XM\|_2)^{\frac{3}{2}}$$

$$\frac{1}{2^3}\left(\|AX\|_2^2 + \|XM\|_2^2 + \|AX\|_2\|XM\|_2\right)^{\frac{3}{2}} + \left(1 - \frac{(3)^{\frac{3}{2}}}{2^3}\right)(\|AX\|_2\|XM\|_2)^{\frac{3}{2}}$$

$$\leq \frac{1}{2^3}(\|AX\|_2 + \|XM\|_2)^3$$

Now, if $q = n$, $n \in \mathbb{N}$

$$\frac{1}{2^n}\left(\|AX\|_2^2 + \|XM\|_2^2 + \|AX\|_2\|XM\|_2\right)^{\frac{n}{2}} + (1 - \frac{3^{\frac{n}{2}}}{2^n})(\|AX\|_2\|XM\|_2)^{\frac{n}{2}}$$

$$\leq \frac{1}{2^n}(\|AX\|_2 + \|XM\|_2)^n$$

We prove in theorem 11 a matrix version of the inequality 11

Theorem 2.8. *Let $A, M, X \in M_n(\mathbb{C})$ such that A and M are positive semidefinite, $p \in R$, $p \geq 1$. Then*

$$\begin{aligned} & r^p\left(|||AX|||^2 + |||XM|||^2 - 2\,|||AX|||\,|||XM|||\right)^p \\ \leq \quad & \left(v\,|||AX|||^2 + (1-v)\,|||XM|||^2\right)^p - \left(\left|\left|\left|A^v X M^{(1-v)}\right|\right|\right|\right)^{2p}. \end{aligned}$$

Proof. based in inequality 4, and by putting $A = A^2$, $M = M^2$

$$\begin{aligned} & \left(\left|\left|\left|A^{2v} X M^{2(1-v)}\right|\right|\right|\right)^m \\ & \quad + r^m\left(\left|\left|\left|A^2X\right|\right|\right|^m + \left|\left|\left|XM^2\right|\right|\right|^m - 2\left|\left|\left|A^2X\right|\right|\right|^m\left|\left|\left|XM^2\right|\right|\right|^m\right) \\ \leq & \left(\left|\left|\left|A^v X M^{(1-v)}\right|\right|\right|\right)^{2m} \\ & \quad + r^m\left(|||AX|||^{2m} + |||XM|||^{2m} - 2\,|||AX|||^{2m}\,|||XM|||^{2m}\right) \\ \leq & \left(v\,|||AX|||^2 + (1-v)\,|||XM|||^2\right)^m. \end{aligned}$$

by lemma 2.6, and by inequality 11. Let $m = 1$.Then

$$\begin{aligned} & r\left(|||AX|||^2 + |||XM|||^2 - 2\,|||AX|||\,|||XM|||\right) \\ \leq \quad & \left(v\,|||AX|||^2 + (1-v)\,|||XM|||^2\right) - \left(\left|\left|\left|A^v X M^{(1-v)}\right|\right|\right|\right)^2. \end{aligned}$$

Let $\Phi : [0, \infty) \to \mathbb{R}$ be a strictly increasing convex function, and applying lemma 1.1 on the previous inequality. Then we have

$$\begin{aligned} & \Phi\left(r\,|||AX|||^2 + |||XM|||^2)\right) - \Phi\left(2r\,|||AX|||\,|||XM|||\right) \\ \leq \quad & \Phi\left(\left(v\,|||AX|||^2 + (1-v)\,|||XM|||^2\right)\right) - \Phi\left(\left(\left|\left|\left|A^v X M^{(1-v)}\right|\right|\right|\right)^2\right). \end{aligned}$$

In particular, when $\Phi(x) = x^p$ ($p \in R$, $p \geq 1$), we have

$$\begin{aligned} & r^p \left(|||AX|||^2 + |||XM|||^{2p} - 2\,|||AX|||\,|||XM||| \right)^p \\ \leq \quad & \left(v\,|||AX|||^2 + (1-v)\,|||XM|||^2 \right)^p - \left(\left|\left|\left| A^v X M^{(1-v)} \right|\right|\right| \right)^{2p}. \end{aligned}$$

□

3. Conclusions

We have introduced other forms of improvement to Young inequalities based on the same manner of evidence in the research [7]. we prove Theorem 2.1 using inequality 2.7 presented in the paper [1], and lemma 1.1. By same way we prove Theorem 2.2. Corollary 2.3 it is new improvement based on theorem 2.2. Theorem 2.4 is too another improvement of Young inequality we prove with corollary 2.3. After that, we use these inequalities that we derive to get new improvement of young inequalities for matrices which we explained in theory 2.5, theory 2.7, and theory 2.8. The use of these improvements may inspire researchers to look for new improvements and reflections of Young and Heinz inequalities for matrices.

Conflicts of interest : We declare that there are no conflicts of interest to disclose.

Data availability : The data and information utilized in this study were gathered from various sources, including books, research articles, and online resources. Due to the diverse and decentralized nature of these sources, specific links or publicly archived datasets cannot be provided. It is important to note that the study relied solely on publicly accessible information and did not involve the use of proprietary or confidential data. We recognize the limitations in offering direct access to the data. Researchers interested in replicating or accessing the data are encouraged to refer to the same sources used in this study for data retrieval.

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