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FURTHER STUDY OF RINGS IN WHICH ESSENTIAL MAXIMAL RIGHT IDEALS ARE GP-INJECTIVE†

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Abstract. In this paper, rings in which essential maximal right ideals are GP-injective are studied. Whether the rings with this condition satisfy von Neumann regularity is the goal of this study. The obtained research results are twofold:

First, it was shown that this regularity holds even when the reduced ring is replaced with π-IFP and NI-ring. Second, it was shown that this regularity also holds even when the maximal right ideal is changed to GW-ideal.

This can be interpreted as an extension of the existing results.

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1. Introduction

In all parts of this paper, R denotes an associative ring with identity, and all modules are unitary. A ring R is called right principally injective (p-injective) if every R-homomorphism from a principal right ideal to R is left multiplication by an element of R. A right R-module M is called right generalized principally injective (briefly right GP-injective) if, for any $0 \neq a \in R$, there exists a positive integer n such that and any right R-homomorphism of into M extends to one of R into M. Clearly, right p-injective modules are right GP-injective, but the converse is not true by [6]. Recall that a ring R is called reduced if R has no non-zero nilpotent elements. Due to Bell [3], a right (or left) ideal I of a ring R is said to have the insertion-of-factors-property (simply, IFP) if $ab \in I$ implies $aRb \in I$ for $a, b \in R$. Also we shall call a ring R an IFP ring if the zero ideal of R has the IFP. R is (von Neumann) regular if for every $a \in R$, there exists some $b \in R$ such that $a = aba$. R is strongly regular if for every $a \in R$, there

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exists some $b \in R$ such that $a = a^2b$. It is well-known that a ring R is strongly regular if and only if R is a reduced regular ring. Recall that a ring R is called π-regular if for every $x \in R$, there exists a positive integer n, depending on x, such that $x^n = x^n y x^n$ for some $y \in R$. Von Neumann regularity of rings whose maximal right ideals are GP-injective has studied in [5, 11, 17, 22, 23, etc.]. Chen and Ding [5] proved that a ring R is von Neumann regular if and only if every proper principal right ideal of R is GP-injective if and only if every essential right ideal of R is GP-injective. Subedi and Buhphang [17] proved that R is a strongly regular ring if and only if R is reduced and every essential maximal right(left) ideals are GP-injective if and only if R is left quasi-duo and every essential maximal right ideal is GP-injective. Recently, Jeong and Kim[11] proved that the following statements are equivalent:

(1) R is strongly regular;

(2) R is a 2-primal rings whose essential maximal right ideals are GP-injective;

(3) R is a right (or left) quasi-duo rings whose essential maximal right ideals are GP-injective.

In this study, more advanced results were derived by using the new characteristic of von Neumann regularity. Concretely, we prove the following details: Let R be a ring in which essential maximal right ideals are GP-injective; The following statements are equivalent;

(1) R is strongly regular;

(2) R is an NI ring ;

(3) R is a π-IFP ring;

(4) R is a semi-IFP I-rings ;

(5) maximal right ideals are GW-ideals.

2. Further study of rings in which essential maximal right ideals are GP-injective

In this paper, we consider rings in which essential maximal right ideals are GP-injective. For any nonempty subset S of R, $r(S)$ and $l(S)$ denote the right annihilator and the left annihilator of $S \in R$, respectively.

We begin with the following notations and definitions.

Notation.

(1) $P(R)$: the prime radical

(2) $J(R)$: the Jacobson radical

(3) $N*(R)$: the upper nilradical

(4) $N(R)$: the set of all nilpotent elements of R.

Note that $P(R) \subseteq N*(R) \subseteq N(R) \subseteq J(R)$.

Definition 2.1. (1) A ring R is called right(resp. left) quasi-duo[21] if every maximal right(resp. left) ideal of R is two-sided ideal.

(2) A ring R is called weakly right(resp. left) duo[20] if for any $a \in R$, there exists a positive integer n such that $a^nR(resp.\ Ra^n)$ is two-sided ideal.

(3) A ring R is called reduced if $N(R) = 0$.

(4) A ring R is called IFP if $ab = 0$ implies $aRb = 0$ for any $a, b \in R$.

(5) A ring R is called 2-primal [4] if $P(R) = N(R)$.

(6) A ring R is called NI [13] if $N^*(R) = N(R)$.

(7) A ring R (possibly without identity) is called π-IFP[7] if $x^m R y^n = 0$ for some positive integers m, n whenever $xy = 0$ for $x,\ y \in R$.

(8) A ring R is called semi-IFP [18] if $a^2 = 0$ for $a \in R$, implies $aRa = 0$.

Narbonne [15] called IFP rings semi-commutative. It is easily checked that reduced rings are IFP rings and IFP rings are 2-primal.

Lemma 2.2 (11, Lemma 2)**.** *Let I be a right ideal a ring R and $a \neq 0 \in I$. If I is GP-injective, then there exists a positive integer n such that $a^n \neq 0$ and $a^n = ca^n$ for some $c \in I$.*

Lemma 2.3 (11, Proposition 3)**.** *Suppose that every essential maximal right ideal of R is GP-injective. Then;*

(1) For a two-sided ideal I of R, if R/I is a reduced ring, then R/I is a strongly regular ring.

(2) If R is right (or left) quasi-duo, then it is reduced.

Recall that a ring R is called 2-primal [4] if $P(R) = N(R)$. Due to Marks [13], a ring R is called NI ring if $N^*(R) = N(R)$. Note that R is NI if $N(R)$ forms an ideal if and only if $R/N(R)$ is reduced. It is well-known that 2-primal rings are NI rings. Using Lemma 2.2 and Lemma 2.3, we obtain the following result which extend known results [22, Theorem 5.1 and Proposition 7], [17, Theorem 2.5] and [11, Theorem 6].

Theorem 2.4. *The following statements are equivalent;*

(1) R is a strongly regular ring.

(2) R is an NI rings in which essential maximal right ideals are GP-injective.

(3) R is an NI rings in which essential maximal left ideals are GP-injective.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clearly valid.

(2) $\Rightarrow$ (1) : Assume that R is NI. Then $R/N(R)$ is reduced ring. By Lemma 2.3, $R/N(R)$ is strongly regular ring. Thus $J(R/N(R)) = 0$, we get $J(R) \subseteq N(R)$, entailing $J(R) = N(R)$. Hence $R/J(R)$ is strongly regular ring. Suppose that $J(R) \neq 0$. Then there exists $0 \neq b \in J(R)$ such that $b^2 = 0$. We claim that $J(R) + l(b) = R$ for any $b \in J(R)$. If not, then there exists $b \in J(R)$ such that $J(R) + l(b) \neq R$. There exists a maximal right ideal K such that $J(R) + l(b) \subseteq K$. First observe that K is an essential right ideal of R. If not, then K is a direct summand of R. So we can write $K = r(e)$ for some $0 \neq e = e^2 \in R$. Since $b \in K$, $eb = 0$, and $e \in l(b) \subseteq K = r(e)$; whence $e = 0$. It is a contradiction. Thus M is right essential in R. Hence it is GP-injective and $b^2 = 0$. By Lemma 2.2, there exists $c \in M$ such that $b = cb$; whence $(1-c)b = 0$ and $1 - c \in l(b) \subseteq K$; which is a contradiction. Therefore, $J(R) = 0$, and so R is a strongly regular ring.

(3) $\Rightarrow$ (1) ; Similarly we can prove (2) $\Rightarrow$ (1) □

Corollary 2.5 (11, Theorem 6)**.** *For a ring R, The following statements are equivalent;*

(1) R is a strongly regular ring.

(2) R is a reduced rings whose essential maximal right ideals are GP-injective.

(3) R is a IFP rings whose essential maximal right ideals are GP-injective.

(4) R is a 2-primal rings whose essential maximal right ideals are GP-injective.

Recall that a ring R is called nil semi-commutative[14] if for any $a, b \in N(R)$, $ab = 0$ implies $aRb = 0$. A ring R is called central IFP[1] if $a, b \in R$, $ab = 0$ implies $aRb \in C(R)$. It is proved that R is a nil semi-commutative(or central IFP) ring, then R is 2-primal by [14, Lemma 2.7] and [1, Theorem 2.8]. A left ideal L of R is called an N-ideal if for every $b \in N(R) \cap L$, implies $bR \subseteq L$. A ring R is NZI[19] if for any $a \in R$, $l(a)$ is an N-ideal of R. Wei proved that IFP rings are NZI and NZI rings are NI [19, Corollary 2.3].

Corollary 2.6. *The following statements are equivalent;*

(1) R is a strongly regular ring.

(2) R is a nil semi-commutative rings whose essential maximal right(or left) ideals are GP-injective.

(3) R is a central IFP rings whose essential maximal right(or left) ideals are GP-injective.

(4) R is a NZI rings whose essential maximal right(or left) ideals are GP-injective.

Recall that a ring R (possibly without identity) is called π-IFP[7] if $x^m R y^n = 0$ for some positive integers m, n whenever $xy = 0$ for x, $y \in R$. A ring R is an abelian if each idempotent is central. It is proved that any π-IFP ring is abelian by [7, Lemma 1.8(1)].

In the following we get the same result as theorem 2.4 with the π-IFP ring in place of the NI ring. Clearly, strongly regular rings are reduced (hence π-IFP).

Theorem 2.7. *The following statements are equivalent ;*

(1) R is a strongly regular ring.

(2) R is a π-IFP rings in which essential maximal left ideals are GP-injective.

(3) R is a π-IFP rings in which essential maximal right ideals are GP-injective.

Proof. Clearly (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

(2) $\Rightarrow$ (1); Let $0 \neq b \in R$ such that $b^2 = 0$. We claim that $Rb + r(bR) = R$. If not, there exists a maximal left ideal K such that $Rb + r(bR) \subseteq K$. First observe that K is an essential left ideal of R. If not, then K is a direct summand of R. So we can write $K = l(e)$ for some $0 \neq e = e^2 \in R$. Since $b \in K = l(e)$, $be = 0$. By hypothesis R is a π-IFP ring and $b^2 = 0$, there exists a positive integer n such that $bRe^n = 0$. Thus $bRe = 0$ and $e \in r(bR) \subseteq K = l(e)$; whence $e = 0$. It is a contradiction. Therefore K is an essential left ideal of R. Thus K is an essential maximal left ideal. By hypothesis K is GP-injective, there exists $c \in K$ such that $b = bc$ by Lemma 2.2. Thus, $b(1 - c) = 0$. By hypothesis, R is a π-IFP ring

and $b^2 = 0$, there exists a positive integer n such that $bR(1-c)^n = 0$. Thus, $(1-c)^n \subseteq r(bR) \subseteq K$. It is a contradiction. Hence, $Rb + r(bR) = R$. Thus, there exist $r \in R$ and $y \in r(bR)$ such that $rb + y = 1$. Therefore $brb = b$ and $b(rb-1) = 0$. Since R is a π-IFP ring and $b^2 = 0$, there exists a positive integer n such that $bR(rb-1)^n = 0$. Thus $br(rb-1)^n = 0$. In case $n = 2k$ for any positive integer k, $br(rb-1)^n = br(1-rb) = 0$, since $brb = b$ and so $br = brrb$. In case $n = 2k-1$ for any positive integer k, $br(rb-1)^n = br(rb-1) = 0$, since $brb = b$ and so $br = brrb$. Thus $b = brb = (brrb)b = 0$. It is a contradiction. Hence b = 0 and so R is reduced. Therefore R is a strongly regular by Corollary 2.5.

(3) $\Rightarrow$ (1); Similarly we can prove (2) $\Rightarrow$ (1) □

Köthe [12], a ring is called an I-ring if each non-nil left (right) ideal contains a nonzero idempotent. Algebraic algebra and π-regular rings are I-rings by [10, Proposition 9.4.1]. It is easy to check that Jacobson radicals of I-rings are nil. Recall that a ring R is called semi-IFP[18] if $a^2 = 0$ for $a \in R$, implies $aRa = 0$.

Notation: We write $N_1(R) = \{a \in R : a^2 = 0\}$.

Lemma 2.8. *If R is a semi-IFP ring, then $N_1(R) \subseteq P(R)$.*

Proof. For any $a \in N_1(R)$ such that $a^2 = 0$. Since R is semi-IFP, $aRa = 0 \in P(R)$. Hence $a \in P(R)$. Therefore $N_1(R) \subseteq P(R)$. □

Lemma 2.9 (9, Theorem 2)**.** *Let R be a ring such that $R/J(R)$ is an I-ring and suppose that idempotents lift modulo $J(R)$. If $J(R)$ contains $N_1(R)$, then $R/J(R)$ is reduced ring.*

Proposition 2.10. *If R is a semi-IFP I-ring, then $R/J(R)$ is reduced ring.*

Proof. Since R is semi-IFP, then $N_1(R) \subseteq P(R) \subseteq J(R)$. The Jacobson radical $J(R)$ of the I-ring R is a nil ideal and the ring $R/J(R)$ also is an I-ring. By lemma 2.9, $R/J(R)$ is reduced ring. □

Lemma 2.11 (11, Theorem 7)**.** . *For a ring R, The following statements are equivalent; (1) R is a strongly regular ring. (2) R is a weakly right duo rings whose essential maximal right ideals are GP-injective. (3) R is a right quasi-duo rings whose essential maximal left ideals are GP-injective.*

By the product of proposition 2.10 and lemma 2.3, we have the following results.

Theorem 2.12. *The following statements are equivalent;*

(1) R is a strongly regular ring.

(2) R is a semi-IFP I-rings in which essential maximal right ideals are GP-injective.

Proof. Clearly (1) $\Rightarrow$ (2). (2) $\Rightarrow$ (1); Assume that R is a semi-IFP I-ring. By Proposition 2.10, $R/J(R)$ is reduced. Also by lemma 2.3, $R/J(R)$ is strongly regular ring. Thus $R/J(R)$ is right quasi-duo, hence R is right quasi-duo. By lemma 2.11, R is a strongly regular ring. □

Since π-regular rings are I-ring, the following corollary follows.

Corollary 2.13. *The following statements are equivalent;*
(1) R is a strongly regular ring.
(2) R is a semi-IFP π-regular rings in which essential maximal right ideals are GP-injective.

Following [24], a left ideal L of a ring R is called a weakly ideal (simply, W-ideal) if for any $0 \neq a \in L$ there exists a positive integer n such that $a^n \neq 0$ and $a^n R \subseteq L$. A right ideal K of a ring R is defined similarly to be a weakly ideal. A left ideal L of a ring R is a generalized weak ideal (GW-ideal) if for all $a \in L$, there exists a positive integer n such that $a^n R \subseteq L$. A right ideal K of R is defined similarly to be a GW-ideal. Clearly, W-ideals are GW-ideals.

Lemma 2.14 (16, Lemma 2.1)**.** *Let R be a ring whose maximal right(or left) ideals are GW-ideals, then $R/J(R)$ is reduced.*

The following result is the extension of [11, Theorem 7] and [17, Theorem 2.15].

Theorem 2.15. *The following statements are equivalent;*
(1) R is a strongly regular ring.
(2) R is a ring in which maximal right(or left) ideals are W-ideals and essential maximal right ideals are GP-injective.
(3) R is a ring in which maximal right(or left) ideals are GW-ideals and essential maximal right ideals are GP-injective.

Proof. It is obtained in a similar way to theorem 2.4. □

Corollary 2.16. *The following statements are equivalent;*
(1) R is a strongly regular ring.
(2) R is a right(or left) quasi-duo rings in which essential maximal right ideals are GP-injective.

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