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RATIONAL PERIOD FUNCTIONS FOR $\Gamma_0^+(3)$ WITH POLES ONLY AT 0

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Abstract. We characterize a rational period function $q(z)$ for $\Gamma_0^+(3)$ which has a pole only at 0.

1. Introduction and statement of results

Let k be an integer and $p \in \{1, 2, 3\}$. For any meromorphic function f on the complex upper half plane $\mathbb{H}$, the usual slash operator is defined by

$$(f|_k\gamma)(z) := (cz + d)^{-k} f(\gamma z) \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

Let $\Gamma_0^+(p)$ be the group generated by the congruence group $\Gamma_0(p)$ and the Fricke involution $W_p := \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$. Let $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $U := TW_p$.

A rational period function $q(z)$ of weight $2k$ for $\Gamma_0^+(p)$ is a rational function satisfying

$$q|_{2k}W_p + q = 0 \tag{1.1}$$

and

$$q|_{2k}U^{n_p-1} + q|_{2k}U^{n_p-2} + \cdots + q|_{2k}U + q = 0, \tag{1.2}$$

where $n_p = \begin{cases} 3, & \text{if } p = 1 \\ 2p & \text{if } p = 2, 3. \end{cases}$

The notion of rational period functions was initiated by Knopp through modular integrals (see [6, 7]). Knopp [7] investigated the location of poles of any rational period functions $q(z)$ for $\Gamma_0^+(1) = SL_2(\mathbb{Z})$ and proved that when $q(z)$ has a pole, $q(z)$ has poles only at 0 or at real quadratic irrationalities. For $p = 2, 3$, the author and Kim [2] proved that when $q(z)$ has a pole, $q(z)$ has poles only at 0 or at real quadratic

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irrationalities. We refer the reader to [3, 4, 5, 6, 7, 9] and [2, 8] for the results related to rational period functions for $\Gamma_0^+(1)$ and $\Gamma_0^+(p)$ $(p = 2, 3)$, respectively.

Knopp [7] found the exact forms of rational period functions for $\Gamma_0^+(1)$ with poles only at 0. Extending the result of Knopp, Oh [8] found the exact forms of rational period functions for $\Gamma_0^+(2)$ with poles only at 0. In the same paper, Oh also remarked the exact forms of rational period functions for $\Gamma_0^+(3)$ with poles only at 0 without proof. In this paper, modifying the proof of Theorem 1.2 in [8] we prove that Oh's assertion is true.

THEOREM 1.1. *Let $q(z)$ be any rational period function of weight $2k$ for $\Gamma_0^+(3)$. If $q(z)$ has poles only at 0, then*

$$q(z) = \begin{cases} c_1(1-(\sqrt{3}z)^{-2k}), & \text{if } k > 1 \\ c_1(1-(\sqrt{3}z)^{-2}) + c_2 z^{-1}, & \text{if } k = 1 \\ c_1(3^{k-1}z^{-1} + z^{-2k+1}) + p_k(z), & \text{if } k \le 0, \end{cases} \tag{1.3}$$

where c_1, c_2 are complex numbers and p_k is a polynomial in z of degree at most $-2k$.

This paper is organized as follows. In Section 2, we prove Theorem 1.1.

2. Proof of Theorem 1.1

Proof of Theorem 1.1:
Case 1 : $k > 0$.
We first consider

$$q(z) = a_l z^{-l} + \cdots + a_1 z^{-1} + b_0 + b_1 z + \cdots + b_m z^m \ (a_l \neq 0, b_m \neq 0), \tag{2.1}$$

with $l \ge 1, m \ge 0$. Applying (1.1) to $q(z)$, we have

$$\begin{aligned} -q(z) &= (\sqrt{3}z)^{-2k} q(\frac{-1}{3z}) \\ &= (-1)^l a_l 3^{l-k} z^{l-2k} + \cdots + (-1)a_1 3^{1-k} z^{1-2k} \\ &+ b_0 3^{-k} z^{-2k} + b_1(-1)3^{-1-k} z^{-1-2k} + \cdots \\ &+ b_m(-1)^m 3^{-m-k} z^{-m-2k}. \end{aligned} \tag{2.2}$$

Comparing the lowest term in (2.2), we have $l = m + 2k$. Note that $U = \begin{pmatrix} \sqrt{3} & -1/\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$, $U^2 = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$, $U^3 = \begin{pmatrix} \sqrt{3} & -2/\sqrt{3} \\ 2\sqrt{3} & -\sqrt{3} \end{pmatrix}$ $U^4 = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$, $U^5 =$

$\begin{pmatrix} 0 & -1/\sqrt{3} \\ \sqrt{3} & -\sqrt{3} \end{pmatrix}$ and $U^5T = W_3$. From (1.2), we have

$$\begin{aligned} 0 &= (q|_{2k}U^5 + q|_{2k}U^4 + \cdots + q|_{2k}U + q)|_{2k}T \\ &= q|_{2k}W_3 + (3z+1)^{-2k}q\Big(\frac{z}{3z+1}\Big) + (2\sqrt{3}z+\sqrt{3})^{-2k}q\Big(\frac{\sqrt{3}z+\frac{1}{\sqrt{3}}}{2\sqrt{3}z+\sqrt{3}}\Big) \\ &+ (3z+2)^{-2k}q\Big(\frac{2z+1}{3z+2}\Big) + (\sqrt{3}z+\sqrt{3})^{-2k}q\Big(\frac{\sqrt{3}z+\frac{2}{\sqrt{3}}}{\sqrt{3}z+\sqrt{3}}\Big) + q(z+1), \end{aligned}$$

which gives from (1.1)

$$\begin{aligned} q(z) &= (3z+1)^{-2k}q\Big(\frac{z}{3z+1}\Big) + (2\sqrt{3}z+\sqrt{3})^{-2k}q\Big(\frac{\sqrt{3}z+\frac{1}{\sqrt{3}}}{2\sqrt{3}z+\sqrt{3}}\Big) \\ &+ (3z+2)^{-2k}q\Big(\frac{2z+1}{3z+2}\Big) + (\sqrt{3}z+\sqrt{3})^{-2k}q\Big(\frac{\sqrt{3}z+\frac{2}{\sqrt{3}}}{\sqrt{3}z+\sqrt{3}}\Big) \\ &+ q(z+1) \\ &= \sum_{j=0}^{l-1}\frac{a_{l-j}(3z+1)^{l-2k-j}}{z^{l-j}} + \sum_{i=0}^{m}\frac{b_i z^i}{(3z+1)^{2k+i}} \\ &+ \sum_{j=0}^{l-1}\frac{a_{l-j}3^{-k}(2z+1)^{l-2k-j}}{(z+1/3)^{l-j}} + \sum_{i=0}^{m}\frac{b_i 3^{-k}(z+1/3)^i}{(2z+1)^{2k+i}} \\ &+ \sum_{j=0}^{l-1}\frac{a_{l-j}(3z+2)^{l-2k-j}}{(2z+1)^{l-j}} + \sum_{i=0}^{m}\frac{b_i(2z+1)^i}{(3z+2)^{2k+i}} \\ &+ \sum_{j=0}^{l-1}\frac{a_{l-j}3^{-k}(z+1)^{l-2k-j}}{(z+2/3)^{l-j}} + \sum_{i=0}^{m}\frac{b_i 3^{-k}(z+2/3)^i}{(z+1)^{2k+i}} \\ &+ \sum_{j=0}^{l-1}a_{l-j}(z+1)^{j-l} + \sum_{i=0}^{m}b_i(z+1)^i. \end{aligned} \tag{2.3}$$

Since $l-2k=m$ and $l>m$, comparing the principal part at ∞ in (2.1) and (2.3), we get

$$b_0 + b_1 z + \cdots + b_m z^m = b_0 + b_1(z+1) + \cdots + b_m(z+1)^m,$$

which gives $m=0$, hence

$$l = 2k \text{ and } q(z) = a_l z^{-l} + \cdots + a_1 z^{-1} + b_0. \tag{2.4}$$

By applying (1.1) to (2.4) and comparing the coefficients, we have
(2.5)

$$b_0 = -\sqrt{3}^l a_l, \quad a_{l-j}(-1)^{l-j}\sqrt{3}^{l-2j} = -a_j \quad \text{for } 1 \le j \le l-1 = 2k-1.$$

In particular, $a_k = (-1)^{k+1}a_k$, so $a_k = 0$ if k is even.

Applying (1.2) to (2.4) leads to

$$\begin{aligned}
& \sum_{j=0}^{2k-1} \frac{a_{2k-j}(-\sqrt{3})^{2k-j}}{(\sqrt{3}z-\sqrt{3})^j} + b_0(\sqrt{3}z-\sqrt{3})^{-2k} \\
+ & \sum_{j=0}^{2k-1} \frac{a_{2k-j}}{(3z-2)^j(z-1)^{2k-j}} + b_0(3z-2)^{-2k} \\
+ & \sum_{j=0}^{2k-1} \frac{a_{2k-j}}{(2\sqrt{3}z-\sqrt{3})^j(\sqrt{3}z-2/\sqrt{3})^{2k-j}} + b_0(2\sqrt{3}z-\sqrt{3})^{-2k} \\
+ & \sum_{j=0}^{2k-1} \frac{a_{2k-j}}{(3z-1)^j(2z-1)^{2k-j}} + b_0(3z-1)^{-2k} \\
+ & \sum_{j=0}^{2k-1} \frac{a_{2k-j}}{(\sqrt{3}z)^j(\sqrt{3}z-1/\sqrt{3})^{2k-j}} + b_0(\sqrt{3}z)^{-2k} \\
(2.6) \quad + & \sum_{j=0}^{2k-1} a_{2k-j}z^{j-2k} + b_0 = 0.
\end{aligned}$$

If $k = 1$, then $q(z) = a_2z^{-2} + a_1z^{-1} + b_0$. It follows from (2.5) that $b_0 = -\sqrt{3}^2 a_2$. Hence

$$q(z) = b_0(1 - (\sqrt{3}z)^{-2}) + a_1z^{-1}.$$

Suppose that $k \ge 2$.

Note that from the partial fraction expansion, we have

$$\frac{1}{z^N(z-\frac{1}{3})^M} = \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_N}{z^N} + \frac{B_1}{z-\frac{1}{3}} + \frac{B_2}{(z-\frac{1}{3})^2} + \cdots + \frac{B_M}{(z-\frac{1}{3})^M},$$

where $A_{N-j} = 3^{M+j}(-1)^M\binom{M+j-1}{M-1}$ $(0 \le j \le N-1)$ and $B_{M-j} = 3^{N+j}(-1)^j\binom{N+j-1}{N-1}$ $(0 \le j \le M-1)$.

By applying the partial fraction expansion to $\frac{a_1}{(\sqrt{3}z)^{2k-1}(\sqrt{3}z-1/\sqrt{3})}$ and $\frac{a_2}{(\sqrt{3}z)^{2k-2}(\sqrt{3}z-1/\sqrt{3})^2}$ on the left hand side of (2.6), we obtain that the coefficient of z^{-2k+2} is $-3^{2-k}a_1 + 3^{-k+2}a_2 + a_{2k-2}$ so that

$$a_{2k-2} + 3^{2-k}a_2 - 3^{2-k}a_1 = 0. \tag{2.7}$$

By (2.5), we have $a_{2k-2} + 3^{-k+2}a_2 = 0$ and so $a_1 = 0$.

If $k = 2$, then $a_2 = 0, a_3 = 0$ and $b_0 = -\sqrt{3}^4 a_4$ by (2.5) and (2.7). Therefore

$$q(z) = b_0(1 - (\sqrt{3}z)^{-4}).$$

Suppose that $k \geq 3$.

We now assume $a_1 = a_2 = \cdots = a_{i-1} = 0$ for $2 \leq i \leq k-1$. By applying the partial fraction expansion on the left hand side of (2.6), we obtain that the coefficient of $z^{-2k+i+1}$ is $3^{-k+i+1}(-1)^{i+1}a_{i+1} + i3^{-k+i+1}(-1)^i a_i + a_{2k-i-1}$ so that

$$0 = a_{2k-i-1} + (-1)^{i+1}3^{-k+i+1}a_{i+1} + i(-1)^i 3^{-k+i+1}a_i.$$

By (2.5), $a_{2k-i-1} + a_{i+1}(-1)^{i+1}3^{-k+i+1} = 0$, which gives $a_i = 0$. Consequently, we have $a_1 = a_2 = \cdots = a_{k-1} = 0$. Note that $a_k = 0$ if k is even.

Therefore, for $k \geq 3$, $q(z)$ has the form

$$q(z) = \begin{cases} b_0(1 - (\sqrt{3}z)^{-2k}), & \text{if } k \text{ is even} \\ b_0(1 - (\sqrt{3}z)^{-2k}) + a_k z^{-k}, & \text{if } k \text{ is odd.} \end{cases}$$

Since $b_0(1-(\sqrt{3}z)^{-2k})$ satisfies (1.1) and (1.2), $q(z) = b_0(1-(\sqrt{3}z)^{-2k}) + a_k z^{-k}$ satisfies (1.1) and (1.2) if and only if $a_k z^{-k}$ satisfies (1.1) and (1.2). Note that z^{-k} satisfies (1.1) only when k is odd. We now show that for odd k, z^{-k} satisfies (1.1) and (1.2) if and only if $k = 1$. For $q(z) = z^{-k}$, the functional equation (1.2) says

$$\begin{aligned} &\frac{-1}{(z-1)^k} + \frac{1}{(3z-2)^k(z-1)^k} + \frac{1}{(2z-1)^k(3z-2)^k} \\ &+ \frac{1}{(3z-1)^k(2z-1)^k} + \frac{1}{z^k(3z-1)^k} + \frac{1}{z^k} = 0 \end{aligned} \tag{2.8}$$

and this is 0 only when $k = 1$. Indeed, we have

$$\frac{1}{(3z-2)^k(z-1)^k} = \sum_{j=1}^{k} \frac{A_j}{(z-1)^j} + \sum_{j=1}^{k} \frac{B_j}{(3z-2)^j} \qquad \text{with } A_j B_j \neq 0,$$

which gives that (2.8) is satisfied only when $k = 1$.

We now consider

$$q(z) = a_l z^{-l} + \cdots + a_1 z^{-1}.$$

Note $q_1(z) := q(z) + 1 - (\sqrt{3}z)^{-2k}$ is a rational period function with poles only at 0. By the same proof in the above, $q_1(z)$ have the form

$$q_1(z) = \begin{cases} 1 - (\sqrt{3}z)^{-2k}, & \text{if } k > 1 \\ 1 - (\sqrt{3}z)^{-2} + a_1 z^{-1}, & \text{if } k = 1 \end{cases},$$

which says

$$q(z) = \begin{cases} 0, & \text{if } k > 1 \\ a_1 z^{-1}, & \text{if } k = 1 \end{cases},$$

Case 2: $k \le 0$.
Let D be the differential operator defined by $Df(z) = \frac{1}{2\pi i}\frac{df}{dz}$. By applying Bol's identity [1] to (1.1) and (1.2), we have

$$\begin{aligned} 0 &= D^{-2k+1}(q|_{2k}W_3(z) + q(z)) = (D^{-2k+1}q)|_{2-2k}W_3(z) + D^{-2k+1}q(z), \\ 0 &= D^{-2k+1}(q|_{2k}U^5(z) + q|_{2k}U^4(z) + \cdots + q|_{2k}U(z) + q(z)) \\ &= (D^{-2k+1}q)|_{2-2k}U^5(z) + (D^{-2k+1}q)|_{2-2k}U^4(z) + \cdots \\ &+ (D^{-2k+1}q)|_{2-2k}U(z) + D^{-2k+1}q(z), \end{aligned}$$

which mean that $q^{(-2k+1)}(z)$ is a rational period function of positive weight $2-2k$. Note that the term $b_1 z^{-1}$ does not occur as the derivative of a rational function. Hence it follows from Case 1 of the proof that we have $q^{(-2k+1)}(z) = b_0(1-(\sqrt{3}z)^{2k-2})$. Integrating $-2k+1$ times, we get

$$q(z) = c(3^{k-1}z^{-1} + z^{-2k+1}) + p_k(z),$$

where c is a complex number and $p_k(z)$ is a polynomial of degree $\le -2k$.

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