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DISCRETE VOLUME OF THE POSET POLYTOPE FOR A VARIANT OF UP-DOWN POSETS

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Abstract. Discrete volumes of poset polytopes for a variant of up-down posets introduced in [4] were studied. We obtained the generating functions for the discrete volumes of poset polytopes for a variant of up-down poset using the characteristic matrices.

1. Introduction

Let r be a positive integer. $[r] := \{1, 2, \cdots, r\}$. For a given bipartite simple graph $G = (V, E)$ with $V = [r]$, the *graph polytope* $P(G)$ is defined as follows:

$$P(G) := \{(x_1, x_2, \cdots, x_r) \in [0, 1]^r \mid ij \in E \text{ implies } x_i + x_j \leq 1\}.$$

A characteristic function $K : [0, 1]^2 \to \mathbb{R}$ (resp. $J : [0, 1]^2 \to \mathbb{R}$) is defined by the following:

$$K(s,t)(\text{resp. } J(s,t)) := \begin{cases} 1, & \text{if } s+t \leq 1(\text{resp. } s+t \geq 1) \\ 0, & \text{elsewhere.} \end{cases}$$

Now, if we let $\phi(x_1, x_2, \cdots, x_r) := \prod_{ij \in E} K(x_i, x_j)$, then it can be seen that $P(G) = \phi^{-1}(1)$. Discrete volume of the polytope P of dimension n is defined as

$$L_P(m) := \#(mP \cap \mathbb{Z}^n).$$

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We call this an *Ehrhart function*. The *Ehrhart series* is an ordinary generating function for the sequence $(L_P(m))_{m\geq 0}$ as follows:

$$Ehr_P(z) = \sum_{m\geq 0} L_P(m) z^m.$$

See [1] for Ehrhart functions and Ehrhart series. Next, we let the characteristic matrix $U(m)$ of the first kind be a matrix of size $m \times m$ with 1 over the anti-diagonal entries or above, and 0 elsewhere. Likewise, we let the characteristic matrix $D(m)$ of the second kind be a matrix of size $m \times m$ with 1 over the anti-diagonal entries or below, and 0 elsewhere. For example,

$$U(4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } D(4) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Their corresponding inverse matrices are as follows:

$$U(4)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \text{ and } D(4)^{-1} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Discrete volumes of graph polytopes are related to the characteristic matrices.

Definition 1.1. For a given square matrix M, we denote $s(M)$ by the sum of all entries of the matrix M. Let u be the column vector all of whose entries are 1. Note that $s(M) = u^t M u$.

Theorem 1.2. *Let L_n be the path with $n(\geq 1)$ vertices. That is, $L_n = ([n], E)$, where $E = \{i(i+1) | i = 1, 2, \ldots, n-1\}$. Then the discrete volume of graph polytope $P(L_n)$ is*

$$L_{P(L_n)}(m) = s(((U(m+1))^{n-1}).$$

Proof.

$$L_{P(L_n)}(m) = \#(mP(L_n) \cap \mathbb{Z}^n) = \#(m\phi^{-1}(1) \cap \mathbb{Z}^n) = \#(\phi^{-1}(1) \cap \frac{1}{m}\mathbb{Z}^n).$$

Let $U(m+1) = (u_{ij}(m))_{0\le i,j\le m}$, where

$$u_{ij}(m) = \begin{cases} 1, & \text{if } i+j \le m \\ 0, & \text{otherwise} \end{cases} .$$

Note that $u_{ij}(m) = K(\frac{i}{m}, \frac{j}{m})$.

$$\begin{aligned} ((U(m+1))^{n-1})_{ij} &= \sum_{0\le i_2,i_3,\cdots,i_{n-1}\le m} u_{ii_2}(m)u_{i_2i_3}(m)\cdots u_{i_{n-1}j}(m) \\ &= \sum_{0\le i_2,i_3,\cdots,i_{n-1}\le m} K(\frac{i}{m},\frac{i_2}{m})K(\frac{i_2}{m},\frac{i_3}{m})\cdots K(\frac{i_{n-1}}{m},\frac{j}{m}) \\ &= \sum_{0\le i_2,i_3,\cdots,i_{n-1}\le m} \phi(\frac{i}{m},\frac{i_2}{m},\frac{i_3}{m},\cdots\frac{i_{n-1}}{m},\frac{j}{m}) \end{aligned}$$

$$s(((U(m+1))^{n-1}) = \sum_{0\le i,j\le m} ((U(m+1))^{n-1})_{ij} = \#(\phi^{-1}(1) \cap \frac{1}{m}\mathbb{Z}^n).$$

□

We introduce poset polytopes, and then derive Ehrhart series and generating functions for chain polytopes in Section 1. In Section 2, we obtain the discrete volume of the poset polytopes for a new poset where the chain and the up-down poset are connected. We also compute and obtain the generating function on the poset polytopes for this new poset. The generating functions are represented in terms of $P_m(x) = det(I + xU(m))$, which we are familiar with and had certain relationship with Chebyshev polynomials. (Refer [3] for the analysis of $P_m(x)(= Q_{m-1}(-x))$'s.) In the final Section we raise some issues for further consideration in the future.

2. Poset Polytopes

Let $S = ([n], \le)$ be a graded poset. Poset polytope of the poset S is defined as following:

$$P(S) = \{(x_1, x_2, \cdots, x_n) \in [0,1]^n | i \le j \implies x_i \le x_j \quad \forall i,j \in [n]\}.$$

One of the obvious posets is chains(totally ordered sets) which are given by

$$C_n = \{[n] | 1 \leq 2 \leq \cdots \leq n\}.$$

Therefore, the poset polytope $P(C_n)$ corresponding to the poset C_n is:

(CP) $\quad P(C_n) = \{(x_1, x_2, \cdots, x_n) \in [0,1]^n | 0 \leq x_1 \leq x_2 \leq \cdots x_n \leq 1\}$

Another kind of posets we are interested in is the up-down poset Z_n given by:

$$Z_n = \{[n] | 1 \leq 2 \geq 3 \leq \cdots \geq n \text{ if } n \text{ is odd } (\leq n \text{ if } n \text{ is even})\}.$$

Similar to the previous case, the corresponding poset polytope $P(Z_n)$ can be defined for this up-down poset. All poset polytopes, like graph polytopes, are subset of $n-$dimensional unit hypercube. Note that every simple bipartite graph can be regarded as a graded poset of rank 1. Now, we compute the discrete volume of $P(C_n)$. In order to compute the discrete volume of poset polytope for the chain C_n of length n represented as in the equation (CP), *we need to do the change of variables to use the idea of graph polytope as below: We let*

$$\tau_i = \begin{cases} 1 - x_i, & \text{if } i \text{ is even,} \\ x_i, & \text{otherwise} \end{cases} .$$

Then the successive inequalities turn into the following:

$$P(C_n) = \{(\tau_1, \tau_2, \cdots, \tau_n) \in [0,1]^n \mid 0 \leq \tau_1 \leq 1 - \tau_2 \leq \cdots \leq \tau_n$$
$$\text{if } n \text{ is odd (or }, 1 - \tau_n \text{ if } n \text{ is even)} \leq 1\}.$$

Note that with this change of variables we have the following :

$$P(C_n) = \psi^{-1}(1), \text{ where } \psi(\tau_1, \tau_2, \cdots, \tau_n) = K(\tau_1, \tau_2) J(\tau_2, \tau_3) K(\tau_3, \tau_4) \cdots .$$

The Ehrhart function for this polytope is as follows.

THEOREM 2.1. *Ehrhart function for the poset polytope $P(C_n)$ is given as following:*

$$L_{P(C_{2k+1})}(m) = \#(mP(C_{2k+1}) \cap \mathbb{Z}^{2k+1}) = s[(U(m+1)D(m+1))^k] = \binom{2k+1+m}{m},$$

$$L_{P(C_{2k})}(m) = \#(mP(C_{2k}) \cap \mathbb{Z}^{2k}) = s[(U(m+1)D(m+1))^{k-1}U(m+1)] = \binom{2k+m}{m}.$$

That is,

$$L_{P(C_n)}(m) = \#(mP(C_n) \cap \mathbb{Z}^n) = \binom{n+m}{m},$$

and

$$Ehr_{P(C_n)}(z) = \frac{1}{(1-z)^{n+1}}$$

Proof. We prove the case $n = 2k$. The other case can be proved similarly.

$$U(m+1)D(m+1) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \cdots & m & m+1 \\ 0 & 1 & \cdots & m-1 & m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 3 \\ 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$= I + 2N + 3N^2 + \cdots + (m+1)N^m = (I-N)^{-2},$$

where the nilpotent matrix $N = (a_{ij}), a_{i,i+1} = 1,$ and 0 elsewhere. Using the same method as in the proof of Theorem 1, it can be shown that the following formula holds.

$$L_{P(C_{2k})}(m) = \#(mP(C_{2k}) \cap \mathbb{Z}^{2k}) = s[(U(m+1)D(m+1))^{k-1}U(m+1)]$$
$$= s((I-N)^{-2(k-1)}U(m+1)).$$

$$(I-N)^{-2(k-1)} = \sum_{0\le i\le m} \binom{-2k+2}{i} N^i = \sum_{0\le i\le m} \binom{2k-3+i}{i} N^i$$
$$= \begin{pmatrix} \binom{2k-3}{0} & \binom{2k-2}{1} & \cdots & \binom{2k+m-4}{m-1} & \binom{2k+m-3}{m} \\ 0 & \binom{2k-3}{0} & \cdots & \binom{2k+m-5}{m-2} & \binom{2k+m-4}{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{2k-2}{1} & \binom{2k-1}{2} \\ 0 & 0 & \cdots & \binom{2k-3}{0} & \binom{2k-2}{1} \\ 0 & 0 & \cdots & 0 & \binom{2k-3}{0} \end{pmatrix}.$$

$$(I-N)^{-2(k-1)}U(m+1)$$
$$= \begin{pmatrix} \binom{2k-3}{0} & \binom{2k-2}{1} & \cdots & \binom{2k+m-4}{m-1} & \binom{2k+m-3}{m} \\ 0 & \binom{2k-3}{0} & \cdots & \binom{2k+m-5}{m-2} & \binom{2k+m-4}{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{2k-2}{1} & \binom{2k-1}{2} \\ 0 & 0 & \cdots & \binom{2k-3}{0} & \binom{2k-2}{1} \\ 0 & 0 & \cdots & 0 & \binom{2k-3}{0} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \binom{2k+m-2}{m} & \binom{2k+m-3}{m-1} & \cdots & \binom{2k-1}{1} & \binom{2k-2}{0} \\ \binom{2k+m-3}{m-1} & \binom{2k+m-4}{m-2} & \cdots & \binom{2k-2}{0} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{2k}{2} & \binom{2k-1}{1} & \cdots & 0 & 0 \\ \binom{2k-1}{1} & \binom{2k-2}{0} & \cdots & 0 & 0 \\ \binom{2k-2}{0} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

From the previous expression we obtain the following:

$$L_{P(C_{2k})}(m) = s[(U(m{+}1)D(m{+}1))^{k-1}U(m{+}1)] = \sum_{i=0}^{m} \binom{2k-1+i}{i} = \binom{2k+m}{m}.$$

Thus, the Ehrhart series of the poset polytope $P(C_n)$ is

$$Ehr_{P(C_n)}(z) = \sum_{m\ge 0} L_{P(C_n)}(m) z^m = \frac{1}{(1-z)^{n+1}}.$$

$\square$

EXAMPLE 2.2. *Ehrhart series of $P(C_4)$ is as follows:*

$$Ehr_{P(C_4)}(z) = \sum_{m\geq 0} L_{P(C_4)}(m)z^m = \sum_{m\geq 0}\binom{4+m}{m}z^m = \frac{1}{(1-z)^5}$$
$$= 1 + 5z + 15z^2 + 35z^3 + 70z^4 + \cdots .$$

3. Main Results

We consider the discrete volume of poset polytope for up-down poset. Let $b(k,m) := s(U(m+1)^{k-1})$. (See [2](A050446) or [5] for more details on this bivariate sequence.) For convenience, $b(0,m)$ is defined as 0. We need a notation about the continued fraction. For a given infinite sequence $(a_n)_{n\geq 0}$ we define

$$H_n(a_0,a_1,\cdots,a_n) := \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\; a_{n-1} + \cfrac{1}{a_n+1}}}}}$$

and

$$F_m(x) := H_m((-1)^1x, (-1)^2x, (-1)^3x, \cdots, (-1)^{m+1}x).$$

LEMMA 3.1. *Let $A = (a_{ij})$ be an invertible matrix of size n. Then*

$$s(\mathrm{adj}(A)) = det\begin{pmatrix} 0 & -u^t \\ u & A \end{pmatrix},$$

where u is an $n-$vector all of its entries 1.

Proof. Let α_i be the $i-$th column vector of the adjugate $\mathrm{adj}(A)$ of the matrix A and $|\alpha_i|$ the column sum of the vector α_i.

$$s(A) = u^t\big(\mathrm{adj}(A)\big)u = \sum_{i=1}^{n} |\alpha_i|.$$

Note that

$$|\alpha_i| = \det(\beta_1, \beta_2, \cdots, \beta_{i-1}, u, \beta_{i+1}, \cdots, \beta_n)$$

where β_i is the $i-$th column vector of the matrix A. The cofactor expansion of the given matrix

$$\begin{pmatrix} 0 & -u^t \\ u & A \end{pmatrix}$$

with respect to the first row gives us the value $\sum_{i=1}^{n} |\alpha_i|$. □

The following theorem comes from the reference [3] *and is related to the poset polytope for the up-down poset. We provide its proof for a clear understanding of what follows.*

THEOREM 3.2. *For fixed m, the generating function associated with the discrete volume sequence $L_{P(L_k)}(m)(k = 0, 1, 2, \cdots)$ of poset polytope $P(L_k)$ of up-down poset is an $F_m(x)$. That is,*

$$F_m(x) = 1 + \sum_{k=0}^{\infty} L_{P(L_k)}(m)x^{k+1} = 1 + \sum_{k=0}^{\infty} s(U(m+1)^k)x^{k+1} = \frac{P_m(x)}{Q_m(x)},$$

where the last expression $\frac{P_m(x)}{Q_m(x)}$ is the reduced rational function so that

$$Q_m(x) = det(I - xU(m+1)) \text{ and } P_m(x) = Q_{m-1}(-x).$$

Proof.

$$F_m^*(x) = \sum_{k=0}^{\infty} L_{P(L_k)}(m)x^k = \sum_{k=0}^{\infty} s(U(m+1)^k)x^k$$

$$= s\Big(\sum_{k=0}^{\infty}(xU(m+1))^k\Big) = s\Big((I - xU(m+1))^{-1}\Big) = \frac{s(\mathrm{adj}(I - xU(m+1)))}{\det(I - xU(m+1))},$$

where $\mathrm{adj}(I - xU(m+1))$ is an adjugate of the matrix $I - xU(m+1)$.

By the previous lemma, the following holds:

$$s(\mathrm{adj}(I - xU(m+1))) = \det \begin{pmatrix} 0 & -u^t \\ u & I - xU(m+1) \end{pmatrix}.$$

In order to get the formula for $F_m(x) = 1 + xF_m^*(x)$ we need the following:

$$\begin{aligned}
&x\det \begin{pmatrix} 0 & -u^t \\ u & I - xU(m+1) \end{pmatrix} + \det(I - xU(m+1)) \\
&= \det \begin{pmatrix} 0 & -u^t \\ xu & I - xU(m+1) \end{pmatrix} + \det \begin{pmatrix} 1 & -u^t \\ 0 & I - xU(m+1) \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & -u^t \\ xu & I - xU(m+1) \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & x \\ 0 & 0 & 0 & \cdots & x & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+x & x \\ 0 & 0 & x & \cdots & 0 & 1+x \end{pmatrix} \\
&= \det(I + xD(m-1)) = \det(I + xU(m-1)) = Q_{m-1}(-x) = P_m(x).
\end{aligned}$$

So we get the desired result:

$$F_m(x) = 1 + xF_m^*(x) = \frac{P_m(x)}{Q_m(x)}.$$

$\square$

Example 3.3. *$F_m(x)$ satisfies the first-order nonlinear recurrence relation:*

$$F_{m+1}(x) = \frac{1}{-x + F_m(-x)} \quad \text{with } F_0(x) = 1.$$

We list first several continued fractions $F_m(x)$.

$$\begin{array}{rcc}
F_0(x) = & \dfrac{1}{1-x} = & \dfrac{P_0(x)}{Q_0(x)} \\
F_1(x) = & \dfrac{1+x}{1-x-x^2} = & \dfrac{P_1(x)}{Q_1(x)} \\
F_2(x) = & \dfrac{1+x-x^2}{1-2x-x^2+x^3} = & \dfrac{P_2(x)}{Q_2(x)} \\
F_3(x) = & \dfrac{1+2x-x^2-x^3}{1-2x-3x^2+x^3+x^4} = & \dfrac{P_3(x)}{Q_3(x)} \\
\vdots & \vdots & \vdots
\end{array}$$

The following result is useful to prove the main results.

THEOREM 3.4. *We let $\alpha_i \quad (0 \leq i \leq m)$ be the i-th column vector of the adjoint matrix $adj(I_{m+1} - xU(m+1))$, and $|\alpha_i|$ its column sum. Then the following holds:*

$$|\alpha_i| = \begin{cases} P_{m-2i}(x) & i \leq \lfloor \frac{m}{2} \rfloor \\ P_{2i-1-m}(-x) & i > \lfloor \frac{m}{2} \rfloor \end{cases} \tag{3.1}$$

where $P_0(x) = 1$ and $P_m(x) = det(I_m + xU(m)) = Q_{m-1}(-x)$ for $m \geq 1$.

Proof. We use the mathematical induction on m. For the case $m = 1$:

$$\text{adj}(I_2 - xU(2)) = \begin{pmatrix} 1 & x \\ x & 1-x \end{pmatrix},$$

$$\begin{cases} |\alpha_0| = 1 + x = P_1(x), \\ |\alpha_1| = 1 = P_0(-x). \end{cases}$$

This says that the formula (3.1) works for the case $m = 1$. Now, we assume that the next formula holds for the case $k < m$.

$$|\alpha_i| = \begin{cases} P_{k-2i}(x) & i \leq \lfloor \frac{k}{2} \rfloor \\ P_{2i-1-k}(-x) & i > \lfloor \frac{k}{2} \rfloor \end{cases} \tag{3.2}$$

We show that the formula (3.2) holds for the case $k = m$. $|\alpha_i|$, which is the i-th column sum of $\mathrm{adj}(I_{m+1} - xU(m+1))$, is as following:

$$|\alpha_i| = \det \begin{pmatrix} 1-x & -x & \cdots & 1 & \cdots & -x & -x \\ -x & 1-x & \cdots & 1 & \cdots & -x & 0 \\ -x & -x & \cdots & 1 & \cdots & 0 & 0 \\ -x & -x & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x & -x & \cdots & 1 & \cdots & 1 & 0 \\ -x & 0 & \cdots & 1 & \cdots & 0 & 1 \end{pmatrix}_{(m+1)\times(m+1)}.$$

Note that the $(m+1)-$column vector u is positioned at the i-th column. Add an x times i-th column to all remaining columns leads to the following:

$$\begin{aligned} &\det \begin{pmatrix} 1 & 0 & \cdots & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & \cdots & 0 & x \\ 0 & 0 & \cdots & 1 & \cdots & x & x \\ 0 & 0 & \cdots & 1 & \cdots & x & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 1+x & x \\ 0 & x & \cdots & 1 & \cdots & x & 1+x \end{pmatrix}_{(m+1)\times(m+1)} \\ = \quad &\det \begin{pmatrix} 1 & 0 & \cdots & 1 & \cdots & 0 & x \\ 0 & 1 & \cdots & 1 & \cdots & x & x \\ 0 & 0 & \cdots & 1 & \cdots & x & x \\ 0 & 0 & \cdots & 1 & \cdots & x & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & x & \cdots & 1 & \cdots & 1+x & x \\ x & x & \cdots & 1 & \cdots & x & 1+x \end{pmatrix}_{m\times m} \end{aligned}$$

Now, u is m-column vector positioned at the $(i-1)$th column. Add $(-x)u$ to all the remaining columns. Then we obtain the next:

$$= \det \begin{pmatrix} 1-x & -x & \cdots & 1 & \cdots & -x & 0 \\ -x & 1-x & \cdots & 1 & \cdots & 0 & 0 \\ -x & -x & \cdots & 1 & \cdots & 0 & 0 \\ -x & -x & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x & 0 & \cdots & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 1 \end{pmatrix}_{m\times m}$$

By cofactor expansion along the last column, we get the following:

$$= \det \begin{pmatrix} 1-x & -x & \cdots & 1 & \cdots & -x & -x \\ -x & 1-x & \cdots & 1 & \cdots & -x & 0 \\ -x & -x & \cdots & 1 & \cdots & 0 & 0 \\ -x & -x & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x & -x & \cdots & 1 & \cdots & 1 & 0 \\ -x & 0 & \cdots & 1 & \cdots & 0 & 1 \end{pmatrix}_{(m-1)\times(m-1)}$$

This determinant is a $(i-1)$th column sum of $\mathrm{adj}(I_{m-1} - xU(m-1))$. Therefore, by the induction assumption, from the formula (3.2) with (k, i) replaced by $(m-2, i-1)$ we get the following equations.

$$|\alpha_i| = \begin{cases} P_{(m-2)-2(i-1)}(x) & i-1 \leq \lfloor \frac{m-2}{2} \rfloor \\ P_{2(i-1)-1-(m-2)}(-x) & i-1 > \lfloor \frac{m-2}{2} \rfloor \end{cases}$$

The right hand side of the previous expression is exactly same as that of the formula (3.1). This completes the proof. $\square$

EXAMPLE 3.5. *Consider the example with the case $m = 4$.*

$$\begin{aligned}
&(|\alpha_0|, |\alpha_1|, |\alpha_2|, |\alpha_3|, |\alpha_4|) \\
&= u^t \mathrm{adj}(I_5 - xU(5)) \\
&= (1 + 2x - 3x^2 - x^3 + x^4, 1 + x - x^2, 1, 1 - x, 1 - 2x - x^2 + x^3) \\
&= (P_4(x), P_2(x), P_0(x), P_1(-x), P_3(-x))
\end{aligned}$$

□

A variant of the up-down poset is defined as follows. (See [4] *and Figure 1 for details.)*

$$A_{s,n} := \{\sigma_s < \sigma_{s-1} < \cdots < \sigma_2 < \sigma_1 < \tau_1 < \tau_2 > \tau_3 < \tau_4 > \cdots < (\text{ or } >)\tau_n\},$$

where $[n + s] = \{\sigma_i\}_{i=1}^s \cup \{\tau_j\}_{j=1}^n$. In other words, the sub-poset $\sigma_i' s$ forms a chain and the orders between $\tau_i' s$ in $A_{s,n}$ change alternatively. Our goal here is to find the discrete volume of poset polytope for variant of updown posets.

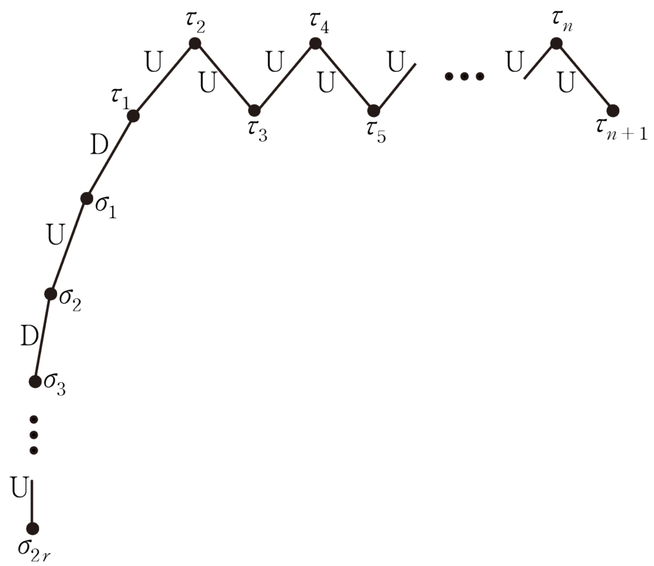

FIGURE 1. A Variant of the up-down poset

We consider the discrete volume of poset polytope for $A_{2r,n+1}$. (See

Figure 1.) The Ehrhart function of the poset polytope for the poset described as $A_{2r,n+1}$ is obtained as the sum of the all entries of the matrix given by

$$\big(U(m+1)D(m+1)\big)^r U(m+1)^n = \big(U(m+1)J\big)^{2r} U(m+1)^n,$$

so that

$$L_{P(A_{2r,n+1})}(m) = \#(mP \cap \mathbb{Z}^{2r+n+1}) = s\Big(\big(U(m+1)J\big)^{2r} U(m+1)^n\Big),$$

where $J = (b_{ij})$ be a $(m+1) \times (m+1)$ square matrix with $b_{i,j} = 1$ $(0 \le i, j \le m)$ if $i + j = m$, 0 for other entries. Similarly, we have

$$\begin{aligned} L_{P(A_{2r+1,n+1})}(m) = \#(mP \cap \mathbb{Z}^{2r+n+2}) &= s\Big(D(m+1)U(m+1)\big)^{r+1} U(m+1)^{n-1}\Big) \\ &= s\Big(\big(JU(m+1)\big)^{2r+2} U(m+1)^{n-1}\Big) \\ &= s\Big(J\big(JU(m+1)\big)^{2r+2} U(m+1)^{n-1}\Big) \\ &= s\Big(\big(U(m+1)J\big)^{2r+1} U(m+1)^{n}\Big) \end{aligned}$$

Note here that $J^{-1} = J$, $\quad D(m+1) = JU(m+1)J$, and

$$(U(m+1)D(m+1) = (U(m+1)J)^2 = I+2N+3N^2+\cdots+(m+1)N^m = (I-N)^{-2}.$$

The following formula is useful.

(3.3)

$$u^t \mathrm{adj}(I - xU(m+1)J) = u^t \begin{pmatrix} (1-x)^m & x(1-x)^{m-1} & x(1-x)^{m-2} & \cdots & x(1-x) & x \\ 0 & (1-x)^m & x(1-x)^{m-1} & \cdots & x(1-x)^2 & x(1-x) \\ 0 & 0 & (1-x)^m & \cdots & x(1-x)^3 & x(1-x)^2 \\ 0 & 0 & 0 & \cdots & x(1-x)^4 & x(1-x)^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (1-x)^m & x(1-x)^{m-1} \\ 0 & 0 & 0 & \cdots & 0 & (1-x)^m \end{pmatrix}$$

$$= ((1-x)^m, (1-x)^{m-1}, (1-x)^{m-2}, \cdots, (1-x), 1).$$

THEOREM 3.6. *Let $g_m(x,y)$ be the bi-variate generating function of $L_{P(A_{s,n})}(m)$ for fixed m as following:*

$$g_m(x,y) = \sum_{s,n\geq 0} L_{P(A_{s,n})}(m)x^s y^n,$$

and let $f_m(x,y)$ be the modified generating function for $L_{P(A_{s,n})}(m)$. That is,

$$f_m(x,y) = 1 + xg_m(x,0) + yg_m(0,y) + xyg_m(x,y).$$

Then $f_m(x,y)$ satisfies the following:

$$f_m(x,y) = \frac{1}{(1-x)^{m+1}} + F_m(y) - 1 + xyg_m(x,y),$$

where $F_m(y)$ is a generating function given in the form of a continued fraction as in Theorem 3.2,

$$g_m(x,y) = \frac{1}{(1-x)^{m+1}\cdot P_{m+1}(-y)}\left(\sum_{i=0}^{m} |w_i|(1-x)^{m-i}\right),$$

and

$$|w_i| = \begin{cases} P_{m-2i}(y) & i \leq \lfloor \frac{m}{2} \rfloor \\ P_{2i-1-m}(-y) & i > \lfloor \frac{m}{2} \rfloor \end{cases}.$$

Proof. (1) Case 1: Both of the Chain and the Up-down poset are empty. For convenience we let

$$f_m(0,0) = 1.$$

(2) Case 2: The Chain is empty, but the Up-down poset is not.

$$f_m(0,y) = 1+yg_m(0,y) = 1+ys[(I-yU(m+1))^{-1}] = \frac{P_m(-y)}{P_{m+1}(y)} = F_m(y).$$

(3) Case 3: The Up-down poset is empty, but the Chain is not.

$$\begin{aligned} f_m(x,0) &= 1 + x g_m(x,0) \\ &= 1 + xs[(I - xU(m+1)J)^{-1}] \\ &= 1 + \frac{x}{(1-x)^{m+1}} \sum_{j=0}^{m} (1-x)^j \\ &= \frac{1}{(1-x)^{m+1}}. \end{aligned}$$

(4) Case 4: Neither of the Chain nor the Up-down poset is empty.

$$\begin{aligned} g_m(x,y) &= \sum_{s,n\geq 0} L_{P(A_{s,n})}(m) x^s y^n \\ &= s\big((I + xU(m+1)J + (xU(m+1)J)^2 + \cdots)(I + yU(m+1) + \big(yU(m+1)\big)^2 + \cdots)\big) \\ &= u^t (I - xU(m+1)J)^{-1} (I - yU(m+1))^{-1} u \\ &= u^t \left(\frac{1}{(1-x)^{m+1}} \operatorname{adj}(I - xU(m+1)J) \frac{1}{P_{m+1}(-y)} \operatorname{adj}(I - yU(m+1)) \right) u \\ &= \frac{1}{(1-x)^{m+1} \cdot P_{m+1}(-y)} \left(u^t \operatorname{adj}(I - xU(m+1)J) \right) \left(\operatorname{adj}(I - yU(m+1)) u \right) \end{aligned}$$

Let $|v_i|$ is i-th column sum of $\operatorname{adj}(I - (xU(m+1)J)$, $|w_j|$ is j-th row sum of $\operatorname{adj}(I - yU(m+1))$. Then the last two factors in the previous expression is changed to the following:

$$\left(u^t \operatorname{adj}(I - xU(m+1)J) \right) \left(\operatorname{adj}(I - yU(m+1)) u \right) = \sum_{i=0}^{m} |v_i||w_i|.$$

By the Formula (3.3), $|v_i| = (1-x)^{m-i} (0 \leq i \leq m)$, and $|w_i|$ is the formula given by Theorem 3.4.

$$\begin{aligned} g_m(x,y) &= \sum_{s,n\geq 0} s\big((U(m+1)J)^s U(m+1)^n\big) x^s y^n \\ &= \frac{1}{(1-x)^{m+1} \cdot P_{m+1}(-y)} \left(\sum_{i=0}^{m} |w_i| (1-x)^{m-i} \right), \end{aligned}$$

where $|w_i| = \begin{cases} P_{m-2i}(y) & i \leq \lfloor \frac{m}{2} \rfloor \\ P_{2i-1-m}(-y) & i > \lfloor \frac{m}{2} \rfloor \end{cases}$. $\square$

EXAMPLE 3.7. *In this example we consider the generating function of the given poset polytope (magnified by a factor of 4, i.e., $m = 4$) for the variant poset. Note that*

$$P_{n+2}(y) = yP_{n+1}(-y) + P_n(y)(n = 0, 1, 2, \cdots)$$

with

$$P_0(y) = 1, P_1(y) = 1 + y.$$

So, we can find other $P_i(y)$'s as follows:

$$\begin{aligned}
P_2(y) &= 1 + y - y^2, \\
P_3(y) &= 1 + 2y - y^2 - y^3, \\
P_4(y) &= 1 + 2y - 3y^2 - y^3 + y^4 \\
|w_0| &= P_4(y) = 1 + 2y - 3y^2 - y^3 + y^4 \\
|w_1| &= P_2(y) = 1 + y - y^2 \\
|w_2| &= P_0(y) = 1 \\
|w_3| &= P_1(-y) = 1 - y \\
|w_4| &= P_3(-y) = 1 - 2y - y^2 + y^3
\end{aligned}$$

This information gives us the generating function for fixed $m = 4$:

$$f_4(x, y) = \frac{1}{(1-x)^5} + \frac{P_4(y)}{P_5(-y)} - 1 + \frac{xy}{(1-x)^5 P_5(-y)} \sum_{i=0}^{4} |w_i|(1-x)^{4-i}$$

where the summation term in $f_4(x, y)$ is:

$$\sum_{i=0}^{4} |w_i|(1-x)^{4-i}$$

$$= P_4(y)(1-x)^4 + P_2(y)(1-x)^3 + P_0(y)(1-x)^2 + P_1(-y)(1-x) + P_3(-y).$$

4. Conclusion

In [4] a generating function on the sequence given by the continuous volume of poset polytope for variant up-down poset according to the length of both chain and up-down poset was found. Here we also obtained the generating functions on the sequence given by the discrete volume of the same poset(of course, according to the length of both chain and up-down poset) using characteristic matrix. However, those results obtained are for the fixed m. In other words, it remains to try the generating functions on m with these results.

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