J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
https://doi.org/10.7468/jksmeb.2023.30.4.429
Volume 30, Number 4 (November 2023), Pages 429–442

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

FUZZY IDEALS IN $\Gamma-$BCK-ALGEBRAS

Arsham Borumand Saeid [a,*], M. Murali Krishna Rao [b] and Rajendra Kumar Kona [c]

Abstract. In this paper, we introduce the concept of fuzzy ideals, anti-fuzzy ideals of $\Gamma-$BCK-algebras. We study the properties of fuzzy ideals, anti-fuzzy ideals of $\Gamma-$BCK-algebras. We prove that if $f^{-1}(\mu)$ is a fuzzy ideal of M, then μ is a fuzzy ideal of N, where $f : M \to N$ is an epimorphism of $\Gamma-$BCK-algebras M and N.

1. Introduction

In 1995, M. Murali Krishna Rao introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, a ternary semiring and a semiring [12]. M. Murali Krishna Rao studied $\Gamma-$semiring as a generalization of a soft semiring [16]. We studied $\Gamma-$BCK-algebra as a generalization of a soft BCK-algebra [3]. As a generalization of a ring, the notion of a $\Gamma-$ring was introduced by Nobusawa in 1964. Sen introduced the notion of a $\Gamma-$semigroup as a generalization of a semigroup [18]. M. Murali Krishna Rao studied a regular $\Gamma-$incline, a $\Gamma-$group, and a $\Gamma-$Field [13, 14, 15]. The important reason for the development of a $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, semigroups, and ternary semirings.

By an algebra (groupoid) we mean a non-empty set G together with a binary multiplication and a some distinguished element 0. Such an algebra is denoted by $(G, \cdot, 0)$. Each such algebra will follow equality axioms and the rule of substitution as well as some other rules. Many of such algebras were inspired by some logical systems. For example, so-called BCK-algebras are inspired by a BCK logic. We have BCK-algebra and BCK positive logic, BCI-algebra and BCI positive logic, positive implicative BCK-algebra and positive implicative logic, implicative BCK-algebra

Received by the editors June 15, 2023. Revised August 15, 2023. Accepted August 23, 2023.
2020 *Mathematics Subject Classification.* 03G25, 06F35, 03E72, 08A35.
Key words and phrases. $(\Gamma-)$BCK-algebra, subalgebra, ideal, fuzzy set, (anti)fuzzy ideal, epimorphism.
*Corresponding author.

and implicative logic and so on. The connection between such algebras and their corresponding logics is much stronger. Therefore one can give a translation procedure which translates all well formed formulas and all theorems of a given logic, into theorems of the corresponding algebra. Iseki and Tanaka, based on these relationships introduced a new class of general algebras named BCK-algebras Every BCI-algebra M satisfy $0 * x = 0$, for all $x \in M$ is a BCK-algebra. Residuated lattices, Boolean algebras, MV-algebras, BE-algebras, Wajsberg algebras, BL-algebras, Hilbert algebras, Heyting algebras, NM-algebras, MTL-algebras, Weak $-R_0$ algebras, etc., can be expressed as particular cases of BCK- algebras. BCK-algebras have been studied by many mathematicians and applied to group theory, functional analysis, probability theory, topology and so on. Ideal theory plays an important role in studying of these algebras. BCK-algebras are the algebraic formulation of the BCK system in combinatory logic, which has applications in the language of functional programming. Hong, S. M., Jun, Y. B. studied anti fuzzy ideals in BCK-algebras. In 1991, Xi applied the concept of fuzzy subsets to BCK-algebras and studied fuzzy BCK-algebras, introduced by Imai and Iseki [7].

Many real-world problems are complicated due to various uncertainties. In addressing them, classical methods may not be the best option. To mention a few, artificial intelligence plays a vital role in dealing with uncertain information by simulating the people's needs comprising of uncertain data. Several theories like Probability, Randomness, Rough sets were introduced. In addressing uncertainty, one of the appropriate theory is the fuzzy set theory. Zadeh in 1965 developed the fuzzy set theory [19]. Many papers on fuzzy sets appeared, showing the importance of the concept and its applications to logic, group theory, ring theory, multi agent systems, machine learning, information processing, real analysis, topology, measure theory, etc. Rosenfeld introduced the fuzzification of algebraic structure, and he introduced the notion of fuzzy subgroups in 1971 [10]. Fuzzy algebraic structures play a vital role in mathematics with wide applications in many branches such as theoretical physics, computer sciences, control engineering, coding theory etc.

This paper aims to introduce the concepts of fuzzy subalgebra, fuzzy ideals, homomorphism and epimorphism in terms of $\Gamma-$BCK- algebras, study some of the properties and relations between them.

2. Preliminaries

In this section, we recall the following definitions and results which are necessary for completeness.

Definition 2.1 ([12]). Let M and Γ be two non-empty sets. Then M is called a $\Gamma-$*semigroup* if there exists a mapping $M \times \Gamma \times M \to M$ (the images of (x, α, y) will be denoted by $x\alpha y$, $x, y \in M, \alpha \in \Gamma$) such that,

$$x\alpha(y\beta z) = (x\alpha y)\beta z, \text{ for all } x, y, z \in M, \alpha, \beta \in \Gamma.$$

Definition 2.2 ([12]). Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then M is said to be $\Gamma-$*semiring* M if there exists a mapping $M \times \Gamma \times M \to M$ (the images of (x, α, y) will be denoted by $x\alpha y$, $x, y \in M, \alpha \in \Gamma$) such that it satisfies,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$.
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation as the usual semiring multiplication.

Definition 2.3 ([12]). A $\Gamma-$semiring M is said to have *zero element* if there exists an element $0 \in M$ such that, $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$. And M is said to be *commutative* $\Gamma-$*semiring* if $x\alpha y = y\alpha x$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.4 ([12]). Let M be a $\Gamma-$semiring. An element $a \in M$ is said to be *idempotent* of M if there exists $\alpha \in \Gamma$, such that $a = a\alpha a$ and a is said to be α *idempotent*. And an element $a \in M$ is said to be *regular element* of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Every element of M is a regular element of M then, M is said to be *regular* $\Gamma-$*semiring*.

Definition 2.5 ([5]). An algebra $(M, *, 0)$ is called a BCK-*algebra* if it satisfies the following axioms
i) $[(x * y) * (x * z)] * (z * y) = 0$,
ii) $(x * (x * y)) * y = 0$,
iii) $x * x = 0$,
iv) $0 * x = 0$,
v) $x * y = y * x = 0$ imply $x = y$ for all $x, y, z \in$ M.
We can define a partial ordering $\leq$ on M by $x \leq y$ if and only if $x * y = 0$.

Theorem 2.6. *In any BCK-algebra* $(X, *, 0)$, *the following hold,*
i) $(x * y) * (y * z) \leq z * y$,
ii) $x * [x * (x * y)] \leq x * y$,
iii) $0 \leq x$,
iv) $x * y = 0$, *if and only if* $x \leq y$,
v) $x \leq y$ *implies* $x * z \leq y * z$ *and* $z * y \leq z * x$,
vi) $(x * y) * z = (x * z) * y$,
vii) $x * y \leq z$, *if and only if* $x * z \leq y$,
viii) $0 * (x * y) = (0 * x) * (0 * y)$,
ix) $(x * y) * x = 0$,
x) $(x * z) * (y * z) \leq x * y$, *for all* $x, y, z \in M$.

Definition 2.7 ([11]). A BCK-algebra M is said to be *commutative* if $y * (y * x) = x * (x * y)$.

Definition 2.8 ([11]). A non-empty subset I of a BCK-algebra M is called a *subalgebra* of M, if $x * y \in I$, $x, y \in I$.

Definition 2.9 ([11]). Let M be a BCK-algebra and I be a non-empty subset of M. Then I is called an *ideal* of M if i) $0 \in I$ ii) $x * y \in I$, $y \in I \Rightarrow x \in I$.

Definition 2.10 ([11]). A BCK-algebra M can be partially ordered by $x \leq y$ if and only if $x * y = 0$, for all $x, y \in M$. This ordering is called a BCK *ordering.*

Definition 2.11 ([11]). Let M and N be BCK-algebras. A map $f : M \to N$ is called a *homomorphism* if $f(x * y) = f(x) * f(y)$, for all $x, y \in M$.

Definition 2.12 ([11]). Let M and N be BCK-algebras and $f : M \to N$ be a homomorphism. Then the set $\{x \in M/f(a) = 0\}$ is called a *kernel* of f and it is denoted by kerf and the set $\{f(x)/x \in M\}$ is called *image* of f and is denoted by $Im(f)$.

Definition 2.13 ([3]). Let M be a set with element 0 and Γ be a non-empty set. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y$, for all $x, y \in M$ and $\alpha \in \Gamma$) satisfies the following axioms:
i)$[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y) = 0$,
ii) $x\alpha y = y\alpha x = 0 \Rightarrow x = y$,
iii) $x\alpha x = 0$,
iv) $0\alpha x = 0$, for all $\alpha, \beta \in \Gamma$, $x, y, z \in M$. Then M is called a $\Gamma-$BCK-algebra.

Note: Let M be a $\Gamma-$BCK-algebra and $\alpha \in \Gamma$. Define a mapping $* : M \times M \to M$ such that $a * b = a\alpha b$ for all $a, b \in M$. Then $(M, *, 0)$ is a BCK-algebra and it is denoted by M_α.

Example 2.14. Let $M = \{0, a, b, c, d, e\}$ and $\Gamma = \{\alpha, \beta, \gamma, \delta, \psi\}$. The ternary operation is defined by the following tables

α	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	a	a	a	a
b	b	b	0	b	b	b
c	c	c	c	0	c	c
d	d	d	d	d	0	d
e	e	e	e	e	e	0

β	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	b	b	b	b
b	b	c	0	c	c	c
c	c	d	d	0	d	d
d	d	e	e	e	0	e
e	e	a	a	a	a	0

γ	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	c	c	c	c
b	b	d	0	d	d	d
c	c	e	e	0	e	e
d	d	a	a	a	0	a
e	e	b	b	b	b	0

δ	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	d	d	d	d
b	b	e	0	e	e	e
c	c	a	a	0	a	a
d	d	b	b	b	0	b
e	e	c	c	c	c	0

ψ	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	e	e	e	e
b	b	a	0	a	a	a
c	c	b	b	0	b	b
d	d	c	c	c	0	c
e	e	d	d	d	d	0

Let $N = \{0, 1, 2, 3\}$ and $\Gamma = \{\alpha, \beta\}$. The ternary operation is defined by the following tables

α	0	1	2	3
0	0	0	0	0
1	1	0	2	2
2	2	3	0	3
3	3	1	1	0

β	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then M and N are $\Gamma-$BCK-algebras.

Example 2.15. Any BCK-algebra $(M, *, 0)$ can be considered as $\Gamma-$BCK-algebra if we choose $\Gamma = \{0\}$ and the ternary operation $x0y$ is defined as $(x * 0) * y$ for all $x, y \in M$.

Let $M = \{0, b_1, b_2, b_3\}$. The binary operation $*$is defined by the following table

$*$	0	b_1	b_2	b_3
0	0	0	0	0
b_1	b_1	0	0	0
b_2	b_2	b_1	0	0
b_3	b_3	b_3	b_3	0

Then M is a BCK-algebra.
If $\Gamma = \{0\}$ the ternary operation $x0y$ is defined by the following table

0	0	b_1	b_2	b_3
0	0	0	0	0
b_1	b_1	0	0	0
b_2	b_2	b_1	0	0
b_3	b_3	b_3	b_3	0

Then M is a $\Gamma-$BCK-algebra.

Definition 2.16 ([3])**.** A non-empty subset I of a $\Gamma-$BCK-algebra M is called a *subalgebra* of M, if $x\alpha y \in I$, for all $x, y \in M$, $\alpha \in \Gamma$.

Note: Let I be a subalgebra of a $\Gamma-$BCK-algebra M. Then $0 \in I$.

Definition 2.17 ([3])**.** Let M be a $\Gamma-$BCK-algebra and I be a non-empty subset of M. Then I is called an *ideal* of M if i) $0 \in I$ ii) $x\alpha y \in I$, $\alpha \in \Gamma$, $y \in I \Rightarrow x \in I$. M and $\{0\}$ are trivial ideals. An ideal I is proper if $I \neq M$.

Definition 2.18. A $\Gamma-$BCK-algebra M is said to be *commutative* if $y\alpha(y\beta x) = x\alpha(x\beta y)$, for all $x, y \in M$, $\alpha, \beta \in \Gamma$.
A $\Gamma-$BCK-algebra M can be partially ordered by $x \leq y$ if and only if $x\alpha y = 0$, for all $\alpha \in \Gamma, x, y \in M$. This ordering is called a $\Gamma-$BCK *ordering.*

Theorem 2.19 ([3])**.** *Let M be a $\Gamma-$BCK-algebra. Then the following are equivalent*
(i) M is a commutative $\Gamma-$BCK-algebra,
(ii) $x \leq y \Rightarrow x = y\alpha(y\beta x)$, for all $\alpha, \beta \in \Gamma, x, y \in M$.

3. Fuzzy Ideals in $\Gamma-$BCK-algebras

In this section, we introduce the concepts of a fuzzy subalgebra, a fuzzy ideal, an anti fuzzy ideal of $\Gamma-$BCK-algebras and and study the properties of (anti) fuzzy ideals of $\Gamma-$BCK-algebras.

Definition 3.1. A fuzzy subset μ of a $\Gamma-$BCK-algebra M is called a *fuzzy subalgebra* of M if $\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}$ and a (anti) fuzzy ideal of M if
i) $\mu(0) \geq \mu(x)(\mu(0) \leq \mu(x))$,
ii) $\mu(x) \geq \min\{\mu(x\alpha y), \mu(y)\}(\mu(x) \leq \max\{\mu(x\alpha y), \mu(y)\})$, for all $x, y \in M, \alpha \in \Gamma$.

Example 3.2. Let $M = \{0, a, b, c, d\}$ and $\Gamma = \{\alpha, \beta, \gamma, \delta\}$.Then ternary operation is defined with the following tables

α	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

β	0	a	b	c	d
0	0	0	0	0	0
a	a	0	b	b	b
b	b	c	0	c	c
c	c	c	d	0	d
d	d	a	a	a	0

γ	0	a	b	c	d
0	0	0	0	0	0
a	a	0	c	c	c
b	b	d	0	d	d
c	c	a	a	0	a
d	d	b	b	b	0

δ	0	a	b	c	d
0	0	0	0	0	a
a	a	0	d	d	d
b	b	a	0	a	a
c	c	b	b	0	b
d	d	c	c	c	0

Let $t_0 < t_1 < t_2 < t_3 < t_4$ such that $t_0, t_1, t_2, t_3, t_4 \in [0, 1]$.
Define $\mu : M \to [0, 1]$ such that $\mu(0) = t_0, \mu(a) = t_1, \mu(b) = t_2, \mu(c) = t_3$, and $\mu(d) = t_4$. By routine verification, μ is a fuzzy subalgebra and an anti fuzzy ideal of the $\Gamma-$BCK-algebra M.

Lemma 3.3. *If μ is a fuzzy subalgebra of a $\Gamma-$BCK-algebra M, then $\mu(0) \geq \mu(x)$, for all $x \in M$.*

Proof. We have $x\alpha x = 0$ for all $\alpha \in \Gamma, x \in M$. $\mu(0) = \mu(x\alpha x) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$. Hence $\mu(0) \geq \mu(x)$. □

Corollary 3.4. *If μ is an anti fuzzy subalgebra of a $\Gamma-$BCK-algebra M, then $\mu(0) \leq \mu(x)$, for all $x \in M$.*

Theorem 3.5. *If μ is a fuzzy ideal of a $\Gamma-$BCK-algebra M, then $x \leq y \Rightarrow \mu(x) \geq \mu(y))$.*

Proof. Let μ be a fuzzy ideal of a $\Gamma-$BCK-algebra M and $x, y \in M$
such that $x \leq y$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} \mu(x) &\geq \min\{\mu(x\alpha y), \mu(y)\} \\ &= \min\{\mu(0), \mu(y)\} \\ &= \mu(y). \end{aligned}$$

□

Lemma 3.6. *Let μ be a fuzzy ideal of a $\Gamma-BCK$-algebra M. If the inequality $x\alpha y \leq z$ $x, y, z \in M, \alpha \in \Gamma$, holds in M then, $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.*

Proof. Let $x, y, z \in M, \alpha \in \Gamma$ and $x\alpha y \leq z$. Then $(x\alpha y)\alpha z = 0$ for all $\alpha \in \Gamma$ and $\mu(x\alpha y) \geq \mu(z)$. Then

$$\begin{aligned} \mu(x) &\geq \min\{\mu(x\alpha y), \mu(y)\} \\ &\geq \min\{\mu(z), \mu(y)\}. \end{aligned}$$

□

Theorem 3.7. *Every fuzzy ideal of a $\Gamma-BCK$-algebra M is a fuzzy subalgebra of M.*

Proof. Let μ be a fuzzy ideal of the $\Gamma-$BCK-algebra M. We have $x\alpha y \leq x$ for all $\alpha \in \Gamma, x, y \in M$. Then

$$\begin{aligned} &\mu(x\alpha y) \geq \mu(x) \\ \Rightarrow\ &\mu(x\alpha y) \geq \mu(x) \geq \min\{\mu(x\alpha y), \mu(y)\} \\ \Rightarrow\ &\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}. \end{aligned}$$

This shows that μ is a fuzzy subalgebra of the $\Gamma-$BCK-algebra M. □

We now give a condition for a fuzzy subalgebra to be a fuzzy ideal.

Theorem 3.8. *Let μ be a fuzzy subalgebra of a $\Gamma-BCK$-algebra M such that, $\mu(x) \geq \min\{\mu(y), \mu(z)\}$ if $x\alpha y \leq z$ for all $\alpha \in \Gamma, x, y, z \in M$. Then μ is a fuzzy ideal of M.*

Proof. We have $\mu(x) \geq \min\{\mu(y), \mu(z)\}$ if $x\alpha y \leq z \Rightarrow \mu(x\alpha y) \geq \mu(z)$ Therefore $\mu(x) \geq \min\{\mu(x\alpha y), \mu(y)\}$. Hence μ is a fuzzy ideal of the $\Gamma-$BCK-algebra M. □

Corollary 3.9. *Every anti fuzzy ideal μ of a $\Gamma-BCK$-algebra M is an anti fuzzy subalgebra of M.*

Definition 3.10. Let μ be a fuzzy subset of a $\Gamma-$BCK-algebra M, the complement of μ is denoted by μ^c and it is defined as $\mu^c(x) = 1 - \mu(x)$, for all $x \in M$.

Theorem 3.11. *A fuzzy subset μ of a $\Gamma-BCK$-algebra M is a fuzzy ideal of M if and only if its complement μ^c is an anti fuzzy ideal of M.*

Proof. Let μ be a fuzzy ideal of M. We have

$$\begin{aligned}
\mu(0) &\geq \mu(x), then - \mu(0) \leq -\mu(x),\\
\text{implies that } 1-\mu(0) &\leq 1-\mu(x), \text{implies } \mu^c(0) \leq \mu^c(x) \text{ and}\\
\mu^c(x) &= 1-\mu(x)\\
\mu(x) &\geq \min\{\mu(x\alpha y), \mu(y)\}\\
-\mu(x) &\leq -\min\{\mu(x\alpha y), \mu(y)\}\\
1-\mu(x) &\leq 1-\min\{\mu(x\alpha y), \mu(y)\}\\
\text{implies that } \mu^c(x) &\leq 1-\min\{1-\mu^c(x\alpha y), 1-\mu^c(y)\}\\
&= \max\{\mu^c(x\alpha y), \mu^c(y)\}.
\end{aligned}$$

Hence μ^c is an anti fuzzy ideal of M. Similarly we can prove the converse. □

Theorem 3.12. *Let μ be an anti fuzzy ideal of a $\Gamma-BCK$-algebra M. Then the set $M_\mu = \{x \in M/\mu(x) = \mu(0)\}$ is an ideal of M.*

Proof. Obviously $0 \in M_\mu$. Let $x\alpha y \in M_\mu$, $y \in M_\mu$, $x \in M$and $\alpha \in \Gamma$. Then $\mu(x\alpha y) = \mu(0)$ and $\mu(y) = \mu(0)$ It follows that

$$\begin{aligned}
\mu(x) &\leq \max\{\mu(x\alpha y), \mu(y)\}\\
&= \max\{\mu(0), \mu(0)\}\\
&= \mu(0).
\end{aligned}$$

We have $\mu(0) \leq \mu(x)$. Therefore $\mu(0) = \mu(x)$. Hence $x \in M_\mu$. □

Definition 3.13. Let μ be a fuzzy subset of a $\Gamma-$BCK-algebra M. Then for $t \in [0,1]$ the lower level t-cut of μ is the set $\mu^t = \{x \in M/\mu(x) \leq t\}$ and the upper level t-cut of μ is the set $\mu_t = \{x \in M/\mu(x) \geq t\}$. Obviously $\mu^1 = M$ and $\mu^t \cup \mu_t = M$ for all $t \in [0,1]$. If $t_1 < t_2$ then $\mu^{t_1} \leq \mu^{t_2}$.

Theorem 3.14. *Let μ be a fuzzy subset of a $\Gamma-BCK$-algebra M. Then μ is an anti fuzzy ideal of M if and only if for each $t \in [0,1]$ $\mu^t \neq \phi$, then lower t-level cut μ^t is an ideal of M.*

Proof. Let μ be an anti fuzzy ideal of M and $t \in [0,1]$. Obviously $0 \in \mu^t$. Let $x\alpha y \in \mu^t, y \in \mu^t, x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}\mu(x) &\leq \max\{\mu(x\alpha y), \mu(y)\}\\ &\leq \max\{t, t\}\\ &= t. \text{ Implies } x \in \mu^t.\end{aligned}$$

Hence μ^t is an ideal of M.

Conversely suppose that μ^t is an ideal of M for all $t \in [0,1]$. Suppose there exists $x_0 \in M$ such that $\mu(0) > \mu(x_0)$. put $t_0 = \frac{1}{2}\{\mu(0) + \mu(x_0)\} \Rightarrow \mu(x_0) < t_0 < \mu(0) < 1$. Then $x_0 \in \mu^{t_0}$, since μ^{t_0} is an ideal of M. Thus $0 \in \mu^{t_0} \Rightarrow \mu(0) \leq t_0$. Which is a contradiction. Hence, $\mu(0) \leq \mu(x)$, for all $x \in M$.

Now we prove that $\mu(x) \leq \max\{\mu(x\alpha y), \mu(y)\}$ for all $x \in M, \alpha \in \Gamma$, if not there exists $x_0, y_0 \in M$ such that $\mu(x) > \max\{\mu(x_0\alpha y_0), \mu(y_0)\}$.
Let $p_0 = \frac{1}{2}\{\mu(x_0) + \max\{\mu(x_0\alpha y_0), \mu(y_0)\}\}$, then $p_0 < \mu(x_0)$ and $0 \leq \max\{\mu(x_0\alpha y_0), \mu(y_0)\}< p_0 < 1$. Thus $p_0 > \mu(x_0\alpha y_0)$ and $p_0 > \mu(y_0)$, implies $x_0\alpha y_0 \in \mu^{p_0}$ and $y_0 \in \mu^{p_0}$. As μ^{p_0} is an ideal of M, it follows that $x_0 \in \mu^{p_0}$ or $\mu(x_0) \leq p_0$. Which is a contradiction. Hence, $\mu(0) \leq \mu(x)$, for all $x \in M$. □

Theorem 3.15. *Let μ be an anti fuzzy ideal of a $\Gamma-$BCK-algebra M. If $x\alpha y \leq z$ then $\mu(x) \leq max\{\mu(y), \mu(z)\}$ for all $\alpha \in \Gamma, x, y, z \in M$.*

Proof. Suppose $x\alpha y \leq z$. Then $\mu((x\alpha y)\alpha z) = 0$, for all $x, y, z \in M, \alpha \in \Gamma$.

$$\begin{aligned}\mu(x\alpha y) &\leq \max\{\mu((x\alpha y)\alpha z), \mu(z)\}\\ &= \max\{\mu(0), \mu(z)\}\\ &\leq \mu(z)\\ \Rightarrow \mu(x\alpha y) &\leq \mu(z).\\ \mu(x) &\leq \max\{\mu(x\alpha y), \mu(y)\}\\ &\leq \max\{\mu(z), \mu(y)\}.\end{aligned}$$

□

Definition 3.16. A mapping $f : M \to N$ of a $\Gamma-$BCK-algebra is called a *homomorphism* if $f(x\alpha y) = f(x)\alpha f(y)$ for all $\alpha \in \Gamma$, $x, y \in M$.
If $f : M \to N$ is a homomorphism of $\Gamma-$BCK-algebras M and N, then $f(0) = 0$. We define, $\mu^f : M \to [0,1]$, $\mu^f(x) = \mu(f(x))$, for all $x, \in M$.

Definition 3.17. Let $f : M \to N$ be a homomorphism of $\Gamma-$BCK-algebras M, N and μ be a fuzzy subset of N. Then f^{-1} is defined as $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in M$.

Theorem 3.18. *Let $f : M \to N$ be a homomorphism of a $\Gamma-BCK$-algebras M, N. If μ is a fuzzy ideal of N, then $f^{-1}(\mu)$ is a fuzzy ideal of M.*

Proof. We have

$$\begin{aligned} f^{-1}(\mu)(x) = \mu(f(x)) &\leq \mu(0) \\ &= \mu(f(0)) \\ &= f^{-1}(\mu)(0) \text{ for all } x \in M \end{aligned}$$

Suppose $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned} f^{-1}(\mu)(x) &= \mu(f(x)) \\ &\geq \min\{\mu((f(x)\alpha f(y)), \mu(f(y))\} \\ &= \min\{\mu((f(x\alpha y)), \mu(f(y))\} \\ &= \min\{f^{-1}(\mu)(x\alpha y), f^{-1}(\mu)(y)\}. \end{aligned}$$

Hence $f^{-1}(\mu)$ is a fuzzy ideal of M. □

Theorem 3.19. *Let $f : M \to N$ be an epimorphism of $\Gamma-BCK$-algebras M, N. If $f^{-1}(\mu)$ is a fuzzy ideal of M, then μ is a fuzzy ideal of N.*

Proof. Suppose $x \in N$, then there exists $a \in M$ such that $f(a) = x$. Then

$$\begin{aligned} \mu(x) = \mu(f(a)) &= f^{-1}(\mu)(a) \\ &\leq f^{-1}(\mu)(0) \\ &= \mu(f(0)) \\ &= \mu(0). \end{aligned}$$

Let $x, y \in N,\ \alpha \in \Gamma$. Then there exist $a, b \in M$ such that $f(a) = x, f(b) = y$,

$$\begin{aligned} \mu(x) = \mu(f(a)) &= f^{-1}(\mu)(a) \\ &\geq \min\{f^{-1}(\mu)(a\alpha b), f^{-1}(\mu)(b)\} \\ &= \min\{\mu(f(a\alpha b)), \mu(f(b))\} \\ &= \min\{(\mu)(f(a)\alpha f(b)), \mu(f(b))\} \\ &= \min\{(\mu)(x\alpha y), \mu(y)\}. \end{aligned}$$

□

Theorem 3.20. *Let $f : M \to N$ be a homomorphism of $\Gamma-BCK$-algebras. If μ is a fuzzy ideal of N, then $\mu^f(x) = \mu(f(x))$ in M is a fuzzy ideal of M.*

Proof. We have $\mu^f(x) = \mu(f(x)) \leq \mu(0) \leq \mu(f(0)) = \mu^f(0)$, for all $x \in M,\ \alpha \in \Gamma$. Then

$$\begin{aligned}\min\{\mu^f(x\alpha y), \mu^f(y)\} &= \min\{\mu(f(x\alpha y)), \mu(f(y))\} \\ &= \min\{(\mu f(x)\alpha\mu f(y)), \mu(f(y))\} \\ &\leq \mu f(x) = \mu^f(x).\end{aligned}$$

Therefore μ^f is a fuzzy ideal of M. □

Theorem 3.21. *Let $f : M \to N$ be a epimorphism of $\Gamma-BCK$-algebras and μ be a fuzzy subset of N. If μ^f is a fuzzy ideal of M, then μ is a fuzzy ideal of N.*

Proof. Let $f : M \to N$ be a epimorphism of $\Gamma-$BCK algebras. For any $x \in N$, there exists $a \in M$ such that $f(a) = x$. Then $\mu(x) = \mu(f(a)) = \mu^f(a) \leq \mu^f(0) = \mu(f(0)) = \mu(0)$.

Let $x, y \in N$, then $f(a) = x$ and $f(b) = y$ for $a, b \in M$. It follows that

$$\begin{aligned}\mu(x) &= \mu(f(a)) = \mu^f(a) \\ &\geq \min\{\mu^f(a\alpha b), \mu^f(b)\} \\ &= \min\{\mu(f(a\alpha b)), \mu(f(b))\} \\ &= \min\{\mu((f(a)\alpha(f(b)))), \mu(f(b))\} \\ &= \min\{\mu((x\alpha y))), \mu(y))\}.\end{aligned}$$

Hence μ is a fuzzy ideal of N. □

4. Conclusion

In this paper, we introduced the concepts of (anti) fuzzy ideals of $\Gamma-$BCK-algebras, which generalizes the study of (anti) fuzzy soft ideals. We studied the properties of fuzzy ideals and anti fuzzy ideals of $\Gamma-$BCK-algebras. We proved that a fuzzy subset of a $\Gamma-$BCK-algebra is a fuzzy ideal if and only if the complement of the fuzzy subset is an anti fuzzy ideal of $\Gamma-$BCK-algebra. If $f : M \to N$ is an epimorphism of a $\Gamma-$BCK-algebra M, and $f^{-1}(\mu)$ is a fuzzy ideal of M, then μ is a fuzzy ideal of N.

Acknowledgement

The authors would like to thank the anonymous reviewers for their constructive suggestion and helpful comments, which enabled us to improve the presentation of our work.

References

1. Akram M.: Spherical fuzzy K-algebras. Journal of Algebraic Hyperstructures and Logical Algebras **2** (2021), no. 3, 85-98. DOI:10.52547/HATEF.JAHLA.2.3.7
2. Akram M., Davvaz B. & Feng F.: Intutionistic fuzzy soft K-algebras. Mathematics in Computer science **7** (2013), no. 3, 353-365. DOI:10.1007/s11786-013-0158-5
3. A. Borumand Saeid, M. Murali Krishna Rao & K.R. Kumar: $\Gamma-$BCK Algebras. J. of Mahani Mathematical Research **11** (2022), no. 3, 133-145. DOI:10.22103/JMMR.2022.19322.1234
4. Dar K.H. & Akram M.: On K-homomorphisms of K-algebras. International Mathematical Forum **2** (2007), no. 46, 2283-2293.
5. Huang Y.S.: BCI-Algebra. Science Press, Beijing, China, (2006).
6. K. Iseki: On BCI-algebras. Kobe University, Mathematics Seminar Notes **8** (1980), no. 1, 125-130 .
7. K. Iseki & S. Tanaka: An introduction to the theory of BCK-algebras. Mathematica Japonica **23** (1978), no. 1, 1-26.
8. Y. Imai & K. Iseki: On axiom systems of propositional calculi, XIV. Proceedings of the Japan Academy **42** (1966) 19-22. DOI:10.3792/pja/1195522169
9. K. Iseki: An algebra related with a propositional calculus. Proceedings of the Japan Academy **42** (1966) 26-29. DOI:10.3792/pja/1195522171
10. J.M. Jie & Y.B. Jun: BCK-Algebras. Kyung Moon Sa Co., Seoul, Republic of Korea, (1994).
11. J. Meng & X.E. Guo: On fuzzy ideals in BCK/BCI-algebras. Fuzzy sets and Systems **149** (2005), no. 3, 509-525. https://doi.org/10.1016/j.fss.2003.11.014
12. M. Murali Krishna Rao: $\Gamma-$semirings-I. Southeast Asian Bull. of Math. **19** (1995), no. 1, 49-54.
13. M. Murali Krishna Rao: $\Gamma-$Field. Disc. Math. General Alg. and Appli. **39** (2019), 125-133. https://doi.org/10.7151/dmgaa.1303
14. M. Murali Krishna Rao: $\Gamma-$Inclines. Bulletin Int. Math. Virtual Inst. **10** (2020), no. 2, 305-314. DOI:10.7251/BIMVI2002305R
15. M. Murali Krishna Rao: $\Gamma-$Group. Bulletin Int. Math. Virtual Inst. **10** (2020), no. 1, 51-58. DOI:10.7251/BIMVI2001051R

16. M. Murali Krishna Rao: A study of $\Gamma-$semiring as a generalization of soft semiring (F, Γ) over M. Bulletin Int. Math. Virtual Inst. **8** (2018), 533-541. DOI:10.7251/BIMVI 1803533R
17. A. Rosenfeld: Fuzzy groups. J. Math. Anal. Appl. **35** (1971), no. 3, 512-517. https://doi.org/10.1016/0022-247X(71)90199-5
18. M.K. Sen: On $\Gamma-$semigroup. Proc. of Inter. Con. of Alg. and its Appl., Decker Publicaiton, New York (1981) 301-308.
19. L.A. Zadeh: Fuzzy sets. Information and control **8** (1965), no. 3, 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[a]Professor: Department of Pure Mathematics, Shahid Bahonar University of Kerman, Faculty of Mathematics and Computer, Kerman, Iran
Email address: arsham@uk.ac.ir

[b]Professor: Department of Mathematics, Sankethika Institute of Tech. and Management, Visakhapatnam, 530 041, India
Email address: mmarapureddy@gmail.com

[c]Professor: Department of Mathematics, GIS, GITAM (Deemed to be University), Visakhapatnam- 530 045, A.P., India
Email address: rkkona1972@gmail.com