J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2023.30.4.357 ISSN(Online) 2287-6081
Volume 30, Number 4 (November 2023), Pages 357–375

A NOTE ON BIPOLAR SOFT SUPRA TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce the concept of bipolar soft supra topological space and provide a characterization of the related concepts of bipolar soft supra closure and bipolar soft supra interior. We also establish a connection between bipolar soft supra topology and bipolar soft topology. Additionally, we present the concept of bipolar soft supra continuous mapping and examine the concept of bipolar soft supra compact topological space. A related result concerning the image of the bipolar soft supra compact space is proved. Finally, we identify the concepts of disconnected (connected) and strongly disconnected (strongly connected) space and derive several results linking them together. Relationships among these concepts are clarified with the aid of examples.

1. Introduction

Soft set theory is used to solve complex, particularly uncertainty, problems when classical methods are insufficient. Molodtsov introduced the soft set theory to eliminate uncertainty [30]. There are several active research studies in the literature regarding soft sets. Soft set theory has been studied for its properties, operations, and applications by [4, 5, 19, 26, 27, 31, 32]. In 2011, Shabir and Naz employed the concept of soft topological space using a soft set definition on an initial universal set with predefined parameters [35]. Theoretical studies of soft topological spaces were explored by [13, 14, 15, 24, 25, 29, 38]. In 1983, Mashhour et al. [28] constructed a broader category, known as the supra topological space, and defined supra closed sets, supra open sets, and supra continuous mappings. Elmonsef and Ramadan [2] took this concept further by applying it to fuzzy topological spaces, proposing a fuzzy supra topological space with fuzzy supra open sets, fuzzy supra

Received by the editors March 14, 2023. Revised August 22, 2023. Accepted August 25, 2023.
2020 *Mathematics Subject Classification.* 54A05, 03E20, 03E75, 54D05.
Key words and phrases. bipolar soft supra topological space, bipolar soft supra closure, bipolar soft supra interior, disconnected space, strongly disconnected space.
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closed sets, and fuzzy supra continuity. This proposal provides a generalisation of fuzzy continuity. Abbas [1] defined the intuitionistic supra fuzzy topological space by using an intuitionistic fuzzy bitopological space, and examined the correlation between the intuitionistic supra fuzzy closure space and the intuitionistic supra fuzzy topological space. Turanlı presented several primary outcomes of intuitionistic fuzzy supra topological space, including connectedness concepts and compactness, [37]. S.A.El-Sheikh and A.M.Abd-El-Latif [16] hybridized supra topological space and soft theory in 2014 to introduce the idea of soft supra topological spaces. Aras [20] proposed intuitionistic fuzzy soft supra topological space as an alternative type of soft supra topological spaces. Various researchers explored the findings regarding soft supra topological spaces [8, 9, 10, 11, 12], including supra soft axioms through soft supra topological spaces with respect to the soft points [3], supra soft compactness and supra soft closure operators [17]. Soft axioms via supra soft topological spaces were studied with respect to the ordinary points [6], and the soft points [21]; the generalization of supra soft topological spaces [22], and the concept of supra soft topological ordered spaces [7]. Ozturk [33] proposed the concept of bipolar soft points and examined some of its key properties. However, the representation of a bipolar soft set $\left(F, \widetilde{E}\right)$ as a combination of bipolar soft points provided in [33] is not valid. In order to rectify this flaw, a redefinition of the bipolar soft point was presented in [23]. Shabir and Naz were the first to use the concept of bipolar soft set by joining the ideas of bipolarity with soft set theory, [34]. A structure of a bipolar soft set is created by two mappings, F and G, where $F : E \to P(X)$ and $G : \neg E \to P(X)$, where $\neg E$ represents the "ot set of E". F expresses positive information and G expresses opposite approximation. Subsequently, Shabir and Bakhtawar [36] commenced the study of bipolar soft topological spaces.

This paper firstly introduces bipolar soft supra topological space and defines the concepts of bipolar soft supra closure and bipolar soft supra interior. In addition, the paper establishes a connection between bipolar soft supra topology and bipolar soft topology. Secondly, the paper presents the concepts of bipolar soft supra continuous mapping with the assistance of bipolar soft supra closure and bipolar soft supra interior notions. This research focuses on the concept of bipolar soft supra-compact topological spaces. We demonstrate that the bipolar soft supra-compact space's image remains intact under bipolar soft supra-continuous mapping. Building on bipolar soft supra topological space, we define the concepts of disconnected (connected) and strongly disconnected (strongly connected) spaces and establish several interconnections between them. We also used examples to clarify the relationships among these concepts.

2. Preliminaries

This section presents the essential definitions for bipolar soft sets. In this paper, X represents the initial universe, $\widetilde{E}$ represents the set of all parameters, and $P(X)$ represents the power set of X.

Definition 2.1 ([27]). Consider E to be a set of parameters, $E = \{e_1, e_2, ..., e_n\}$. We define the 'not set' of E as $\neg E = \{\neg e_1, \neg e_2, ..., \neg e_n\}$, denoted by $\neg E$. Here, for each i, $\neg e_i = \textit{not } e_i$.

Definition 2.2 ([34]). A mapping $F : \widetilde{E} = E \cup \neg E \to P(X)$ defines a bipolar soft set $\left(F, \widetilde{E}\right)$ over X in which $F(e) \cap F(\neg e) = \varnothing$ holds for every $e \in \widetilde{E}$.

Definition 2.3 ([34]). Consider two bipolar soft sets over X, $\left(F, \widetilde{E}\right)$ and $\left(G, \widetilde{E}\right)$. We refer to $\left(F, \widetilde{E}\right)$ as a bipolar soft subset of $\left(G, \widetilde{E}\right)$ if and only if $F(e) \subset G(e)$ for every $e \in E$ and $G(\neg e) \subset F(\neg e)$ for each $\neg e \in \neg E$. This relationship is denoted by $\left(F, \widetilde{E}\right) \subset \left(G, \widetilde{E}\right)$, meaning that $\left(F, \widetilde{E}\right)$ is a bipolar soft subset of $\left(G, \widetilde{E}\right)$. We refer to $\left(F, \widetilde{E}\right)$ as bipolar soft equal to $\left(G, \widetilde{E}\right)$ if and only if $\left(F, \widetilde{E}\right)$ is a bipolar soft subset of $\left(G, \widetilde{E}\right)$ and $\left(G, \widetilde{E}\right)$ is a bipolar soft subset of $\left(F, \widetilde{E}\right)$.

Definition 2.4 ([34]). Consider two bipolar soft sets $\left(F, \widetilde{E}\right)$ and $\left(G, \widetilde{E}\right)$ over the universe set X. Their bipolar soft union results in a bipolar soft set $\left(H, \widetilde{E}\right)$ over X where $H(e) = (F \cup G)(e) = F(e) \cup G(e)$ for all $e \in E$, and $H(\neg e) = (F \cup G)(\neg e) = F(\neg e) \cap G(\neg e)$ for all $\neg e \in \neg E$.

Definition 2.5 ([34]). Consider two bipolar soft sets $\left(F, \widetilde{E}\right)$ and $\left(G, \widetilde{E}\right)$ over the universe set X. The bipolar soft intersection of $\left(F, \widetilde{E}\right)$ and $\left(G, \widetilde{E}\right)$ is then transformed to a bipolar soft set $\left(H, \widetilde{E}\right)$ over the domain X. Specifically, $H(e) = (F \cap G)(e) = F(e) \cap G(e)$ for each $e \in E$ and $H(\neg e) = (F \cap G)(\neg e) = F(\neg e) \cup G(\neg e)$ for each $\neg e \in \neg E$.

Definition 2.6 ([34]). Let $\left(F, \widetilde{E}\right)$ be a bipolar soft set over X. Then bipolar soft complement of $\left(F, \widetilde{E}\right)$, denoted by $\left(F, \widetilde{E}\right)^c = \left(F^c, \widetilde{E}\right)$, where $F^c : \widetilde{E} \to P(X)$ is a mapping defined by $F^c(e) = F(\neg e)$ for each $e \in E$ and $F^c(\neg e) = F(e)$ for each $\neg e \in \neg E$.

Definition 2.7 ([34])**.** A bipolar soft set $\left(F, \widetilde{E}\right)$ over X is considered a null bipolar soft set, denoted by $\left(\Phi, \widetilde{E}\right)$, if $F(e) = \varnothing$ for each $e \in E$ and $F(\neg e) = X$ for each $\neg e \in \neg E$.

Definition 2.8 ([34])**.** A bipolar soft set $\left(F, \widetilde{E}\right)$ over X is said to be an *absolute bipolar soft set*, denoted by $\left(\widetilde{X}, \widetilde{E}\right)$, if $F(e) = X$ for each $e \in E$ and $F(\neg e) = \varnothing$ for each $\neg e \in \neg E$.

Definition 2.9 ([36])**.** Let $\widetilde{\tau}$ be the collection of bipolar soft sets over X with $\widetilde{E}$. Then $\widetilde{\tau}$ is said to be a *bipolar soft topology* on X if

1) $\left(\Phi, \widetilde{E}\right), \left(\widetilde{X}, \widetilde{E}\right)$ belong to $\widetilde{\tau}$;
2) the union of any number of bipolar soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$;
3) the intersection of any two bipolar soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triplet $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ is called a *bipolar soft topological space* over X. The members of $\widetilde{\tau}$ are said to be *bipolar soft open sets* in X.

Definition 2.10 ([36])**.** Let $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ be a bipolar soft topological space over X. A soft set $\left(F, \widetilde{E}\right)$ over X is said to be a *bipolar soft closed set* in X, if its complement $(F, E)^c$ belongs to $\widetilde{\tau}$.

3. Bipolar Soft Supra Topology

Let $BSS\left(X, \widetilde{E}\right)$ be a family of all bipolar soft sets over X with parameters in $\widetilde{E}$. This section defines key concepts such as bipolar soft supra topology, bipolar soft supra open set, a bipolar soft supra closed set, bipolar soft supra interior, bipolar soft supra continuous, bipolar soft supra continuous, bipolar soft supra open subcover, bipolar soft supra subspace, disconnected space (connected space), strongly disconnected space (strongly connected space) with some properties and relation between them.

Definition 3.1. A family $\widetilde{\tau}$ of $BSS's$ on X is called a *bipolar soft supra topology* ($BSST$ for short) on X if

(1) $\left(\Phi, \widetilde{E}\right), \left(\widetilde{X}, \widetilde{E}\right)$ belong to $\widetilde{\tau}$,
(2) $\widetilde{\tau}$ is closed under arbitrary union.

Then $\left(X,\widetilde{\tau},\widetilde{E}\right)$ is called *bipolar soft supra topological spaces* ($BSSTS$ for short).

Definition 3.2. Let $\left(X,\widetilde{\tau},\widetilde{E}\right)$ be a bipolar soft supra topological space. Then each member of $\widetilde{\tau}$ is called a *bipolar soft supra open set*.

Definition 3.3. Let $\left(X,\widetilde{\tau},\widetilde{E}\right)$ be a bipolar soft supra topological space. A bipolar soft set $\left(F,\widetilde{E}\right)$ over X is said to be a *bipolar soft supra closed set*, if its complement $\left(F,\widetilde{E}\right)^c$ belongs to $\widetilde{\tau}$.

Let $\mathfrak{F}$ be a family of bipolar soft supra closed sets.

Proposition 3.4. *Let* $\left(X,\widetilde{\tau},\widetilde{E}\right)$ *be a bipolar soft supra topological space. Then*
(1) $\left(\Phi,\widetilde{E}\right),\left(\widetilde{X},\widetilde{E}\right)$ *are bipolar soft supra closed sets,*
(2) $\mathfrak{F}$ *is closed under arbitrary intersection.*

Proof. The proof is obtained from the definition of a bipolar soft supra closed set. □

Remark 3.5. Let $\mathfrak{F}$ be a family that meets above the proposition conditions. Thus the complement of elements of $\mathfrak{F}$ form a bipolar soft supra topology.

Example 3.6. Let $X=\{x_1,x_2,x_3\}$ and $E=\{e_1,e_2\}$. The bipolar soft sets $\left(F_1,\widetilde{E}\right),\left(F_2,\widetilde{E}\right)$ on X are defined as follows:

$$\begin{aligned}F_1(e_1) &= X, F_1(\neg e_1)=\varnothing, F_1(e_2)=\{x_1,x_2\}, F_1(\neg e_2)=\{x_3\},\\ F_2(e_1) &= \{x_1,x_2\}, F_2(\neg e_1)=\{x_3\}, F_2(e_2)=\{x_3\}, F_2(\neg e_2)=\varnothing.\end{aligned}$$

Then $\widetilde{\tau}=\left\{\left(\Phi,\widetilde{E}\right),\left(\widetilde{X},\widetilde{E}\right),\left(F_1,\widetilde{E}\right),\left(F_2,\widetilde{E}\right)\right\}$ is a bipolar soft supra topology on X.

Example 3.7. Let us consider Example 3.6. Next, we define the complements of $\left(F_1,\widetilde{E}\right),\left(F_2,\widetilde{E}\right)$ on X as:

$$\begin{aligned}(F_1)^c(e_1) &= \varnothing, (F_1)^c(\neg e_1)=X, (F_1)^c(e_2)=\{x_3\}, (F_1)^c(\neg e_2)=\{x_1,x_2\},\\ (F_2)^c(e_1) &= \{x_3\}, (F_2)^c(\neg e_1)=\{x_1,x_2\}, (F_2)^c(e_2)=\varnothing, (F_2)^c(\neg e_2)=\{x_3\}.\end{aligned}$$

Thus $\mathfrak{F}=\left\{\left(\Phi,\widetilde{E}\right),\left(\widetilde{X},\widetilde{E}\right),\left(F_1,\widetilde{E}\right)^c,\left(F_2,\widetilde{E}\right)^c\right\}$ is a family of bipolar soft supra closed sets.

Proposition 3.8. *Let* $\left(X,\widetilde{\tau},\widetilde{E}\right)$ *be a bipolar soft supra topological space over* X. *Then*

$$\begin{aligned}\tau_e &= \left\{F(e) : \left(F,\widetilde{E}\right)\in\widetilde{\tau}\right\},\\ \tau_{\neg e} &= \left\{F(\neg e) : \left(F,\widetilde{E}\right)\in\widetilde{\tau}\right\}\end{aligned}$$

define two supra topologies on X *for each* $e\in\widetilde{E}$.

Proof. It is clear. □

It is important to note that τ_e and $\tau_{\neg e}$ are independent of each other.

Remark 3.9. It is evident that bipolar soft supra topology is a natural generalisation of supra soft topology.

Remark 3.10. A bipolar supra topological space is considered as a supra soft topological space that depends on only one parameter.

Example 3.11. Let $X=\{x_1,x_2,x_3\}$ and $E=\{e_1,e_2\}$. The bipolar soft sets $\left(F_1,\widetilde{E}\right),\left(F_2,\widetilde{E}\right)$ on X are defined as follows:

$$\begin{aligned}F_1(e_1) &= \{x_1\}, F_1(\neg e_1)=\{x_2,x_3\}, F_1(e_2)=\{x_2,x_3\}, F_1(\neg e_2)=\{x_1\},\\ F_2(e_1) &= \{x_2,x_3\}, F_2(\neg e_1)=\{x_1\}, F_2(e_2)=\{x_1\}, F_2(\neg e_2)=\varnothing.\end{aligned}$$

Hence $\widetilde{\tau}=\left\{\left(\Phi,\widetilde{E}\right),\left(\widetilde{X},\widetilde{E}\right),\left(F_1,\widetilde{E}\right),\left(F_2,\widetilde{E}\right)\right\}$ is a bipolar soft supra topology on X.

$$\begin{aligned}\tau_{e_1} &= \{F_1(e_1)=\{x_1\}, F_2(e_1)=\{x_2,x_3\},\varnothing,X\},\\ \tau_{e_2} &= \{F_1(e_2)=\{x_2,x_3\}, F_2(e_2)=\{x_1\},\varnothing,X\},\\ \tau_{\neg e_1} &= \{F_1(\neg e_1)=\{x_2,x_3\}, F_2(\neg e_1)=\{x_1\},\varnothing,X\},\\ \tau_{\neg e_2} &= \{F_1(\neg e_2)=\{x_1\}, F_2(\neg e_2)=\varnothing,\varnothing,X\}\end{aligned}$$

are supra topologies on X.

Proposition 3.12. *If* $\widetilde{\tau}$ *is a bipolar soft supra topology on* X, *then*

$$\beta_{\widetilde{\tau}}=\left\{\left(G,\widetilde{E}\right)\in BSS\left(X,\widetilde{E}\right):\forall\left(F,\widetilde{E}\right),\ \left(F,\widetilde{E}\right)\in\widetilde{\tau}\Rightarrow\left(G,\widetilde{E}\right)\cap\left(F,\widetilde{E}\right)\in\widetilde{\tau}\right\}$$

is a bipolar soft topology on X *and* $\beta_{\widetilde{\tau}}\subset\widetilde{\tau}$.

Proof. 1. $\left(F,\widetilde{E}\right)\in\widetilde{\tau}\Rightarrow\left(\Phi,\widetilde{E}\right)\cap\left(F,\widetilde{E}\right)=\left(\Phi,\widetilde{E}\right)$ and $\left(\widetilde{X},\widetilde{E}\right)\cap\left(F,\widetilde{E}\right)=\left(F,\widetilde{E}\right)\in\widetilde{\tau}$. Hence $\left(\Phi,\widetilde{E}\right),\left(\widetilde{X},\widetilde{E}\right)\in\beta_{\widetilde{\tau}}$.

2. Let $\left\{\left(G_i,\widetilde{E}\right): i\in I\right\}\subset\beta_{\widetilde{\tau}}$ and $\left(F,\widetilde{E}\right)\in\widetilde{\tau}$. Since
$\left(\underset{i\in I}{\cup}\left(G_i,\widetilde{E}\right)\right)\cap\left(F,\widetilde{E}\right)=\underset{i\in I}{\cup}\left(\left(G_i,\widetilde{E}\right)\cap\left(F,\widetilde{E}\right)\right)\in\widetilde{\tau}$,
so we have $\underset{i\in I}{\cup}\left(G_i,\widetilde{E}\right)\in\beta_{\widetilde{\tau}}$.
3. Let $\left(G_1,\widetilde{E}\right)$, $\left(G_2,\widetilde{E}\right)\in\beta_{\widetilde{\tau}}$ and $\left(F,\widetilde{E}\right)\in\widetilde{\tau}$. Then for each $e\in\widetilde{E}$,

$$\begin{aligned}&\left(\left[\left(G_1,\widetilde{E}\right)\cap\left(G_2,\widetilde{E}\right)\right]\cap\left(F,\widetilde{E}\right)\right)(e)\\&=\left[\left(G_1\cap G_2\right)(e),\left(G_1\cup G_2\right)(\neg e)\right]\cap\left[F(e),F(\neg e)\right]\\&=\left(\left[\left(G_1\cap G_2\right)\cap F\right](e),\left[\left(G_1\cup G_2\right)\cup F\right](\neg e)\right)\\&=\left(\left[G_1\cap\left(G_2\cap F\right)\right](e),\left[G_1\cup\left(G_2\cup F\right)\right](\neg e)\right).\end{aligned}$$

So we have

$$\left[\left(G_1,\widetilde{E}\right)\cap\left(G_2,\widetilde{E}\right)\right]\cap\left(F,\widetilde{E}\right)=\left(G_1,\widetilde{E}\right)\cap\left[\left(G_2,\widetilde{E}\right)\cap\left(F,\widetilde{E}\right)\right].$$

Since $\left(G_2,\widetilde{E}\right)\cap\left(F,\widetilde{E}\right)\in\widetilde{\tau}$ and $\left(G_1,\widetilde{E}\right)\in\beta_{\widetilde{\tau}}$, $\left(G_1,\widetilde{E}\right)\cap\left[\left(G_2,\widetilde{E}\right)\cap\left(F,\widetilde{E}\right)\right]\in\widetilde{\tau}$. So $\left(G_1,\widetilde{E}\right)\cap\left(G_2,\widetilde{E}\right)\in\beta_{\widetilde{\tau}}$. Thus $\beta_{\widetilde{\tau}}$ is a bipolar soft topology on X.

If $\left(G,\widetilde{E}\right)\in\beta_{\widetilde{\tau}}$, since $\left(\widetilde{X},\widetilde{E}\right)\in\widetilde{\tau}$ and $\left(G,\widetilde{E}\right)\cap\left(\widetilde{X},\widetilde{E}\right)=\left(G,\widetilde{E}\right)\in\widetilde{\tau}$, $\left(G,\widetilde{E}\right)\in\widetilde{\tau}$ is obtained. □

Proposition 3.13. *Consider two bipolar soft supra topological spaces* $\left(X,\widetilde{\tau_1},\widetilde{E}\right)$ *and* $\left(X,\widetilde{\tau_2},\widetilde{E}\right)$ *defined over the same universe* X. *Then, the intersection of these spaces* $\left(X,\widetilde{\tau_1}\cap\widetilde{\tau_2},\widetilde{E}\right)$ *is also a bipolar soft supra topological space over* X.

Proof. The proof is straightforward. □

Remark 3.14. The union of two bipolar soft supra topologies on X may not be a bipolar soft supra topology on X.

Definition 3.15. The $s-cl\left(F,\widetilde{E}\right)$, which stands for the bipolar soft supra closure of $\left(F,\widetilde{E}\right)\in BSS$, is defined as the intersection of all bipolar soft supra closed sets that contain $\left(F,\widetilde{E}\right)$.

Proposition 3.16. *Let* $\left(X,\widetilde{\tau},\widetilde{E}\right)$ *be a bipolar soft supra topological space on* X, $\left(F_1,\widetilde{E}\right)$ *and* $\left(F_2,\widetilde{E}\right)\in BSS's$. *Then*

(1) $s-cl\left(\left(\Phi,\widetilde{E}\right)\right)=\left(\Phi,\widetilde{E}\right)$,

(2) $\left(F_1,\widetilde{E}\right) \subset s-cl\left(F_1,\widetilde{E}\right)$,

(3) $\left(F_1,\widetilde{E}\right)$ *is a bipolar soft supra closed set if and only if* $\left(F_1,\widetilde{E}\right) = s-cl\left(F_1,\widetilde{E}\right)$,

(4) $s-cl\left(F_1,\widetilde{E}\right) = s-cl\left(s-cl\left(F_1,\widetilde{E}\right)\right)$,

(5) $\left(F_1,\widetilde{E}\right) \subset \left(F_2,\widetilde{E}\right)$ *implies that* $s-cl\left(F_1,\widetilde{E}\right) \subset s-cl\left(F_2,\widetilde{E}\right)$,

(6) $s-cl\left(F_1,\widetilde{E}\right) \cup s-cl\left(F_2,\widetilde{E}\right) = s-cl\left(\left(F_1,\widetilde{E}\right) \cup \left(F_2,\widetilde{E}\right)\right)$.

Proof. The proof is obtained from the definition of bipolar soft supra closure. □

Example 3.17. Let $X = \{x_1,x_2,x_3\}$ and $E = \{e_1,e_2\}$. The bipolar soft sets $\left(F_1,\widetilde{E}\right)$,$\left(F_2,\widetilde{E}\right)$ on X are defined as follows:

$$\begin{aligned} F_1(e_1) &= \{x_2\}, F_1(\neg e_1) = \{x_1,x_3\}, F_1(e_2) = \{x_3\}, F_1(\neg e_2) = \{x_1\}, \\ F_2(e_1) &= \{x_1,x_3\}, F_2(\neg e_1) = \varnothing, F_2(e_2) = \{x_1,x_2\}, F_2(\neg e_2) = \varnothing. \end{aligned}$$

Hence $\widetilde{\tau} = \left\{\left(\Phi,\widetilde{E}\right), \left(\widetilde{X},\widetilde{E}\right), \left(F_1,\widetilde{E}\right), \left(F_2,\widetilde{E}\right)\right\}$ is a bipolar soft supra topology on X. Let us consider

$$F(e_1) = \{x_1\}, F(\neg e_1) = \{x_2,x_3\}, F(e_2) = \{x_1\}, F(\neg e_2) = \{x_3\}.$$

Then $s-cl\left(F,\widetilde{E}\right) = \left(F_1,\widetilde{E}\right)^c$ is obtained.

Definition 3.18. The bipolar soft supra interior of $\left(F,\widetilde{E}\right) \in BSS$, denoted by $s-int\left(F,\widetilde{E}\right)$, is defined as the union of all bipolar soft supra open sets that are included in $\left(F,\widetilde{E}\right)$.

Proposition 3.19. *Let* $\left(X,\widetilde{\tau},\widetilde{E}\right)$ *be a bipolar soft supra topological space on* X, $\left(F_1,\widetilde{E}\right)$ *and* $\left(F_2,\widetilde{E}\right) \in BSS's$. *Then*

(1) $s-int\left(\left(\widetilde{X},\widetilde{E}\right)\right) = \left(\widetilde{X},\widetilde{E}\right)$,

(2) $s-int\left(F_1,\widetilde{E}\right) \subset \left(F_1,\widetilde{E}\right)$,

(3) $\left(F_1,\widetilde{E}\right)$ *is a bipolar soft supra open set if and only if* $s-int\left(F_1,\widetilde{E}\right) = \left(F_1,\widetilde{E}\right)$,

(4) $s-int\left(F_1,\widetilde{E}\right) = s-int\left(s-int\left(F_1,\widetilde{E}\right)\right)$,

(5) $\left(F_1,\widetilde{E}\right) \subset \left(F_2,\widetilde{E}\right)$ *implies that* $s-int\left(F_1,\widetilde{E}\right) \subset s-int\left(F_2,\widetilde{E}\right)$,

(6) $s-int\left(F_1,\widetilde{E}\right) \cap s-int\left(F_2,\widetilde{E}\right) \neq s-int\left(\left(F_1,\widetilde{E}\right) \cap \left(F_2,\widetilde{E}\right)\right)$.

Proof. The proof is obtained from the definition of bipolar soft supra interior. □

Example 3.20. Let $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1\}$. The bipolar soft sets $\left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right), \left(F_3, \widetilde{E}\right)$ on X are defined as follows:

$$\begin{aligned} F_1(e_1) &= \{x_2, x_3\}, F_1(\neg e_1) = \{x_1\}, \\ F_2(e_1) &= \{x_2\}, F_2(\neg e_1) = \{x_4\} \\ F_3(e_1) &= \{x_2, x_3\}, F_3(\neg e_1) = \varnothing. \end{aligned}$$

Hence $\widetilde{\tau} = \left\{\left(\Phi, \widetilde{E}\right), \left(\widetilde{X}, \widetilde{E}\right), \left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right), \left(F_3, \widetilde{E}\right)\right\}$ is a bipolar soft supra topology on X. Let us consider

$$\begin{aligned} F(e_1) &= \{x_2, x_3, x_4\}, F(\neg e_1) = \{x_1\}, \\ G(e_1) &= \{x_2\}, G(\neg e_1) = \varnothing. \end{aligned}$$

Then $s\!-\!int\left(F, \widetilde{E}\right) = \left(F_1, \widetilde{E}\right)$ and $s\!-\!int\left(G, \widetilde{E}\right) = \left(F_2, \widetilde{E}\right)$. So $s\!-\!int\left(F, \widetilde{E}\right) \cap s - int\left(G, \widetilde{E}\right) = \left(H, \widetilde{E}\right)$, where

$$H(e_1) = \{x_2\}, H(\neg e_1) = \{x_1, x_4\}.$$

But $\left(F, \widetilde{E}\right) \cap \left(G, \widetilde{E}\right) = \left(M, \widetilde{E}\right)$, where

$$M(e_1) = \{x_2\}, M(\neg e_1) = \{x_1\},$$

and so $s - int\left(\left(F, \widetilde{E}\right) \cap \left(G, \widetilde{E}\right)\right) = \left(\Phi, \widetilde{E}\right)$. This implies that $s - int\left(F, \widetilde{E}\right) \cap s - int\left(G, \widetilde{E}\right) \neq s - int\left(\left(F, \widetilde{E}\right) \cap \left(G, \widetilde{E}\right)\right)$.

Definition 3.21 ([23])**.** Let $\left(F, \widetilde{E}\right)$ be a bipolar soft set over X. Then $\left(F, \widetilde{E}\right)$ is said to be *bipolar soft point* if there exist $x, y \in X$, (It is possible to take $x = y$), $e \in E$ and $\neg e \in \neg E$ such that

$$x_e^y(e') = \begin{cases} \varnothing, & \text{if } e \neq e' \\ \{x\}, & \text{if } e = e'. \end{cases}$$

and

$$x_e^y(\neg e') = \begin{cases} X, & \text{if } e \neq e' \\ X \setminus \{x, y\}, & \text{if } e = e'. \end{cases}$$

A bipolar soft point can be denoted briefly by x_e^y.

Remark 3.22. Every bipolar soft set can be written as a union of bipolar soft points.

Theorem 3.23. *Let* $\left(F, \widetilde{E}\right)$ *be a bipolar soft set and* x_e^y *be a bipolar soft point of* $\left(X, \widetilde{\tau}, \widetilde{E}\right)$. *Then* $x_e^y \in s - cl\left(F, \widetilde{E}\right)$ *if and only if* $\left(F, \widetilde{E}\right) \cap \left(G, \widetilde{E}\right) \neq \left(\Phi, \widetilde{E}\right)$ *for each bipolar supra soft open set* $\left(G, \widetilde{E}\right)$ *that contains* x_e^y.

Proof. It is clear. □

Definition 3.24 ([18])**.** Let $\left(X, \widetilde{E}\right)$ and $\left(Y, \widetilde{E'}\right)$ be two bipolar soft sets. If $f : X \to Y$ is an injective mapping, $\varphi : \widetilde{E} \to \widetilde{E'}$ is a mapping such that $\varphi(\neg e) = \neg\varphi(e)$, then $(f, \varphi) : \left(X, \widetilde{E}\right) \to \left(Y, \widetilde{E'}\right)$ is called a *bipolar soft mapping* of bipolar soft sets.

Definition 3.25 ([18])**.** Let $(f, \varphi) : \left(X, \widetilde{E}\right) \to \left(Y, \widetilde{E'}\right)$ be a bipolar soft mapping of bipolar soft sets and $\left(F, \widetilde{E}\right)$ be a bipolar soft set over X. Then image of the bipolar soft set $\left(F, \widetilde{E}\right)$ under the mapping (f, φ) is a bipolar soft set in Y given as, for each $e' \in \widetilde{E'}$,

$$\begin{aligned} (f, \varphi)\left(\left(F, \widetilde{E}\right)\right)(e') &= f\left(\bigcup_{e \in \varphi^{-1}(e') \cap E} (F(e))\right), \\ (f, \varphi)\left(\left(F, \widetilde{E}\right)\right)(\neg e') &= f\left(\bigcap_{\neg e \in \varphi^{-1}(\neg e') \cap \neg E} (F(\neg e))\right). \end{aligned}$$

Definition 3.26 ([18])**.** Let $(f, \varphi) : \left(X, \widetilde{E}\right) \to \left(Y, \widetilde{E'}\right)$ be a bipolar soft mapping of bipolar soft sets and $\left(G, \widetilde{E'}\right)$ be a bipolar soft set over Y. Then inverse image of the bipolar soft set $\left(G, \widetilde{E'}\right)$ under the mapping (f, φ) is a bipolar soft set in X given as, for each $e \in \widetilde{E}$,

$$\begin{aligned} (f, \varphi)^{-1}\left(\left(G, \widetilde{E'}\right)\right)(e) &= f^{-1}(G(\varphi(e))), \\ (f, \varphi)^{-1}\left(\left(G, \widetilde{E'}\right)\right)(\neg e) &= f^{-1}(G(\varphi(\neg e))). \end{aligned}$$

Definition 3.27. Let $\left(X, \widetilde{\tau}, \widetilde{E}\right)$,$\left(Y, \widetilde{\tau'}, \widetilde{E'}\right)$ be two bipolar soft supra topological spaces and $(f, \varphi) : \left(X, \widetilde{\tau}, \widetilde{E}\right) \to \left(Y, \widetilde{\tau'}, \widetilde{E'}\right)$ be a bipolar soft mapping. Then the mapping (f, φ) is defined to be a bipolar soft supra continuous if for each bipolar soft set BSS in $\widetilde{\tau'}$, its inverse image is a bipolar soft set in $\widetilde{\tau}$.

Theorem 3.28. *Let* $\left(X, \widetilde{\tau}, \widetilde{E}\right)$, $\left(Y, \widetilde{\tau'}, \widetilde{E'}\right)$ *be two bipolar soft supra topological spaces and* $(f, \varphi) : \left(X, \widetilde{\tau}, \widetilde{E}\right) \to \left(Y, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft mapping. Then the followings are equivalent:*

(1) (f,φ) *is a bipolar soft supra continuous mapping,*

(2) For each bipolar soft supra closed set $\left(H,\widetilde{E'}\right)$ *on* Y, $(f,\varphi)^{-1}\left(\left(H,\widetilde{E'}\right)\right)$ *is a bipolar soft supra closed set on* X,

(3) $(f,\varphi)\left(s-cl\left(F,\widetilde{E}\right)\right)\subset s-cl\left((f,\varphi)\left(F,\widetilde{E}\right)\right), \forall\left(F,\widetilde{E}\right)\in\left(X,\widetilde{\tau},\widetilde{E}\right)$,

(4) $s{-}cl\left((f,\varphi)^{-1}\left(H,\widetilde{E'}\right)\right)\subset (f,\varphi)^{-1}\left(s-cl\left(H,\widetilde{E'}\right)\right), \forall\left(H,\widetilde{E'}\right)\in\left(Y,\widetilde{\tau'},\widetilde{E'}\right)$,

(5) $(f,\varphi)^{-1}\left(s-int\left(H,\widetilde{E'}\right)\right)\subset s{-}int\left((f,\varphi)^{-1}\left(H,\widetilde{E'}\right)\right), \forall\left(H,\widetilde{E'}\right)\in\left(Y,\widetilde{\tau'},\widetilde{E'}\right)$.

Proof. It is clear. □

Definition 3.29. Let $\left(X,\widetilde{\tau},\widetilde{E}\right)$ be a bipolar soft supra topological space. A family $\left\{\left(F_i,\widetilde{E}\right):i\in I\right\}$ of bipolar soft supra open sets in X is called a *bipolar soft supra open cover* of X, if it satisfies the following condition:

$$\tilde{\underset{i}{\cup}}\left(F_i,\widetilde{E}\right)=\left(\widetilde{X},\widetilde{E}\right)$$

This means that

$$\underset{i}{\cup}(F_i)(e)=X, \underset{i}{\cap}(F_i)(\neg e)=\varnothing$$

is satisfied for each $e\in\widetilde{E}$.

Definition 3.30. A family $\left\{\left(F_i,\widetilde{E}\right):i\in J\subset I\right\}$ is called a *bipolar soft supra open subcover* of X, i.e. $\tilde{\underset{i\in J}{\cup}}\left(F_i,\widetilde{E}\right)=\left(\widetilde{X},\widetilde{E}\right)$ is satisfied.

Definition 3.31. Let $\left(X,\widetilde{\tau},\widetilde{E}\right)$ be a bipolar soft supra topological space. $\left(X,\widetilde{\tau},\widetilde{E}\right)$ is called a *bipolar soft supra compact topological space* if every bipolar soft supra open cover of X has a finite subcover.

Now we can constitute two supra topologies as $\widetilde{\tau}=(\tau_0,\tau_c)$ on $\left(X,\widetilde{\tau},\widetilde{E}\right)$ where τ_0 is a family of all soft open sets and τ_c is a family of all soft closed sets.

Theorem 3.32. *If* $\left(X,\widetilde{\tau},\widetilde{E}\right)$ *is bipolar soft supra compact topological space, then* (X,τ_0,E) *is a soft supra compact topological space. However, the topological space* $(X,\tau_c,\neg E)$ *is not soft supra compact topological space.*

Proof. Let $\left(X,\widetilde{\tau},\widetilde{E}\right)$ be a bipolar soft supra compact topological space and the family of soft sets $\{(F_i,E):i\in I\}$ be a soft open covering of (X,τ_0,E). Let us add soft open sets $(F_i,\neg E)$ to this family. Since $F_i(\neg e)\subset X\backslash F_i(e)$ and $\underset{i}{\cup}F_i(e)=\varnothing$, $\underset{i}{\cap}F_i(\neg e)=\varnothing$ are obtained. Thus the family $\{(F_i,E\cup\neg E)\}_{i\in I}$ is a open covering of

$\left(X,\widetilde{\tau},\widetilde{E}\right)$. Since $\left(X,\widetilde{\tau},\widetilde{E}\right)$ is a bipolar soft supra compact topological space, there exists finite subcovering of this covering such that $\{(F_{i_k}, E\cup\neg E)\}_{k=\overline{1,n}}$. Hence the family $\{(F_{i_k}, E)\}_{k=\overline{1,k}}$ is a finite subcovering of (X,τ_0,E). □

The subsequent example demonstrates that $(X,\tau_c,\neg E)$ fails to qualify as a soft supra compact topological space.

Example 3.33. Let $X=\{x_1,x_2,x_3\}$ and $E=\{e_1,e_2\}$. The bipolar soft open sets $\left(F_1,\widetilde{E}\right),\left(F_2,\widetilde{E}\right)$ on X are defined as follows:

$$\begin{aligned}
F_1(e_1) &= \{x_1\}, F_1(\neg e_1)=\{x_2\},\\
F_1(e_2) &= \{x_1,x_2\}, F_1(\neg e_2)=\{x_3\},\\
F_2(e_1) &= \{x_3\}, F_2(\neg e_1)=\{x_1\},\\
F_2(e_2) &= \{x_2\}, F_2(\neg e_2)=\{x_1\}.
\end{aligned}$$

Altough $(F_1,\neg E)$, $(F_2,\neg E)$ is a covering of $(X,\tau_c,\neg E)$, but the bipolar soft open sets $\left(F_1,\widetilde{E}\right),\left(F_2,\widetilde{E}\right)$ do not cover of $\left(X,\widetilde{\tau},\widetilde{E}\right)$.

Theorem 3.34. *Let $\left(X,\widetilde{\tau},\widetilde{E}\right)$, $\left(Y,\widetilde{\tau'},\widetilde{E'}\right)$ be two bipolar soft supra topological spaces, $f:X\to Y$ be an injective mapping and $(f,\varphi):\left(X,\widetilde{\tau},\widetilde{E}\right)\to\left(Y,\widetilde{\tau'},\widetilde{E'}\right)$ be a bipolar soft supra continuous mapping. If $\left(X,\widetilde{\tau},\widetilde{E}\right)$ is a bipolar soft supra compact topological space, then $(f,\varphi)\left(\left(X,\widetilde{\tau},\widetilde{E}\right)\right)$ is also a compact bipolar soft supra topological space.*

Proof. It is clear. □

Definition 3.35. Let $\left(X,\widetilde{\tau},\widetilde{E}\right)$ be a bipolar soft supra topological space and $Y\subset X$. Then the family

$$\widetilde{\tau}_Y=\left\{\widetilde{Y}\cap\left(F,\widetilde{E}\right):\left(F,\widetilde{E}\right)\in\widetilde{\tau}\right\}$$

is called a *bipolar soft supra topology* on Y and $\left(Y,\widetilde{\tau}_Y,\widetilde{E}\right)$ is called a *bipolar soft supra subspace* of $\left(X,\widetilde{\tau},\widetilde{E}\right)$.

Theorem 3.36. *Let $\left(Y,\widetilde{\tau}_Y,\widetilde{E}\right)$ be a bipolar soft supra subspace of $\left(X,\widetilde{\tau},\widetilde{E}\right)$. Then $\left(F,\widetilde{E}\right)$ is a bipolar soft supra closed subset of $\left(Y,\widetilde{\tau}_Y,\widetilde{E}\right)$ if and only if there exists a bipolar soft supra closed subset $\left(G,\widetilde{E}\right)$ of $\left(X,\widetilde{\tau},\widetilde{E}\right)$ such that $\left(F,\widetilde{E}\right)=\left(G,\widetilde{E}\right)\cap\widetilde{Y}$.*

Proof. It is clear. □

Definition 3.37. Let $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ be a bipolar soft supra topological space. A space $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ is defined as a disconnected space if there exist two sets $\left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right) \in \widetilde{\tau}$ such that $F_1(e) \cup F_2(e) = X$ and $F_1(e) \cap F_2(e) = \varnothing$ for each $e \in E$. If there are no such sets $\left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right) \in \widetilde{\tau}$, then the space is said to be a connected space.

Definition 3.38. Let $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ be a bipolar soft supra topological space. If there exist $\left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right) \in \widetilde{\tau}$ such that $\left(F_1, \widetilde{E}\right) \cup \left(F_2, \widetilde{E}\right) = \left(\widetilde{X}, \widetilde{E}\right)$ and $\left(F_1, \widetilde{E}\right) \cap \left(F_2, \widetilde{E}\right) = \left(\Phi, \widetilde{E}\right)$, then $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ is said to be a *strongly disconnected space.* Otherwise, this space is said to be a *strongly connected space.*

Remark 3.39. A strongly disconnected space is a disconnected space. However, the opposite does not hold.

Example 3.40. Let $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2\}$. The bipolar soft open sets $\left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right)$ on X are defined as follows:

$$\begin{aligned}
F_1(e_1) &= \{x_1, x_2\}, F_1(\neg e_1) = \{x_3\}, \\
F_1(e_2) &= \{x_2, x_4\}, F_1(\neg e_2) = \{x_1\}, \\
F_2(e_1) &= \{x_3, x_4\}, F_2(\neg e_1) = \{x_2\}, \\
F_2(e_2) &= \{x_1, x_3\}, F_2(\neg e_2) = \{x_4\}.
\end{aligned}$$

Then $\widetilde{\tau} = \left\{\left(\Phi, \widetilde{E}\right), \left(\widetilde{X}, \widetilde{E}\right), \left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right)\right\}$ is a bipolar soft supra topology on X and $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ is disconnected space. Since $\left(F_1, \widetilde{E}\right) \cup \left(F_2, \widetilde{E}\right) = \left(\widetilde{X}, \widetilde{E}\right)$ and $\left(F_1, \widetilde{E}\right) \cap \left(F_2, \widetilde{E}\right) \neq \left(\Phi, \widetilde{E}\right)$, it is not strongly disconnected space.

Remark 3.41. A connected space is also a strongly connected space.

Example 3.42. Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. The bipolar soft open sets $\left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right)$ on X are defined as follows:

$$\begin{aligned}
F_1(e_1) &= \{x_1, x_2\}, F_1(\neg e_1) = \{x_3\}, \\
F_1(e_2) &= \{x_2, x_3\}, F_1(\neg e_2) = \varnothing, \\
F_2(e_1) &= \{x_1, x_3\}, F_2(\neg e_1) = \{x_2\}, \\
F_2(e_2) &= \{x_1, x_2\}, F_2(\neg e_2) = \{x_3\}.
\end{aligned}$$

Then $\widetilde{\tau} = \left\{\left(\Phi, \widetilde{E}\right), \left(\widetilde{X}, \widetilde{E}\right), \left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right)\right\}$ is a bipolar soft supra topology on X and $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ is connected space. It is also strongly connected space.

Theorem 3.43. *Let* $\left(X,\widetilde{\tau},\widetilde{E}\right)$ *be a bipolar soft supra topological space. Then* $\left(X,\widetilde{\tau},\widetilde{E}\right)$ *is a disconnected space if and only if there exist two bipolar soft supra closed sets* $\left(G_1,\widetilde{E}\right),\left(G_2,\widetilde{E}\right)$ *with* $G_1(\neg e)\neq\varnothing, G_2(\neg e)\neq\varnothing$ *for some* $e\in E$ *such that* $G_1(\neg e)\cup G_2(\neg e)=X$ *and* $G_1(\neg e)\cap G_2(\neg e)=\varnothing$ *for each* $\neg e\in\neg E$.

Proof. Suppose that $\left(X,\widetilde{\tau},\widetilde{E}\right)$ is a disconnected space. Then there exist $\left(F_1,\widetilde{E}\right)$, $\left(F_2,\widetilde{E}\right)\in\widetilde{\tau}$ such that $F_1(e)\cup F_2(e)=X$, $F_1(e)\cap F_2(e)=\varnothing$ and $F_1(e)\neq\varnothing$, $F_2(e)\neq\varnothing$ for each $e\in E$. Since $\left(F_1^c,\widetilde{E}\right),\left(F_2^c,\widetilde{E}\right)$ are bipolar soft supra closed sets, $\varnothing\neq F_1^c(\neg e)=F_1(e)$, $\varnothing\neq F_2^c(\neg e)=F_2(e)$ and $F_1^c(\neg e)\cup F_2^c(\neg e)=X$, $F_1^c(\neg e)\cap F_2^c(\neg e)=\varnothing$ are satisfied.

Conversely, suppose that there exist two bipolar soft supra closed sets $\left(G_1,\widetilde{E}\right)$, $\left(G_2,\widetilde{E}\right)$ with $G_1(\neg e)\neq\varnothing, G_2(\neg e)\neq\varnothing$ for some $e\in E$ such that $G_1(\neg e)\cup G_2(\neg e)=X$ and $G_1(\neg e)\cap G_2(\neg e)=\varnothing$ for each $\neg e\in\neg E$. Then $\left(G_1^c,\widetilde{E}\right),\left(G_2^c,\widetilde{E}\right)$ are bipolar soft supra open sets with $\varnothing\neq G_1^c(e)=G_1(\neg e)$, $\varnothing\neq G_2^c(e)=G_2(\neg e)$ for some $e\in E$ such that $G_1^c(e)\cup G_2^c(e)=X$ and $G_1^c(e)\cap G_2^c(e)=\varnothing$ for each $e\in E$. Thus $\left(X,\widetilde{\tau},\widetilde{E}\right)$ is a disconnected space. □

Theorem 3.44. *Let* $\left(X,\widetilde{\tau},\widetilde{E}\right)$ *be a bipolar soft supra topological space. Then* $\left(X,\widetilde{\tau},\widetilde{E}\right)$ *is a strongly disconnected space if and only if there exist two bipolar soft supra closed sets* $\left(G_1,\widetilde{E}\right),\left(G_2,\widetilde{E}\right)$ *such that* $\left(G_1,\widetilde{E}\right)\cup\left(G_2,\widetilde{E}\right)=\left(\widetilde{X},\widetilde{E}\right)$ *and* $\left(G_1,\widetilde{E}\right)\cap\left(G_2,\widetilde{E}\right)=\left(\Phi,\widetilde{E}\right)$.

Proof. Suppose that $\left(X,\widetilde{\tau},\widetilde{E}\right)$ is a strongly disconnected space. Then there exist $\left(F_1,\widetilde{E}\right),\left(F_2,\widetilde{E}\right)\in\widetilde{\tau}$ such that $\left(F_1,\widetilde{E}\right)\cup\left(F_2,\widetilde{E}\right)=\left(\widetilde{X},\widetilde{E}\right)$ and $\left(F_1,\widetilde{E}\right)\cap\left(F_2,\widetilde{E}\right)=\left(\Phi,\widetilde{E}\right)$. Since $\left(G_1,\widetilde{E}\right)=\left(F_1,\widetilde{E}\right)^c$ and $\left(G_2,\widetilde{E}\right)=\left(F_2,\widetilde{E}\right)^c$ are bipolar soft supra closed sets, $G_1(e)\cup G_2(e)=F_1^c(e)\cup F_2^c(e)=F_1(\neg e)\cup F_2(\neg e)$ for each $e\in E$. Also from the condition $\left(F_1,\widetilde{E}\right)\cap\left(F_2,\widetilde{E}\right)=\left(\Phi,\widetilde{E}\right)$,

$$\begin{aligned}F_1(e)\cap F_2(e) &= \varnothing,\\ F_1(\neg e)\cup F_2(\neg e) &= X=G_1(e)\cup G_2(e)\end{aligned}$$

for each $e\in E$. Thus

$$G_1(\neg e)\cap G_2(\neg e)=F_1^c(\neg e)\cap F_2^c(\neg e)=F_1(e)\cap F_2(e)=\varnothing,$$

i.e. $\left(G_1, \widetilde{E}\right) \cup \left(G_2, \widetilde{E}\right) = \left(\widetilde{X}, \widetilde{E}\right)$ is obtained. Also,

$$G_1(e) \cap G_2(e) = F_1^c(e) \cap F_2^c(e) = F_1(\neg e) \cap F_2(\neg e).$$

Since $\left(F_1, \widetilde{E}\right) \cup \left(F_2, \widetilde{E}\right) = \left(\widetilde{X}, \widetilde{E}\right)$,

$$F_1(e) \cup F_2(e) = X, F_1(\neg e) \cap F_2(\neg e) = \varnothing$$

is satisfied for each $e \in E$, i.e. $G_1(e) \cap G_2(e) = \varnothing$. Also since

$$G_1(\neg e) \cup G_2(\neg e) = F_1^c(\neg e) \cup F_2^c(\neg e) = F_1(e) \cup F_2(e) = X,$$

$\left(G_1, \widetilde{E}\right) \cap \left(G_2, \widetilde{E}\right) = \left(\Phi, \widetilde{E}\right)$ is obtained.

Conversely let $\left(G_1, \widetilde{E}\right), \left(G_2, \widetilde{E}\right)$ be two bipolar soft supra closed sets such that $\left(G_1, \widetilde{E}\right) \cup \left(G_2, \widetilde{E}\right) = \left(\widetilde{X}, \widetilde{E}\right)$ and $\left(G_1, \widetilde{E}\right) \cap \left(G_2, \widetilde{E}\right) = \left(\Phi, \widetilde{E}\right)$. Then $\left(F_1, \widetilde{E}\right) = \left(G_1, \widetilde{E}\right)^c$, $\left(F_2, \widetilde{E}\right) = \left(G_2, \widetilde{E}\right)^c$ are bipolar soft supra open sets and $\left(F_1, \widetilde{E}\right) \cup \left(F_2, \widetilde{E}\right) = \left(\widetilde{X}, \widetilde{E}\right)$ and $\left(F_1, \widetilde{E}\right) \cap \left(F_2, \widetilde{E}\right) = \left(\Phi, \widetilde{E}\right)$ are immediately obtained. □

Proposition 3.45. *If $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ is a disconnected space, then the topological space $\left(X, (\tau_0)_e\right)$ is disconnected, but the topological space $\left(X, (\tau_c)_e\right)$ is not disconnected for each $e \in \widetilde{E}$.*

Example 3.46. We consider Example 3.6. Then $\left(X, (\tau_0)_e\right)$ is disconnected, but $\left(X, (\tau_c)_e\right)$ is not disconnected.

Proposition 3.47. *If $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ is strongly disconnected, then the topological space $\left(X, (\tau_0)_e\right)$, $\left(X, (\tau_c)_e\right)$ are disconnected spaces.*

Proof. It is clear. □

Theorem 3.48. *Let $\left(X, \widetilde{\tau}, \widetilde{E}\right)$ be disconnected space and $\left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right)$ are bipolar soft supra open sets which satisfying disconnected condition. If $\left(Y, \widetilde{\tau}_Y, \widetilde{E}\right)$ is a bipolar soft supra connected, then $Y \subset F_1(e)$ or $Y \subset F_2(e)$ for each $e \in E$.*

Proof. Let $\left(F_1, \widetilde{E}\right), \left(F_2, \widetilde{E}\right)$ be two bipolar soft supra open sets such that

$$F_1(e) \cup F_2(e) = X,\ F_1(e) \cap F_2(e) = \varnothing,\ F_1(e) \neq \varnothing, F_2(e) \neq \varnothing$$

for each $e \in E$. Then $\left(F_1, \widetilde{E}\right) \cap \widetilde{Y}$ and $\left(F_2, \widetilde{E}\right) \cap \widetilde{Y}$ are bipolar soft supra open sets on $\left(Y, \widetilde{\tau}_Y, \widetilde{E}\right)$. For each $e \in E$,

$$\begin{aligned} Y \cap [F_1(e) \cup F_2(e)] &= [Y \cap F_1(e)] \cup [Y \cap F_2(e)] \ldots(1) \\ Y \cap [F_1(e) \cap F_2(e)] &= [Y \cap F_1(e)] \cap [Y \cap F_2(e)] \ldots(2) \end{aligned}$$

are obtained. Since $\left(Y, \widetilde{\tau}_Y, \widetilde{E}\right)$ is a bipolar soft supra connected, $Y \cap F_1(e) = \varnothing$ or $Y \cap F_2(e) = \varnothing$ for each $e \in E$.

If $Y \cap F_1(e) = \varnothing$ for each $e \in E$, then from (1) $Y \cap F_2(e) = Y$. Thus $Y \subset F_2(e)$ is obtained.

If $Y \cap F_2(e) = \varnothing$ for each $e \in E$, then from (2) $Y \cap F_1(e) = Y$. Thus $Y \subset F_1(e)$ is obtained. $\square$

4. Conclusion

We have introduced bipolar soft supra topological space and defined the concepts of bipolar soft supra closure and bipolar soft supra interior. We have also established a connection between bipolar soft supra topology and bipolar soft topology. In addition, we have presented the concepts of bipolar soft supra continuous mapping with the assistance of bipolar soft supra closure and bipolar soft supra interior notions. We have focused on the concept of bipolar soft supra-compact topological spaces and demonstrated that the bipolar soft supra-compact space's image remains intact under bipolar soft supra-continuous mapping. In the end, we have defined the concepts of disconnected (connected) and strongly disconnected (strongly connected) spaces and establish several interconnections between them by using examples to clarify the relationships among these concepts.

References

1. S.E. Abbas: Intuitonistic supra fuzzy topological spaces. Chaos Solitions and Fractals **21** (2004), 1205-1214. `https://doi.org/10.1016/j.chaos.2003.12.070`
2. M.E. Abd-Elmonsef & A. Ramadan: On fuzzy supra topological spaces. Indian J. Pure Appl. Math. **18** (1987), 322-329.
3. M. Abo-Elhamayed & T.M. Al-Shami: Supra homeomorphism in supra topological ordered spaces. Facta Universitatis, Series: Mathematics and Informatics **31** (2016), no. 5, 1091-1106. `doi:10.22190/FUMI1605091A`
4. U. Acar, F. Koyuncu & B. Tanay: Soft sets and soft rings. Comput. Math. Appl. **59** (2010), 3458-3463. `https://doi.org/10.1016/j.camwa.2010.03.034`
5. M. Akram & F. Feng: Soft intersection Lie algebras. Quasigroups and Related Systems **21** (2013), 1-10.
6. T.M. Al-Shami & M.E. El-Shafei: Two types of separation axioms on supra soft topological spaces. Demonstratio Mathematica **52** (2019), no. 1, 147-165. `https://doi.org/10.1515/dema-2019-0016`

7. T.M. Al-Shami & M.E. El-Shafei: On supra soft topological ordered spaces. Arab Journal of Basic and Applied Sciences **26** (2019), no. 1, 433-445. https://doi.org/10.1080/25765299.2019.1664101
8. T.M. Al-Shami & I. Alshammari: Rough sets models inspired by supra-topology structures. Artificial Intelligence Review **56** (2023), 6855–6883. https://doi.org/10.1007/s10462-022-10346-7
9. T.M. Al-Shami, J.C.R. Alcantud & A.A. Azzam: Two New Families of Supra-Soft Topological Spaces Defined by Separation Axioms. Mathematics **10** (2022), no. 23, 4488. https://doi.org/10.3390/math10234488
10. B.A. Asaad, T.M. Al-Shami & El-Sayed A. Abo-Tabl: Applications of some operators on supra topological spaces. Demonstratio Mathematica **53** (2020), no. 1, 292-308. https://doi.org/10.1515/dema-2020-0028
11. T.M. Al-Shami: Homeomorphism and Quotient Mappings in Infrasoft Topological Spaces. Journal of Mathematics, vol. (2021), Article ID 3388288, 10 pages. https://doi.org/10.1155/2021/3388288
12. J.C.R. Alcantud, T.M. Al-Shami & A.A. Azzam: Caliber and Chain Conditions in Soft Topologies. Mathematics **9** (2021), no. 19, 2349. https://doi.org/10.3390/math9192349
13. S. Bayramov & C. Gunduz: Soft locally compact spaces and soft paracompact spaces. J. Math. System Sci. **3** (2013), 122-130.
14. S. Bayramov & C. Gunduz Aras: On intuitionistic fuzzy soft topolgical spaces. TWMS J. Pure Appl. Math. **5** (2014), no. 1, 66-79.
15. N. Çağman, S.Karataş & S. Enginoğlu: Soft topology. Comput. Math. Appl. **62** (2011), no. 1, 351-358. https://doi.org/10.1016/j.camwa.2011.05.016
16. S.A. El-Sheikh & A.M. Abd El-latif: Decompositions of some types of supra soft sets and soft continuity. Int. J. Math. Trends and Tech. **9** (2014), no. 1, 37-56. DOI:10.14445/22315373/IJMTT-V9P504
17. M.E. El-Shafei, M. Abo-Elhamayel & T.M. Al-Shami: Further notions related to new operators and compactness via supra soft topological spaces. International Journal of Advances in Mathematics **2019** (2019), no. 1, 44-60.
18. A. Fadel & S.C. Dzul-Kifli: Bipolar soft functions. AIMS Mathematics **6** (2021), no. 5, 4428-4446. doi:10.3934/math.2021262
19. F. Feng, Y.B. Jun & X. Zhao: Soft semirings. Comput. Math. Appl. **56** (2008), 2621-2628. https://doi.org/10.1016/j.camwa.2008.05.011
20. C. Gunduz Aras: A study on intuitionistic fuzzy soft supra topological spaces. Proc. of the Inst. of Math. and Mech. **44** (2018), no. 2, 187-197.
21. C. Gunduz Aras & S. Bayramov: Results of some separation axioms in supra soft topological spaces. TWMS Journal of Applied and Engineering Mathematics **9** (2019), no. 1, 3479-3486.

22. C. Gunduz Aras & S. Bayramov: Separation axioms in supra soft topological spaces. Filomat **32** (2018), no. 10, 3479-3486. `https://doi.org/10.2298/FIL1810479G`
23. C.G. Aras, T.M. Al-shamib, A. Mhemdic & S. Bayramov: Local compactness and paracompactness on bipolar soft topological spaces. Journal of Intelligent & Fuzzy Systems **43** (2022), 6755–6763. `DOI: 10.3233/JIFS-220834`
24. C. Gunduz Aras & S. Bayramov: On the Tietze extension theorem in soft topological spaces. Proc. of the Inst. of Math. and Mech. **43** (2017), no. 1, 105-115.
25. S. Hussain & B. Ahmad: Some properties of soft topological spaces. Comput. Math. Appl. **62** (2011), no. 11, 4058-4067. `https://doi.org/10.1016/j.camwa.2011.09.051`
26. P.K. Maji & A.R. Roy: An application of soft sets in a decision making problem. Comput. Math. Appl. **44** (2002), 1077-1083. `https://doi.org/10.1016/S0898-1221(02)00216-X`
27. P.K. Maji, R. Biswas & A.R. Roy: Soft Set Theory. Comput. Math. Appl. **45** (2003), 555-562. `https://doi.org/10.1016/S0898-1221(03)00016-6`
28. A.S. Mashhour, A.A. Allam, F.S. Mahmoud & F.H. Khedr: On supra topological spaces. Indian J. Pure App. Math. **14** (1983), no. 4, 502-510.
29. W.K. Min: A note on soft topological spaces. Comput. Math. Appl. **62** (2011), no. 9, 3524-3528. `https://doi.org/10.1016/j.camwa.2011.08.068`
30. D. Molodtsov: Soft set theory-First results. Comput. Math. Appl. **37** (1999), no. 4/5, 19-31.
31. S. Oztunc, S. Aslan & H. Dutta: Categorical structures of soft groups. Soft Comput. **25** (2021), 3059-3064. `https://doi.org/10.1007/s00500-020-05362-0`
32. S. Oztunc: Some Properties of Soft Categories. International Journal of Modeling and Optimization **6** (2016), no. 2, 91-95. `DOI:10.7763/IJMO.2016.V6.510`
33. T.Y. Ozturk: On bipolar soft points. TWMS J. Appl. Eng. Math. **10** (2020), no. 4, 877-885.
34. M. Shabir &], M. Naz: On bipolar soft sets. (2013), Retrieved from `https://arxiv.org/abs/1303.1344v1`.
35. M. Shabir & M. Naz: On soft topological spaces. Comput. Math. Appl. **61** (2011), 1786-1799. `https://doi.org/10.1016/j.camwa.2011.02.006`
36. M. Shabir & A. Bakhtawar: Bipolar soft connected, bipolar soft disconnected and bipolar soft compact spaces. Songklanakarin J. Sci. Technol. **39** (2017), no. 3, 359-371.
37. N. Turanlı: An overwiev of intuitionistic fuzzy supra topological spaces. Hacettepe J. of Math. Statistics **32** (2003), 17-26. Retrieved from `https://dergipark.org.tr/en/pub/hujms/issue/55282/759933`
38. I. Zorlutuna, M. Akdag, W.K. Min & S. Atmaca: Remarks on soft topological spaces. Ann. Fuzzy Math. Inform. **3** (2012), no. 2, 171-185.

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