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ON THE RATIONAL COHOMOLOGY OF MAPPING SPACES AND THEIR REALIZATION PROBLEM

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Abstract. Let $f : X \to Y$ be a map between simply connected CW-complexes of finite type with X finite. In this paper, we prove that the rational cohomology of mapping spaces $\mathrm{map}(X, Y; f)$ contains a polynomial algebra over a generator of degree N, where $N = \max\{i, \pi_i(Y) \otimes \mathbb{Q} \neq 0\}$ is an even number. Moreover, we are interested in determining the rational homotopy type of $\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f)$ and we deduce its rational cohomology as a consequence. The paper ends with a brief discussion about the realization problem of mapping spaces.

1. Introduction

In this paper all spaces are assumed to be simply connected of the homotopy type of a CW-complex and are of finite type over $\mathbb{Q}$, i.e., have finite dimensional rational cohomology in each degree.

Given two topological spaces X and Y, let $\mathrm{map}(X, Y)$ (resp. $\mathrm{map}_*(X, Y)$) denote the space of all free (resp. pointed) continuous functions with the compact-open topology. The space $\mathrm{map}(X, Y)$ is generally disconnected with path-components corresponding to the set of free homotopy classes of maps. We write $\mathrm{map}(X, Y; f)$ for the path-component containing a given map $f : X \to Y$. Whenever X is a finite CW-complex and Y is a CW-complex of finite type over $\mathbb{Q}$, then any path component of both $\mathrm{map}(X, Y)$ and $\mathrm{map}_*(X, Y)$ are nilpotent CW-complexes of finite type over $\mathbb{Q}$ and in particular, it can be rationalized in the classical sense [10].

A fundamental problem is to determine the rational homotopy type and the rational cohomology of the mapping spaces $\mathrm{map}(X, Y; f)$. Generally speaking about cohomology is delicate invariant and difficult to compute. We use methods from rational homotopy theory to compute these groups. As is well known, the homotopy theory of rational spaces, i.e., spaces whose homotopy groups are vector spaces over $\mathbb{Q}$, is equivalent to the homotopy theory of minimal differential graded commutative algebras over $\mathbb{Q}$. More precisely, there

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is an equivalence between the homotopy category of rational spaces and the homotopy category of minimal algebras. Similarly, the rational homotopy type of a continuous map between spaces is the same as the algebraic homotopy class of the corresponding morphism between models.

The study of the rational homotopy type of $\mathrm{map}(X, Y; f)$ was initiated by A. Haefliger [8] who describes its Sullivan model. Following his work, J. M. Møller and M. Raussen determine the rational homotopy type of the components of $\mathrm{map}(X, \mathbb{S}^n)$ in terms of the cohomology algebra $H^*(X;\mathbb{Q})$ and the Sullivan model of $\mathbb{S}^n$ [15]. Recently, the rational homotopy type of mapping spaces seem to be well described in terms of the theory of L_∞-algebra [2,3]. By using this machinery and others, U. Buijs and A. Murillo have explicitly described the rational homotopy of free and pointed mapping spaces between spheres [4]. In their foundational paper *Rationalized evaluation subgroups of a map I: Sullivan models, derivations and G-sequences* [12] (see also [11]), G. Lupton and S. B. Smith gave an elegant formula to determine the rational homotopy type of mapping spaces in terms of derivations. Inspired by their work, we prove the following results.

Theorem 1.1. *Let $f : X \to Y$ be a map between simply connected CW-complexes of finite type in which X is finite. Moreover, assume that $N = \max\{i, \pi_i(Y) \otimes \mathbb{Q} \neq 0\}$ is an even number. Then, the rational cohomology algebra of $\mathrm{map}(X, Y; f)$ contains a polynomial algebra over a generator of degree N.*

Moreover, we determine the rational homotopy type of $\mathrm{map}(X, Y; f)$ in the particular case of choosing $X = \mathbb{S}^n$ and $Y = \mathbb{C}P^m$, and we prove the following result.

Theorem 1.2. *The mapping space $\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f)$ has the following rational homotopy type*:

- $\mathbb{C}P^m$, *if n odd and $n \geq 2m+1$,*
- $\mathbb{C}P^m \times K(\mathbb{Q}, 2m-n+1)$, *if n odd and $n < 2m+1$,*
- $\mathbb{C}P^{m-\frac{n}{2}} \times \mathbb{S}^{2m+1}$, *if n even and $n < 2m+1$,*
- $\mathbb{C}P^m$, *if n even and $n > 2m+1$.*

As a consequence, we compute the rational cohomology of the mapping space $\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f)$. We also deduce the rational homotopy type of the pointed mapping spaces $\mathrm{map}_*(\mathbb{S}^n, \mathbb{C}P^m; f)$.

We mention that Theorem 1.2 might be known, or easily deduced by specialists. However, to my knowledge, it has not been made explicit in the literature.

The paper is organized as follows. In Section 2 we review some basic facts for the theory of Sullivan minimal models and derivations. Section 3 contains the proof of Theorem 1.1, Theorem 1.2 and their consequences. Motivated by our results, we conclude this paper with a brief discussion about the realization problem of mapping spaces.

2. Sullivan minimal models and derivations

This section cannot provide and is not intended to give an introduction to the theory of Sullivan minimal models. We expect the reader to have gained a certain familiarity with necessary concepts for example from [7]. We merely recall some tools and aspects which play a larger role in the paper. For our purposes, we recall the following.

Definition. A commutative differential graded algebra (cdga) is a graded algebra $A = \oplus_{i\geq 0} A^i$ with a differential $d : A^i \to A^{i+1}$ such that $d^2 = 0$, $xy = (-1)^{ij}\, yx$, and $d\,(xy) = d\,(x)\, y + (-1)^i\, xd\,(y)$ for all $x \in A^i$ and $y \in A^j$. A cdga (A, d) is called simply connected if $H^0(A, d) = \mathbb{Q}$ and $H^1(A, d) = 0$. Let $f : (A, d) \to (B, d)$ be a morphism of cdga's. It is called a quasi-isomorphism if $H^*(f) : H^*(A, d) \to H^*(B, d)$ is an isomorphism.

A simply connected commutative graded algebra A is free if it is of the form

$$\Lambda V = S(V^{\mathrm{even}}) \otimes E(V^{\mathrm{odd}}),$$

where $V^{\mathrm{even}} = \oplus_{i\geq 1} V^{2i}$ and $V^{\mathrm{odd}} = \oplus_{i\geq 1} V^{2i+1}$. Moreover if V admits a homogeneous basis $\{x_i\}_{i\in I}$ indexed by a well ordered set I such $dx_j \in \Lambda(\{x_i\})_{i<j}$, we say that $(\Lambda V, d)$ is a Sullivan algebra. Further, it is said to be minimal if $dV \subset \Lambda^{\geq 2} V$. If there is a quasi-isomorphism $f : (\Lambda V, d) \to (A, d)$, where $(\Lambda V, d)$ is a Sullivan minimal algebra, then we say that $(\Lambda V, d)$ is a Sullivan minimal algebra of (A, d).

To each simply connected topological space X of finite type, D. Sullivan associates in a functorial way a cdga $A_{\mathrm{PL}}(X)$ of piecewise linear forms on X such that $H_*(A_{\mathrm{PL}}(X)) \cong H^*(X; \mathbb{Q})$ [18]. A Sullivan minimal model of X is a Sullivan minimal model of $A_{\mathrm{PL}}(X)$. Moreover, the rational homotopy type of X is completely determined by its Sullivan minimal model $(\Lambda V, d)$. In particular, there are isomorphisms

$$H^*\,(\Lambda V, d) \cong H^*\,(X; \mathbb{Q}) \text{ as commutative graded algebras,}$$

$$V \cong \pi_*(X) \otimes \mathbb{Q} \text{ as graded vector spaces.}$$

Sullivan minimal models behave nicely with respect to fibrations. Recall that the KS-model for a rational fibration $F \to E \xrightarrow{p} B$ is a short exact sequence:

$$(\Lambda W, d_W) \to (\Lambda W \otimes \Lambda V, D) \to (\Lambda V, d_V)$$

of cdga, with $(\Lambda W, d_W)$ and $(\Lambda V, d_V)$ are the Sullivan minimal models for B and F, respectively ([7], Proposition 15.5). The differential D satisfies: $D(w) = d_W(w)$ for $w \in W$ and $D(v) - d_V(v) \in \Lambda^+ W \otimes \Lambda V$ for $v \in V$. The cdga $(\Lambda W \otimes \Lambda V, D)$ is a Sullivan model for the total space E but is not in general minimal.

Definition. A fibration $F \to E \xrightarrow{p} B$ is called rationally trivial when

$$E \simeq_{\mathbb{Q}} B \times F,$$

or equivalently if the differential D of the total space E satisfies: $D(w) = d_W(w)$ and $D(v) = d_V(v)$.

Now, we recall the following definition which we use later.

Definition. Let $\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ be a morphism of cdga's. Define a φ-derivation θ of degree n to be a linear map $\theta : \Lambda W \to \Lambda V$ that reduces degree by n such that $\theta(xy) = \theta(x)\varphi(y) + (-1)^{n|x|}\varphi(x)\theta(y)$. Let $\mathrm{Der}_n(\Lambda W, \Lambda V; \varphi)$ denote the vector space of φ-derivations of degree n for $n > 0$. When $n = 1$ we require additionally that $d_V \circ \theta = -\theta \circ d_W$. Define a linear map $\delta : \mathrm{Der}_n(\Lambda W, \Lambda V; \varphi) \to \mathrm{Der}_{n-1}(\Lambda W, \Lambda V; \varphi)$ by $\delta(\theta) = d_V \circ \theta - (-1)^n \theta \circ d_W$.

Note that $(\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi), \delta) := \oplus_n \mathrm{Der}_n(\Lambda W, \Lambda V; \varphi)$ is a chain complex. There is an isomorphism of graded vector spaces

$$\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi) \cong \mathrm{Hom}_*(W, \Lambda V).$$

In view of the preceding remark, we denote by (y, x) the unique φ-derivation sending an element $y \in W$ to $x \in \Lambda V$ and the other generators to zero.

Now, we recall how to determine the rational homotopy type of mapping spaces. The following result is due to G. Lupton and S. B. Smith (see [12], Theorem 2.1).

Theorem 2.1. *Let $f : X \to Y$ be a map between simply connected CW-complexes of finite type with X finite and $\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ its KS-model. Then for $i \geq 2$,*

$$\pi_i(map(X, Y; f)) \otimes \mathbb{Q} \cong H_i(Der(\Lambda W, \Lambda V; \varphi)) \text{ as graded vector spaces,}$$

and also

$$\pi_i(map_*(X, Y; f)) \otimes \mathbb{Q} \cong H_i\left(Der\left(\Lambda W, \widetilde{\Lambda V}; \widetilde{\varphi}\right)\right) \text{ as graded vector spaces,}$$

where $\widetilde{\varphi} : \Lambda W \to \widetilde{\Lambda V}$ is the dga map which agrees with φ in positive degrees and vanishes in degree zero.

Notice that the same result was proved in [3] by U. Buijs and A. Murillo.

As an overriding hypothesis, we assume that all spaces appearing in this paper are *rational simply connected.*

3. Proofs of our results

For the proofs, we use the machinery of rational homotopy theory as described in Section 2. We use more precisely the theory of Sullivan minimal models and the description of the rational homotopy groups of mapping spaces in terms of derivations (see Theorem 2.1).

The following result plays a key role in the proof of our results.

Proposition 3.1. *Let $f : X \to Y$ be a map in which X is finite and Y is π-finite* ($\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$). *Further, if*

$$\pi_i(Y) \otimes \mathbb{Q} = \begin{cases} 0 & \text{for } i > M, \\ \mathbb{Q}^r & \text{for } i = M. \end{cases}$$

Then, we obtain

$$\pi_i(map(X, Y; f)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{for } i > M, \\ \mathbb{Q}^r & \text{for } i = M. \end{cases}$$

Proof. Let $\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ be the KS-model of f. Further, since $\mathrm{Hom}(W, \mathbb{Q}) \cong \pi_*(Y) \otimes \mathbb{Q}$ and from our assumption, we get that W is concentrated in degrees $\leq M$ and $\dim W^M = r$. On other hand, we see that

$$\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi) \cong \mathrm{Hom}(W, \Lambda V).$$

It follows that $\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$ is concentrated in degrees $\leq M$. Therefore, we can deduce

$$H_i(\mathrm{Der}(\Lambda W, \Lambda V; \varphi)) = 0 \text{ for } i > M.$$

By applying Theorem 2.1, we obtain

$$\pi_i(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} \cong 0 \text{ for } i > M. \tag{1}$$

Furthermore, in degree M, for each $\theta \in \mathrm{Hom}(W^M, \mathbb{Q})$, we obtain a derivation in $\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$ of degree M by setting $\theta(W^M) = 1$ and extending as a derivation. Any such derivation is a δ-cycle, since the element of W^M, as the last stage of generators, do not occur in the differential of any other generators. There are no non-zero boundaries of degree M, since $\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$ is zero in degree $M+1$ and higher. So the vector space W^M persists to homology and we have

$$H_M(\mathrm{Der}(\Lambda W, \Lambda V; \varphi)) \cong \mathrm{Hom}\left(W^M, \mathbb{Q}\right).$$

Thus from Theorem 2.1, we deduce that

$$\pi_M(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q} \cong \mathrm{Hom}\left(W^M, \mathbb{Q}\right). \tag{2}$$

Now, we combine (1) and (2) to achieve the proof. □

This result, up to shifting degree, is due to G. Lupton and S. B. Smith for the case $X = Y$ and $f = id$ (see [14], Proposition 2.3).

Now, we are ready to prove our main first result.

Proof of Theorem 1.1. First, since $\pi_*(Y) \otimes \mathbb{Q}$ is finite dimensional, thus

$$\pi_*(\mathrm{map}(X, Y; f)) \otimes \mathbb{Q}$$

is also finite dimensional by applying Proposition 3.1. Next, let $(\Lambda U, d)$ be the Sullivan minimal model of $\mathrm{map}(X, Y; f)$ with U is finite dimensional. So, by using our assumption and again Proposition 3.1 we can write

$$U = U_{\leq N-1} \oplus U_N,$$

where U_N denotes the graded vector space spanned by elements of degrees N. Second, we appeal to some results of S. Halperin concerning elliptic Sullivan minimal models. To any Sullivan minimal model $(\Lambda V, d)$ with V is finite dimensional, there is an associated pure minimal model, denoted $(\Lambda V, d_\sigma)$, which is defined by adjusting the differential d to d_σ as follows: We set $d_\sigma = 0$ on each even degree generator of V, and on each odd degree generator $v \in V$, we set $d_\sigma(v)$ equal to the part of $d(v)$ contained in $\Lambda(V^{\text{even}})$. One checks that this defines a differential d_σ on ΛV, and thus we obtain a pure Sullivan model $(\Lambda V, d_\sigma)$. Applying all this to the Sullivan minimal model $(\Lambda(U_{\leq N-1} \oplus U_N), d)$. For degree reasons and an easy computation shows that every element x in U_N is a non-exact d-cocycle. So for each $n \geq 1$, we obtain

$$[x^n] \text{ is non null in } H^*\left(\Lambda(U_{\leq N-1} \oplus U_N), d_\sigma\right).$$

Thus Proposition 32.4 in [7] shows that $H^*(\Lambda U, d)$ contains a polynomial algebra $\mathbb{Q}[x]$, where $|x| = N$. □

Note that the condition max $\pi_*(Y) \otimes \mathbb{Q}$ even in Theorem 1.1 is sufficient but not necessary. For example:

Example 3.2. Let n be an odd number. From Theorem 1.2, it follows that

$$\operatorname{map}(\mathbb{S}^n, \mathbb{C}P^m; f) \simeq_{\mathbb{Q}} \mathbb{C}P^m \times K(\mathbb{Q}, 2m - n + 1), \text{ if } n < 2m + 1.$$

Now by applying the Kunneth formula, we deduce that the rational cohomology of $\operatorname{map}(\mathbb{S}^n, \mathbb{C}P^m; f)$ contains a polynomial algebra over degree $2m-n+1$ though max $\pi_*(Y) \otimes \mathbb{Q} =$ max $\pi_*(\mathbb{C}P^m) \otimes \mathbb{Q} = 2m + 1$.

Even if max $\pi_*(Y) \otimes \mathbb{Q}$ is an even number, it may hold that the pointed mapping space $\operatorname{map}_*(X, Y; f)$ does not contains a polynomial algebra.

Example 3.3. Let us consider the mapping space $\operatorname{map}_*(K(\mathbb{Q}, 3), K(\mathbb{Q}, 6); f)$. We denote by

$$\varphi : (\Lambda(y_6), 0) \to (\Lambda(z_3), 0)$$

the KS-model of $f : K(\mathbb{Q}, 3) \to K(\mathbb{Q}, 6)$. In both models, subscripts denote degrees. For degree reasons, we have

$$\operatorname{Der}_*(\Lambda W, \widetilde{\Lambda V}; \widetilde{\varphi}) = \mathbb{Q}\langle (y, z) \rangle.$$

Furthermore, it holds that

$$\begin{aligned} H_i(\operatorname{Der}(\Lambda W, \widetilde{\Lambda V}; \widetilde{\varphi})) &\cong \pi_i(\operatorname{map}_*(K(\mathbb{Q}, 3), K(\mathbb{Q}, 6); f)) \otimes \mathbb{Q} \\ &\cong \mathbb{Q} \end{aligned}$$

in degree 3 and zero otherwise. So, we must have

$$\operatorname{map}_*(K(\mathbb{Q}, 3), K(\mathbb{Q}, 6); f)) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 3).$$

Finally, we conclude that $\operatorname{map}_*(K(\mathbb{Q}, 3), K(\mathbb{Q}, 6); f)$ does not contains a polynomial algebra.

To phrase our next result, we recall a well known numerical invariant of the homotopy type of spaces. The rational Lusternik-Schnirelmann category of X, $\mathrm{cat}_0(X)$, is the least integer n such that X can be covered by $(n+1)$ open subsets contractible in X. This is an invariant of the rational homotopy type of X, and this and its elementary properties are presented in detail in [5, 7].

Corollary 3.4. *Under the hypothesis of Theorem 1.1, map$(X, Y; f)$ has infinite rational Lusternik-Schnirelmann category.*

Proof. From Theorem 1.1, we deduce that $\mathrm{map}(X, Y; f)$ has infinite rational cohomology. Next, we use ([7], Proposition 32.4) to complete the proof. □

We next describe a situation in which we can be assured of a trivial fibration. It will be a key point in the proof of Theorem 1.2. Recall that fibrations with fibre in the homotopy type of F are obtained, up to fibre homotopy equivalence, as a pull-back of the universal fibration [17]

$$F \to B\mathrm{aut}_1^*(F) \overset{p_\infty}{\to} B\mathrm{aut}_1(F),$$

where $\mathrm{aut}_1(F) := \mathrm{map}(F, F; id)$, $\mathrm{aut}_1^*(F) := \mathrm{map}_*(F, F; id)$ and B is the Dold-Lashof functor from monoids to topological spaces [6].

Proposition 3.5. *Let $\xi : K(\mathbb{Q}, n) \to E \overset{p}{\to} B$ be a fibration of simply connected CW-complexes. The following are equivalent*:

(1) *ξ is trivial,*
(2) *$H^{n+1}(B; \mathbb{Q}) = 0$.*

Proof. Let $h : B \to B\mathrm{aut}_1(K(\mathbb{Q}, n))$ denote the classifying map of the fibration ξ which make the following homotopy commutative diagram:

$$\begin{tikzcd}
K(\mathbb{Q},n) \arrow[d] & K(\mathbb{Q},n) \arrow[d] \\
E \arrow[r] \arrow[d, "p"] & B\mathrm{aut}_1^*(K(\mathbb{Q},n)) \arrow[d, "p_\infty"] \\
B \arrow[r, "h"] & B\mathrm{aut}_1(K(\mathbb{Q},n))
\end{tikzcd}$$

Now, it is well known that $B\mathrm{aut}_1(K(\mathbb{Q}, n)) \simeq_{\mathbb{Q}} K(\mathbb{Q}, n+1)$ (see [9], Example 10). Moreover, we see that the rational homotopy set of classifying maps from a space B is giving as (see for example [1], Example 4.1)

$$\begin{aligned}[B, B\mathrm{aut}_1(K(\mathbb{Q}, n))] &= [B, K(\mathbb{Q}, n+1)] \\ &= H^{n+1}(B; \mathbb{Q}).\end{aligned}$$

This means that the classifying map h is trivial if and only if $H^{n+1}(B; \mathbb{Q}) = 0$, as needed. □

The KS-model of a rational fibration is the main technical tool that is used in the sequel. We now give an algebraic description of condition (1) above. It is easy to see that the fibration $\xi : K(\mathbb{Q}, n) \to E \xrightarrow{p} B$ is trivial if and only if its KS-model is given by

$$(\Lambda W, d_W) \to (\Lambda W \otimes \Lambda(v_n), D) \to (\Lambda(v_n), 0),$$

where $D(w) = d_W(w)$ for $w \in W$ and $D(v_n) = 0$.

In the remainder of this section, we will use

$$\mathrm{map}_*(X, Y; f) \to \mathrm{map}(X, Y; f) \xrightarrow{\omega} Y$$

to denote the evaluation fibration at the base point $x_0 \in X$, $\omega(g) = g(x_0)$.

We turn now our attention to the rational homotopy type of mapping spaces.

Proof of Theorem 1.2. To proof our theorem, we will denote the KS-model of $f : \mathbb{S}^n \to \mathbb{C}P^m$ by

$$\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V).$$

For degree reasons, we will distinguish different cases.

Case 1. n odd.

Case 1.1: $n \geq 2m + 1$.

The KS-model $\varphi : (\Lambda(x_2, y_{2m+1}), d_W) \to (\Lambda(z_n), 0)$ is given on generators by $\varphi(x) = 0$ and $\varphi(y) = \alpha z$ for some $\alpha \in \mathbb{Q}$ (it may be zero). Here, subscripts indicating degrees and the differential d_W is defined as follows: $d_W(x) = 0$ and $d_W(y) = x^{m+1}$. So, the vector space $\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$ of derivations of positive degree is spanned by $(x, 1)$ and $(y, 1)$. Direct computation shows that the differential δ in $\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$ is given by $\delta(x, 1) = \delta(y, 1) = 0$. Then, the homology is of rank 1 in degrees 2 and $2m + 1$, and zero otherwise. Thus, $\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f)$ has homotopy groups concentrated in degrees 2 and $2m+1$ (the correct homotopy groups as $\mathbb{C}P^m$). Hence, by using the evaluation fibration

$$\mathrm{map}_*(\mathbb{S}^n, \mathbb{C}P^m; f) \to \mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f) \xrightarrow{\omega} \mathbb{C}P^m,$$

we deduce that

$$\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f) \simeq_{\mathbb{Q}} \mathbb{C}P^m.$$

Case 1.2: $n < 2m + 1$.

By using the same notation as above, a standard check shows that $\varphi(x) = \varphi(y) = 0$. In the style of the above, the vector space $\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$ is spanned by: $(x, 1)$, $(y, 1)$ and (y, z). Moreover, we have

$$\delta(x, 1) = \delta(y, 1) = \delta(y, z) = 0.$$

It follows that $\mathrm{H}_*(\mathrm{Der}(\Lambda W, \Lambda V; \varphi))$ has rank 1 in degrees 2, $2m - n + 1$ and $2m + 1$, and is trivial in all other degrees. Hence from Theorem 2.1, we can write the Sullivan model of $\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f)$ as

$$(\Lambda(a_2, b_{2m-n+1}, c_{2m+1}), D).$$

Now, the KS-model of the evaluation fibration gives that $Da = 0$ and $Dc = a^{m+1}$. Further, to determine the differential D on b, we can construct the following KS-model

$$(\Lambda(a_2, c_{2m+1}), d_W) \to (\Lambda(a_2, b_{2m-n+1}, c_{2m+1}), D) \to (\Lambda(b_{2m-n+1}), 0)$$

whose geometric realization is

$$\xi : K(\mathbb{Q}, 2m-n+1) \to \mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f) \to \mathbb{C}P^m.$$

Clearly, we have $H^{2m-n+2}(\mathbb{C}P^m; \mathbb{Q}) = 0$. Then, from Proposition 3.5, we deduce that ξ is trivial and further $Db = 0$. Consequently, we have proved that

$$\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f) \simeq_{\mathbb{Q}} \mathbb{C}P^m \times K(\mathbb{Q}, 2m-n+1).$$

Case 2. *n* even.
Let denote the KS-model of $f : \mathbb{S}^n \to \mathbb{C}P^m$ by

$$\varphi : (\Lambda(x_2, y_{2m+1}), d_W) \to (\Lambda(u_n, v_{2n-1}), d_V),$$

where $d_W(x) = 0$, $d_W(y) = x^{m+1}$, $d_V(u) = 0$ and $d_V(v) = u^2$.
Case 2.1: $2 < n < 2m+1$.
Without loss of generality, we have $\varphi(x) = \varphi(y) = 0$. Furthermore, a basis for $\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$ is spanned by $(x, 1)$ and $(y, u^{r_0} v^{r_1})$ for some $r_0, r_1 \geq 0$. It is clear that the only non-bounding δ-cycles are $(x, 1)$, $(y, 1)$ and (y, u). This implies that the Sullivan model of $\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f)$ takes the form

$$(\Lambda(a_2, b_{2m-n+1}, c_{2m+1}), D),$$

where $Da = 0$ and $Dc = a^{m+1}$. It remains to determine the differential Db. Since $H^{2m-2n+2}(\mathbb{C}P^m; \mathbb{Q}) \neq 0$ and by using an argument similar as before, we get that $Db = a^{m-\frac{n}{2}+1}$. Now, make a change of KS-basis, replacing c by $\overline{c} = c - a^{\frac{n}{2}} b$. This gives a quasi-isomorphic KS-extension

$$\left(\Lambda(a, b, \overline{c}), \overline{D}\right) \to (\Lambda(a, b, c), D)$$

in which $\overline{D}a = Da$, $\overline{D}b = Db$ and $\overline{D}\overline{c} = 0$. Clearly, it follows that

$$\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f) \simeq_{\mathbb{Q}} \mathbb{C}P^{m-\frac{n}{2}} \times \mathbb{S}^{2m+1}.$$

Case 2.2: $n > 2m+1$.
In this case a Sullivan model of $\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f)$ is again computed via the methods above. Note that the details are very similar with some minor differences, due to the fact that the derivation (y, u) has negative degree and this not contribute to homology. A straightforward computation shows that

$$\pi_i(\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f)) \otimes \mathbb{Q} \cong \mathbb{Q} \text{ in degrees 2 and } 2m+1.$$

Finally, by using the evaluation fibration, we deduce that

$$\mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f) \simeq_{\mathbb{Q}} \mathbb{C}P^m.$$

Case 2.3: $n = 2$.
Here returning to the KS-model φ, we have $\varphi(x) = u$ and $\varphi(y) = u^{m-1}v$. Now, it is evident to see that

$$D(x,1) = -(m+1)(y, u^m),$$

it follows that

$$D\left[(x,1) + (m+1)(y, u^{m-2}v)\right] = 0.$$

A straightforward computation shows that the homology of $\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$ is concentrated in degrees 2, $2m-1$ and $2m+1$. Next, to avoid the repetition, an arguments similar to those explained in Case 2.1 prove that

$$\mathrm{map}\left(\mathbb{S}^2, \mathbb{C}P^m; f\right) \simeq_{\mathbb{Q}} \mathbb{C}P^{m-1} \times \mathbb{S}^{2m+1}.$$

Summarizing all of the above cases finishes the proof of our theorem. □

We finish this section with two immediate consequences of Theorem 1.2.

Corollary 3.6. *The rational cohomology of map* $(\mathbb{S}^n, \mathbb{C}P^m; f)$ *is*

- $\mathbb{Q}[a_2]/a^{m+1}$, *if n odd and $n \geq 2m+1$,*
- $\mathbb{Q}[a_2]/a^{m+1} \otimes \mathbb{Q}[b_{2m-n+1}]$, *if n odd and $n < 2m+1$,*
- $\mathbb{Q}[a_2]/a^{m-\frac{n}{2}+1} \otimes \Lambda(b_{2m+1})$, *if n even and $n < 2m+1$,*
- $\mathbb{Q}[a_2]/a^{m+1}$, *if n even and $n > 2m+1$.*

Proof. It follows directly from Kunneth formula and Theorem 1.2. □

By taking into account Theorem 1.2 and the KS-model of the evaluation fibration

$$\mathrm{map}_*(\mathbb{S}^n, \mathbb{C}P^m; f) \to \mathrm{map}(\mathbb{S}^n, \mathbb{C}P^m; f) \xrightarrow{\omega} \mathbb{C}P^m,$$

we can immediately deduce the following result.

Corollary 3.7. *The mapping space* $\mathrm{map}_*(\mathbb{S}^n, \mathbb{C}P^m; f)$ *has the rational homotopy type of*

- **, if n odd and $n \geq 2m+1$,*
- $K(\mathbb{Q}, 2m-n+1)$, *if n odd and $n < 2m+1$,*
- $K\left(\mathbb{Q}, m-\frac{n}{2}\right)$, *if n even and $n < 2m+1$,*
- **, if n even and $n > 2m+1$.*

We note that Corollary 3.7 can also be obtained by using Theorem 2.1.

4. Realization problem of mapping spaces

Mapping spaces are at the foundations of homotopy theory and appear in the literature dating back, at least, to Hurewicz's definition of the homotopy groups in the 1930s. These spaces were studied by several authors and have led many interesting results in homotopy theory (see [16] for a survey). With this long and extensive history, it is surprising that the question of realizability of mapping spaces has never been addressed. We cannot resist making the following question.

Question 4.1. Is every simply connected CW-complex Z can be realized as mapping spaces $\mathrm{map}(X, Y)$ for some simply connected CW-complexes X and Y?

If X is contractible, i.e., $X \simeq_{\mathbb{Q}} *$, we get $\mathrm{map}(X, Y; f) \simeq_{\mathbb{Q}} Y$. So in particular, our question is trivial if we do not place any restrictions on the space X.

Now, we examine various observations related to the realizability of mapping spaces. First, by Theorem 1.1, we show that every finite CW-complex Z cannot be realized as $\mathrm{map}(X, Y; f)$ when max $\pi_*(Y) \otimes \mathbb{Q}$ is an even number. Further, Theorem 1.2 suggests a several realization results. It would be interesting to know whether there are other examples in which the question is true. Now, let's move to $\mathrm{map}(X, Y; f)$ when $X = Y$ and $f = id$.

Proposition 4.2. *Every H-space can be realized as map$(X, X; id)$ for some space X.*

Proof. This follows from ([13], Theorem 3.6). □

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References

[1] M. Amann and V. Kapovitch, *On fibrations with formal elliptic fibers*, Adv. Math. **231** (2012), no. 3-4, 2048–2068. https://doi.org/10.1016/j.aim.2012.07.022

[2] U. Buijs, Y. Félix, and A. Murillo, *L_∞ rational homotopy of mapping spaces*, Rev. Mat. Complut. **26** (2013), no. 2, 573–588. https://doi.org/10.1007/s13163-012-0105-z

[3] U. Buijs and A. Murillo, *The rational homotopy Lie algebra of function spaces*, Comment. Math. Helv. **83** (2008), no. 4, 723–739. https://doi.org/10.4171/CMH/141

[4] U. Buijs and A. Murillo, *Rational Homotopy type of free and pointed mapping spaces between spheres*, Proceedings of Ukrainian Mathematical Society. **6** (2013), 130–139.

[5] O. Cornea, G. Lupton, J. Oprea, and D. Tanré, *Lusternik-Schnirelmann category*, Mathematical Surveys and Monographs, 103, Amer. Math. Soc., Providence, RI, 2003. https://doi.org/10.1090/surv/103

[6] A. Dold and R. K. Lashof, *Principal quasi-fibrations and fibre homotopy equivalence of bundles*, Illinois J. Math. **3** (1959), 285–305. http://projecteuclid.org/euclid.ijm/1255455128

[7] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, 205, Springer, New York, 2001. https://doi.org/10.1007/978-1-4613-0105-9

[8] A. Haefliger, *Rational homotopy of the space of sections of a nilpotent bundle*, Trans. Amer. Math. Soc. **273** (1982), no. 2, 609–620. https://doi.org/10.2307/1999931

[9] M. R. Hilali and A. Zaim, *Spaces realized and non-realized as Dold-Lashof classifying spaces*, J. Geom. Symmetry Phys. **50** (2018), 29–56. https://doi.org/10.7546/jgsp-50-2018-29-56

[10] P. J. Hilton, G. Mislin, and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Mathematics Studies, No. 15, Notas de Matemática, No. 55., North-Holland, Amsterdam, 1975.

[11] G. Lupton and S. B. Smith, *Rationalized evaluation subgroups of a map and the rationalized G-sequence*, arXiv.math/0309432, 2003.

[12] G. Lupton and S. B. Smith, *Rationalized evaluation subgroups of a map. I. Sullivan models, derivations and G-sequences*, J. Pure Appl. Algebra **209** (2007), no. 1, 159–171. https://doi.org/10.1016/j.jpaa.2006.05.018

[13] G. Lupton and S. B. Smith, *Criteria for components of a function space to be homotopy equivalent*, Math. Proc. Cambridge Philos. Soc. **145** (2008), no. 1, 95–106. https://doi.org/10.1017/S0305004108001175

[14] G. Lupton and S. B. Smith, *The universal fibration with fibre X in rational homotopy theory*, J. Homotopy Relat. Struct. **15** (2020), no. 2, 351–368. https://doi.org/10.1007/s40062-020-00258-0

[15] J. M. Møller and M. Raussen, *Rational homotopy of spaces of maps into spheres and complex projective spaces*, Trans. Amer. Math. Soc. **292** (1985), no. 2, 721–732. https://doi.org/10.2307/2000242

[16] S. B. Smith, *The homotopy theory of function spaces*: *a survey*, arXiv.1009.0804.

[17] J. D. Stasheff, *A classification theorem for fibre spaces*, Topology **2** (1963), 239–246. https://doi.org/10.1016/0040-9383(63)90006-5

[18] D. P. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. No. **47** (1977), 269–331.

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