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THE FLOW-CURVATURE OF CURVES IN A GEOMETRIC SURFACE

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Abstract. For a fixed parametrization of a curve in an orientable two-dimensional Riemannian manifold, we introduce and investigate a new frame and curvature function. Due to the way of defining this new frame as being the time-dependent rotation in the tangent plane of the standard Frenet frame, both these new tools are called flow.

1. Introduction

A recent and exciting area of study in geometric analysis is the theory of geometric flows. The *curve shortening flow* is the simplest of them, and the survey [2] is already 20 years old. Remember that the well-known curvature of plane curves is the primary geometric device in this last flow. In order to restart this problem, it seems necessary to look for variations of the curvature, or in the language of [11], for *deformations* of the typical curvature.

This brief note's proposal is to introduce such a deformation into generic orientable two-dimensional Riemannian geometry rather than plane Euclidean geometry. To be more specific, our framework is a triple (M^2, g, γ) made up of a smooth curve γ on the orientable Riemannian surface (M^2, g), which we refer to as a geometric surface. By rotating the Frenet frame of the standard Frenet theory for γ at an angle exactly determined by the parameter t of γ, we distort it and this new frame will be referred to as the flow-frame; it produces a new curvature function k^f called flow-curvature which is always lower than the usual curvature. A curve with a vanishing k^f will be called a flow-geodesic of the metric g. Following the expression of this new function, we highlight its impact on the Walker-Fermi derivative.

The final section focuses on the plane $\mathbb{R}^2$ and some open subsets where we illustrate a few curves. We conclude by noting that the Euclidean variant of the curvature defined here is the contents of the paper [6], the spherical variant is covered in the paper [8] while the hyperbolic geometry is treated in [7] in the

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hyperboloid model. We have developed also the Lorentzian counterpart for the spacelike curves in [5].

2. The flow-frame and the flow-curvature

The setting of this paper is a *geometric surface*, i.e., ([13, p. 322]) a smooth, orientable two-dimensional Riemannian manifold (M^2, g). Due to its orientability M carries an almost complex structure J; in fact J is integrable and for an arbitrary point $p \in M$ we consider $J_p : T_pM \to T_pM$ as being the multiplication with the complex unit $i \in \mathbb{C}$. Let ∇ be the Levi-Civita connection of g.

Fix also a smooth curve $\gamma : I \subseteq \mathbb{R} \to M$ which we suppose to be *regular*: $\gamma'(t) \in T_{\gamma(t)}M \setminus \{0\}$. Let $\mathfrak{X}(\gamma)$ be the $C^\infty(I)$-module of vector fields along γ, i.e., smooth maps $X : I \to TM$ with $X(t) \in T_{\gamma(t)}M$ for all $t \in I$. It follows the *unit tangent vector field* $T \in \mathfrak{X}(\gamma)$ with:

$$T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}. \tag{2.1}$$

Therefore, the *Frenet frame* of γ is $\mathcal{F} := \left(\begin{smallmatrix} T \\ N := J(T) \end{smallmatrix}\right) \in \mathfrak{X}(\gamma) \times \mathfrak{X}(\gamma)$.

The Riemannian geometry of γ is described by its *geodesic curvature* $k : I \to \mathbb{R}$ provided by the *Frenet equations*:

$$\nabla_{T(t)}\mathcal{F}(t) = \begin{pmatrix} 0 & k(t) \\ -k(t) & 0 \end{pmatrix} \mathcal{F}(t) = k(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{F}(t) \tag{2.2}$$

which means:

$$k(t) := \frac{g(\nabla_{\gamma'(t)}T(t), N(t))}{\|\gamma'(t)\|} = \frac{g(\nabla_{\gamma'(t)}\gamma'(t), J(\gamma'(t)))}{\|\gamma'(t)\|^3}. \tag{2.3}$$

Recall also the pair (g, J) yields *the symplectic form* $\Omega(\cdot, \cdot) := g(\cdot, J\cdot)$ and whence:

$$k(t) := \frac{\Omega(\nabla_{\gamma'(t)}\gamma'(t), \gamma'(t))}{\|\gamma'(t)\|^3}.$$

The generic expression of an Euclidean rotation $R(u) \in SO(2)$ is:

$$R(u) := \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -R'(0) \in so(2). \tag{2.4}$$

This short note defines a new frame and correspondingly a new curvature function for γ:

Definition 2.1. The *flow-frame* of γ consists in the pair of vector fields $(E_1^f, E_2^f) \in \mathfrak{X}(\gamma) \times \mathfrak{X}(\gamma)$ given by:

$$\mathcal{F}^f(t) := \begin{pmatrix} E_1^f \\ E_2^f \end{pmatrix}(t) = R(t)\mathcal{F}(t) = \begin{pmatrix} \cos t T(t) - \sin t N(t) \\ \sin t T(t) + \cos t N(t) \end{pmatrix} \tag{2.5}$$

the letter f being the initial of the word "flow". The *flow-curvature* of γ is the smooth function $k^f : I \to \mathbb{R}$ given by *the flow-equations* as the flow-variant of the equation (2.3):

$$k^f(t) := \frac{g(\nabla_{\gamma'(t)} E_1^f(t), E_2^f(t))}{\|\gamma'(t)\|}. \tag{2.6}$$

We note as main result:

Proposition 2.2. *The expression of the flow-curvature is*:

$$k^f(t) = k(t) - \frac{1}{\|\gamma'(t)\|} = \frac{g(\nabla_{\gamma'(t)} T(t) - N(t), N(t))}{\|\gamma'(t)\|} < k(t). \tag{2.7}$$

If the decomposition of the vector field $X \in \mathfrak{X}(\gamma)$ with respect to the Frenet frame is $X(t) = A(t)T(t) + B(t)N(t)$ with the smooth functions $A, B : I \to \mathbb{R}$ then its decomposition with respect to the flow-frame is:

$$X(t) = [A(t)\cos t - B(t)\sin t]E_1^f(t) + [B(t)\cos t + A(t)\sin t]E_2^f(t). \tag{2.8}$$

In the particular case when γ is a flow-geodesic, i.e., $k^f \equiv 0$ the Frenet equations (2.2) *become*:

$$\nabla_{\gamma'(t)} \mathcal{F}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{F}(t). \tag{2.9}$$

A flow-geodesic having the constant speed $v > 0$ is a Riemannian circle of radius v.

Proof. Fix a local chart (x, y) on M and let $\mathcal{E} := \binom{E_1}{E_2}$ be the oriented g-orthonormal frame obtained by applying the Gram-Schmidt process to the local frame $\{\partial_x, \partial_y\}$. The *angular function* of γ in this chart is the smooth function $\theta : I \to M$ given by the following identity in which ^{T} is the transpose operator:

$$\mathcal{F}^T(t) = \mathcal{E}^T \cdot R(\theta(t)). \tag{2.10}$$

The frame $\mathcal{E}$ defines the *connection form* $\omega_2^1 \in \Omega^1(M)$ by:

$$\nabla E_2 = \omega_2^1 \otimes E_1 \to \omega_2^1(U) = g(\nabla_U E_2, E_1), \forall U \in \mathfrak{X}(M). \tag{2.11}$$

Then the local expression of the curvature is the Liouville formula (Corollary 4.6 of [13, p. 351]):

$$k(t) = \frac{1}{\|\gamma'(t)\|}[\theta'(t) - \omega_2^1(\gamma'(t))], \tag{2.12}$$

since in the frame $\mathcal{E}$ the rotation J is exactly $R'(0) \in so(2)$. Now, the relation (2.10) becomes:

$$\mathcal{F} = R(\theta(t))^T \mathcal{E} = R(-\theta(t))\mathcal{E} \tag{2.13}$$

and hence the *flow-angular function*, i.e., the function θ^f corresponding to the frame $\mathcal{F}^f$ is given by:

$$\mathcal{F}^f(t) = R(-\theta^f(t))\mathcal{E} = R(t)R(-\theta(t))\mathcal{E} = R(t - \theta(t))\mathcal{E} \to \theta^f(t) = \theta(t) - t. \tag{2.14}$$

In conclusion, since the formula (2.12) holds also for the "flow" case we get:

$$k^f(t) = \frac{1}{\|\gamma'(t)\|}[(\theta^f)'(t) - \omega_2^1(\gamma'(t))] = k(t) - \frac{1}{\|\gamma'(t)\|}.$$

The decomposition (2.8) and the identity (2.9) follows immediately. A Riemannian circle of radius r in a Riemannian geometry is a 2-Frenet curve having the constant curvature $k = \frac{1}{r}$ conform [9]. □

Remark 2.3. i) Fix a smooth function $\Omega : I \to \mathbb{R}$. If in the given definition we use the general rotation $R(\Omega(t))$ instead of $R(t)$ we define then the Ω-frame and the Ω-curvature $k^{\Omega}(t) = k(t) - \frac{\Omega'(t)}{\|\gamma'(t)\|}$ since $\theta^{\Omega}(t) = \theta(t) - \Omega(t)$.

ii) An important tool in dynamics is the Fermi-Walker derivative, i.e., the map ([9]) $\nabla^{FW} : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$:

$$\begin{aligned} \nabla^{FW}(X) &:= \nabla_{\gamma'}X + \|r'(\cdot)\|k[g(X,N)T - g(X,T)N] \\ &= \frac{d}{dt}X + \|r'(\cdot)\|k[X^\flat(N)T - X^\flat(T)N] \end{aligned} \tag{2.15}$$

with $X^\flat$ the differential 1-form dual to X with respect to the metric g. In a matrix form we can express this as follows:

$$\nabla_C^{FW} = \nabla_{\gamma'} - \|r'\|k \begin{vmatrix} (\cdot)^\flat(T) & (\cdot)^\flat(N) \\ T & N \end{vmatrix} = \frac{d}{dt} + \|r'\|k \begin{vmatrix} T & (\cdot)^\flat(T) \\ N & (\cdot)^\flat(N) \end{vmatrix}. \tag{2.16}$$

It is natural to make here a remark concerning *rotation-minimizing fields* $X \in \mathfrak{X}(\gamma)$, i.e., fields satisfying:

$$\nabla_{\gamma'(t)}X(t) = \lambda(t)T(t), \quad g(X(t), T(t)) = 0$$

for a smooth function $\lambda = \lambda(t)$. Then the Fermi-Walker derivative of such X is also parallel with the tangent vector field T:

$$\nabla^{FW}X(t) = [\lambda(t) + \|r'(t)\|k(t)g(X(t), N(t))]T(t).$$

Calculating the Fermi-Walker derivative on our frames we get:

$$\nabla^{FW}(T) = \nabla^{FW}(N) = 0, \quad \nabla^{FW}(E_1^f) = -E_2^f, \quad \nabla^{FW}(E_2^f) = E_1^f \tag{2.17}$$

which means that $-\nabla^{FW}$ acts on the frame $\mathcal{F}^f$ as J on $\mathcal{F}$. With the matrix notation we can express these relations as:

$$\nabla^{FW}(\mathcal{F}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \nabla^{FW}(\mathcal{E}) = R\left(\frac{\pi}{2}\right)\mathcal{E} \tag{2.18}$$

and the Fermi-Walker derivative can be expressed in terms of k_f as:

$$\nabla^{FW}(X) = \nabla_{\gamma'}X + (1 + \|r'\|k_f)[X^\flat(N)T - X^\flat(T)N]. \tag{2.19}$$

Also, we can define *the flow-Fermi-Walker* derivative as:

$$\begin{aligned} \nabla^{fFW}(X) &:= \nabla_{\gamma'}X + \|r'(\cdot)\|k_f[X^\flat(N)T - X^\flat(T)N] \\ &= \nabla^{FW}(X) + T \wedge_g N(X) \end{aligned} \tag{2.20}$$

with the skew-symmetric endomorphism $\wedge \in so(2)$ defined by:

$$\begin{cases} X \wedge_g Y := g(X, \cdot)Y - g(Y, \cdot)X = (X^1Y^2 - X^2Y^1)R\left(\frac{\pi}{2}\right), \\ X = X^1E_1 + X^2E_2, \quad Y = Y^1E_1 + Y^2E_2. \end{cases}$$

Then:

$$\nabla^{fFW}(\mathcal{F}) = R\left(-\frac{\pi}{2}\right)\mathcal{F}, \quad \nabla^{fFW}(\mathcal{E}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.21}$$

As in the usual case, if $V, W \in \mathfrak{X}(\gamma)$ are flow-Fermi-Walker fields, i.e., with zero flow-Fermi-Walker derivative, then the value $g(V, W) \in \mathbb{R}$ is constant along γ.

iii) Suppose that γ is of unit speed on the interval $[0, L > 0]$. We have then a Gauss-Bonnet type formula:

$$\theta(L) - \theta(0) - L = \int_0^L k^f(s)ds + \int_\gamma \omega_2^1, \tag{2.22}$$

where the second integral is a curvilinear one.

iv) The term $\omega_2^1(\gamma'(t))$ from the formulae of the curvatures can be expressed in terms of Christoffel symbols of the metric g as:

$$\omega_2^1(\gamma'(t)) = \|\gamma'(t)\|[\Gamma_{12}^1(\gamma(t))\cos\theta(t) + \Gamma_{22}^1(\gamma(t))\sin\theta(t)]. \tag{2.23}$$

v) The Liouville formula (2.12) was generalized for the Euclidean plane by using an arbitrary vector field $U \in \mathfrak{X}(\gamma)$. We follow this idea in the present Riemannian setting and then suppose that $U \in \mathfrak{X}(\gamma)$ is of unit norm with respect to g. Then there exists the *U-angular function* $\theta^U : I \to M$ such that $U = (\cos\theta^U)E_1 + (\sin\theta^U)E_2$. Then we define the *U-curvature* of γ through:

$$k^U(t) := (\theta^U)'(t) - \omega_2^1(U(t)). \tag{2.24}$$

Therefore: $k^T = k$ and $k^N(t) = \theta'(t) - \omega_2^1 \circ J(T(t))$.

3. Examples

Example 3.1. Due to Proposition 2.2 we consider M an open subset of $\mathbb{R}^2$ and fix the Cartesian circle $\mathcal{C}(M_0(x_0, y_0), R > 0)$: $(x - x_0)^2 + (y - y_0)^2 = R^2$ which we will consider in three geometries:

a) In the Euclidean geometry with the global and constant frame $\mathcal{E} = \{E_1 = (1, 0), E_2 = (0, 1)\}$ we have $\omega_2^1 \equiv 0$. For the given circle we use the trigonometric parametrization: $\gamma(t) = Re^{it}$ since the translations are Euclidean isometries. Hence $\theta(t) = t + \frac{\pi}{2}$ and then:

$$k^f(t) = \frac{1}{R} - \frac{1}{R} = 0.$$

In conclusion, the circles are flow-geodesics of $\mathbb{E}^2 := (\mathbb{R}^2, g_{eucl})$ and the Gauss-Bonnet identity (2.22) holds with $L = 2\pi$. Let us remark that in the paper [3] there are determined all Finsler metrics of Randers type for which the Riemannian part is a scalar multiple of the Euclidean metric, on an open subset of the Euclidean plane, whose geodesics are circles. Also, in the paper [1] the notion of *entropy* is introduced for the Euclidean ovals and it is proved that the oval with maximum entropy is the circle; hence is an open problem to develop the flow variant of the entropy.

b) In the Poincaré upper half plane model of the hyperbolic geometry $\mathbb{H}^2 := \left(\mathbb{C}_+ = \{z \in \mathbb{C}; Im\ z > 0\}, ds = \frac{|dz|}{Im\ z}\right)$. Hence $y_0 > R > 0$ and x_0 again can be zero since the horizontal translations are hyperbolic isometries. With the global frame $\mathcal{E} = \{y\partial_x, y\partial_y\}$ we have:

$$\omega_2^1(x,y) = -\frac{dx}{y} \tag{3.1}$$

and for the parametrization $\gamma(t) = (0, y_0) + Re^{it}$ it results:

$$\begin{cases} \|\gamma'(t)\| = ds(T(t)) = \frac{R}{y_0 + R\sin t}, \ T(t) = (y_0 + R\sin t)(-\sin t, \cos t), \\ \omega_2^1(T(t)) = \sin t, \quad \theta(t) = t + \frac{\pi}{2}. \end{cases} \tag{3.2}$$

Then:

$$\begin{cases} k(t) = \frac{y_0 + R\sin t}{R} - \sin t = \frac{y_0}{R} = constant > 1, \\ k^f(t) = \frac{y_0}{R} - \frac{y_0 + R\sin t}{R} = -\sin t. \end{cases} \tag{3.3}$$

It is worth noting that this last curvature is an *universal* one, i.e., does not depend on the pair (y_0, R). With $R \to y_0$ in the first relation of (3.3) we obtain that the circles tangent to the real axis have $k = constant = 1$ and then there are horocycles of $\mathbb{H}^2$. The unit speed horocycles are flow-geodesics of $\mathbb{H}^2$.

c) More generally than the hyperbolic metric suppose that we have a *conformal Euclidean metric* $g = E(z)|dz|^2$ on M. Then, for the parametrization $\gamma(t) = (x_0, y_0) + Re^{it}$ and the global frame $\mathcal{E} = \{\frac{\partial_x}{\sqrt{E}}, \frac{\partial_y}{\sqrt{E}}\}$ we have the same function $\theta = \theta(t)$ as above and:

$$\begin{cases} \|\gamma'(t)\| = R\sqrt{E(\gamma(t))}, \quad T(t) = \frac{1}{\sqrt{E(\gamma(t))}}(-\sin t, \cos t), \\ \omega_2^1(T(t)) = \frac{-1}{2[E(\gamma(t))]^{\frac{3}{2}}}\left(E_x(\gamma(t))\cos t + E_y(\gamma(t))\sin t\right) \\ \qquad = \frac{-g_{eucl}(\nabla_{g_{eucl}} E(\gamma(t)),\ e^{it})}{2[E(\gamma(t))]^{\frac{3}{2}}} \end{cases} \tag{3.4}$$

with $\nabla_{g_{eucl}} E = (E_x, E_y)$ the Euclidean gradient vector field of the smooth function E. We obtain:

$$\begin{cases} k(t) = \frac{1}{R\sqrt{E(\gamma(t))}} + \frac{1}{2[E(\gamma(t))]^{\frac{3}{2}}}[E_x(\gamma(t))\cos t + E_y(\gamma(t))\sin t], \\ k^f(t) = \frac{1}{2[E(\gamma(t))]^{\frac{3}{2}}}[E_x(\gamma(t))\cos t + E_y(\gamma(t))\sin t]. \end{cases} \tag{3.5}$$

We point out that the first relation (3.5) results also directly from the Minding formula of [10, p. 474].

For example, if $M = \mathbb{R}^2$ and $E(z) = \frac{1}{1+|z|^2}$ we have the famous Hamilton's cigar soliton ([4, p. 3339]) which is a steady Ricci soliton. Therefore:

$$\begin{cases} k(t) = \frac{\sqrt{1+R^2}}{R} + \frac{1}{\sqrt{1+R^2}}(R + x_0\cos t + y_0\sin t), \\ k^f(t) = \frac{1}{\sqrt{1+R^2}}(R + x_0\cos t + y_0\sin t). \end{cases} \tag{3.6}$$

If the circle $\mathcal{C}$ is centered in the origin $O \in \mathbb{R}^2$ we have that both curvatures are constant:

$$k \equiv \frac{\sqrt{1+R^2}}{R} + \frac{R}{\sqrt{1+R^2}} \geq 2, \quad k^f \equiv \frac{R}{\sqrt{1+R^2}} \in \left(0, \sqrt{\frac{R}{2}}\right]. \tag{3.7}$$

Example 3.2. In the same model of plane hyperbolic geometry as above let γ be the positive vertical axis $\gamma(t) = (0, t)$ for $t \in (0, +\infty)$. Then:

$$\|\gamma'(t)\| = \frac{1}{t}, \quad T(t) = \gamma(t), \quad \omega_2^1(T(t)) = 0, \quad \theta(t) = constant = \frac{\pi}{2} \tag{3.8}$$

and hence $k \equiv 0$ while $k^f(t) = -t < 0$. The unit speed parametrization of the positive vertical axis is $\gamma(s) = (0, 0, e^s)$ and the unit speed Euclidean curve having the curvature $k(s) = -s$ is the Cornu type spiral:

$$Cornu(s) = \sqrt{\pi}\left(C\left(\frac{s}{\sqrt{\pi}}\right), -S\left(\frac{s}{\sqrt{\pi}}\right)\right) \tag{3.9}$$

for $C = C(\cdot)$ and $S = S(\cdot)$ the Fresnel integrals.

Example 3.3. In this example we solve a kind of *inverse problem*, i.e., we determine a Riemannian metric with two prescribed families of flow-geodesics. We search for an already diagonal one:

$$g = A^2(x, y)dx^2 + B^2(x, y)dy^2 \tag{3.10}$$

and the first curve is "the meridian" $\gamma : y = constant = y_0$. Then:

$$\|\gamma'(t)\| = A(t, y_0), \quad k(t) = -\frac{A_y}{AB}(t, y_0) \tag{3.11}$$

which yields:

$$k^f(t) = -\frac{1}{A(t, y_0)}\left(\frac{A_y}{B}(t, y_0) + 1\right) \tag{3.12}$$

making γ a flow-geodesic if and only if $A_y(x, y_0) = -B(x, y_0) < 0$; then the function $A(\cdot, y_0)$ is a decreasing one. For example, if the metric is a warped one ($A = 1$ and $B = B(x)$), then γ is a unit speed geodesic; hence $k^f \equiv -1$.

Suppose now that the curve is "the parallel" $\gamma : x = constant = x_0$. Then:

$$\|\gamma'(t)\| = B(x_0, t), \quad k(t) = \frac{B_x}{AB}(x_0, t) \tag{3.13}$$

which yields:

$$k^f(t) = \frac{1}{B(x_0, t)}\left(\frac{B_x}{A}(x_0, t) - 1\right) \tag{3.14}$$

making γ a flow-geodesic if and only if $B_x(x_0, y) = A(x_0, y) > 0$; then the function $B(x_0, \cdot)$ is an increasing one.

The Riemannian metric g provided by:

$$A(x, y) := \cos(x + y), \quad B(x, y) := \sin(x + y) \tag{3.15}$$

is of Chebyshev type and it have "the meridians" and "the parallels" as flow-geodesics. With the global frame $\mathcal{E} = \{\frac{\partial_x}{A}, \frac{\partial_y}{B}\}$ we obtain the connection form $\omega^1_2(x, y) = dx + dy = d(x + y)$ and hence the Gaussian curvature vanishes; so g is a flat Riemannian metric on the manifold $M = \{(x, y) \in \mathbb{R}^2; 0 < x + y < \frac{\pi}{2}\}$. For more details on this metric we use the formalism of the book [12, p. 378]: if the metric g is $(\omega^1)^2 + (\omega^2)^2$ then we have two invariants, J and K, provided by the structural equations:

$$d\omega^1 = J\omega^1 \wedge \omega^2, \quad d\omega^2 = K\omega^1 \wedge \omega^2. \tag{3.16}$$

For our Chebysev metric g from (3.15) it results:

$$J(x, y) = \frac{1}{\cos(x + y)} > 0, \quad K(x, y) = \frac{1}{\sin(x + y)} > 0. \tag{3.17}$$

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