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GEOMETRY OF A SEMI-SYMMETRIC RECURRENT METRIC CONNECTION

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Abstract. In the present paper, we study a semi-symmetric recurrent metric connection and verify its various geometric properties.

1. Introduction

Let $M^n = (M^n, g)$ be a Riemannian manifold of dimension n with a metric tensor g. A linear connection ∇ on M^n satisfies

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,
(ii) $\nabla_X (fY) = (Xf)Y + f\nabla_X Y$,

where f, g are smooth functions on M^n and X, Y, Z are smooth vector fields on M^n. The torsion tensor T of ∇ is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

If the torsion tensor T vanishes, then ∇ is said to be symmetric, otherwise it is non-symmetric. If the metric tensor g of M^n satisfies $\nabla g = 0$, then ∇ is said to be a metric connection, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In particular, a nonsymmetric connection ∇ is called semi-symmetric if the torsion tensor T of ∇ satisfies

$$T(X,Y) = u(Y)X - u(X)Y,$$

where u is a 1-form on M^n. An important research work was carried out on the Riemannian manifold equipped with a semi-symmetric metric connection in [13]. In fact, Yano [13] proved that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection whose curvature tensor vanishes. On the other hand Agashe and Chafle [1] introduced the idea of a semi-symmetric non-metric connection on a Riemannian manifold and

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this was further developed by Binh, De and Sengupta [8]. Later on such a connection on a Riemannian manifold equipped with certain geometric structures was extensively studied by several authors [2, 3, 6]. In 2008, Tripathi showed a unified theory of connection which unifies the concepts of various (non-)metric connections such as semi-symmetric (non-)metric connections in [12]. Furthermore, as a particular case he mentioned a semi-symmetric recurrent metric connection which has recently been studied in [14]. Considering this aspect we are motivated to study such a connection. This paper is organized as follows:

Section 2 is devoted to verifying the symmetries and identities of curvature with respect to the semi-symmetric recurrent metric connection under certain conditions.

In Section 3, we investigate a weakly symmetric manifold equipped with the semi-symmetric recurrent metric connection whose Ricci tensor vanishes.

In Section 4, we consider a Riemannian manifold with semi-symmetric recurrent metric connection whose associated vector field is concurrent, and study some semisymmetry conditions on such a manifold.

2. Semi-symmetric recurrent metric connection and its curvature properties

In [12], a linear connection $\bar{\nabla}$ on a Riemannian manifold $M^n = (M^n, g)$ is defined as

$$\bar{\nabla}_X Y = \nabla_X Y - u(X)Y, \tag{1}$$

where ∇ denotes the Levi-Civita connection and u is a 1-form on M^n. Using (1), the torsion tensor $\bar{T}$ of M^n with respect to the connection $\bar{\nabla}$ is given by

$$\bar{T}(X, Y) = u(Y)X - u(X)Y. \tag{2}$$

Further, using (1), we have

$$(\bar{\nabla}_X g)(Y, Z) = 2u(X)g(Y, Z). \tag{3}$$

A linear connection $\bar{\nabla}$ defined by (1) is called a semi-symmetric recurrent metric connection [12] (briefly, SSRM connection). For instance, we can find a non-trivial SSRM connection on a product manifold as follows:

Example. Let $M^n = (M^n, g_{M^n})$ be a Riemannian manifold. Then we have a standard product Riemannian manifold M^{n+1} of M^n with S^1. Since S^1 has a nowhere vanishing vector field, we can choose such a vector field U tangent to S^1 at each point in $M^n \times S^1$ and so we obtain a non-trivial 1-form u associated with U on $M^{n+1} = (M^{n+1}, g)$ by $g(U, X) = u(X)$. Then we have a non-trivial SSRM connection $\bar{\nabla}$ in M^{n+1} by setting $\bar{\nabla}_X Y = \nabla_X Y - u(X)Y$.

Analogous to the definition of curvature tensor R, Ricci tensor r, scalar curvature s and Weyl curvature tensor W, we define the curvature tensor $\bar{R}$, Ricci tensor $\bar{r}$, scalar curvature $\bar{s}$ and Weyl curvature tensor $\bar{W}$ with respect to SSRM connection $\bar{\nabla}$ by

$$\bar{R}(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V) = g(\bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z, V), \tag{4}$$

$$\bar{r}(Y,Z) = \sum_{i=1,\dots,n} \bar{R}(e_i,Y,Z,e_i), \tag{5}$$

$$\bar{s} = \sum_{i=1,\dots,n} \bar{r}(e_i,e_i)$$

and

$$\begin{aligned}\bar{W}(X,Y,Z,V) = \bar{R}(X,Y,Z,V) &- \frac{\bar{s}}{2n(n-1)} g\bullet g(X,Y,Z,V)\\ &- \frac{1}{n-2}(\bar{r} - \frac{\bar{s}}{n}g)\bullet g(X,Y,Z,V),\end{aligned}$$

where $\{e_i\}_{i=1,\dots,n}$ is an orthonormal frame. Here the symbol $\bullet$ is the Nomizu-Kulkarni product of symmetric (0,2)-tensors generating a curvature type tensor:

$$\begin{aligned}h\bullet k(X,Y,Z,W) = h(X,Z)k(Y,W) + h(Y,W)k(X,Z) - h(X,W)k(Y,Z)\\ - h(Y,Z)k(X,W).\end{aligned}$$

Note that $W = 0$ if and only if $M^n = (M^n, g)$ is conformally flat. The Weyl curvature tensor depends only on the conformal class of $M^n = (M^n, g)$. Moreover, it satisfies the curvature symmetries and so we can treat it as a conformal curvature tensor. In particular, the Weyl curvature tensor is traceless.

A Riemannian manifold $M^n = (M^n, g)$ is called Einstein with respect to $\bar{\nabla}$ if the Ricci tensor $\bar{r}$ with respect to $\bar{\nabla}$ is proportional to the metric tensor g on M^n, i.e., $\bar{r} = \frac{\bar{s}}{n}g$. Concerning the symmetries and identities of curvature with respect to SSRM connection, we obtain the following:

Theorem 2.1. *Let $M^n = (M^n, g)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$. If the* 1*-form u in* (1) *is closed, then we have*

(i) $\bar{R}(X,Y,Z,W) = -\bar{R}(Y,X,Z,W)$,
(ii) $\bar{R}(X,Y,Z,W) = -\bar{R}(X,Y,W,Z)$,
(iii) $\bar{R}(X,Y,Z,W) = \bar{R}(Z,W,X,Y)$,
(iv) $\bar{R}(X,Y,Z,W) + \bar{R}(Y,Z,X,W) + \bar{R}(Z,X,Y,W) = 0$,
(v) $\bar{r}(Y,Z) = \bar{r}(Z,Y)$,
(vi) *M^n is Einstein with respect to $\bar{\nabla}$ if and only if M^n is Einstein.*

Proof. From (4), it follows that (i) holds true. By virtue of (1) and (4), one can see [12]

$$\begin{aligned}\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) &- 2du(X,Y)g(Z,W)\\ &- \frac{1}{2}g(U,U)g(X,W)g(Y,Z) + \frac{1}{2}g(U,U)g(X,Z)g(Y,W),\end{aligned} \tag{6}$$

where U is a vector field given by $g(U,X) = u(X)$. Taking account of (6) and $R(X,Y,Z,W) + R(X,Y,W,Z) = 0$, we obtain

$$\bar{R}(X,Y,Z,W) + \bar{R}(X,Y,W,Z) = -4du(X,Y)g(Z,W),$$

which yields from $du = 0$ that (ii) is valid. From (6), we immediately get

$$\bar{R}(X,Y,Z,W) - \bar{R}(Z,W,X,Y) = -2du(X,Y)g(Z,W) + 2du(Z,W)g(X,Y).$$

By the help of $du = 0$, the above identity yields that (iii) holds. It follows from (6) and the first Bianchi identity that

$$\begin{aligned} &\bar{R}(X,Y,Z,W) + \bar{R}(Y,Z,X,W) + \bar{R}(Z,X,Y,W) \\ = &- 2du(X,Y)g(Z,W) - 2du(Y,Z)g(X,W) - 2du(Z,X)g(Y,W). \end{aligned}$$

By virtue of $du = 0$, the last identity yields that (iv) is valid. Taking account of both (5) and (6), we have

$$\bar{r}(Y,Z) = r(Y,Z) - \frac{1}{2}(n-1)g(U,U)g(Y,Z) + 2du(Y,Z). \tag{7}$$

Considering $du = 0$ and (7), we have $\bar{r}(Y,Z) = \bar{r}(Z,Y)$ and so (v) holds true. Again from (7) and $du = 0$, it follows immediately that if M^n is Einstein with respect to $\bar{\nabla}$, then M^n is Einstein, and vice versa. Therefore (vi) is valid too. This completes the proof of Theorem 2.1. □

Concerning the scalar curvature $\bar{s}$ with respect to $\bar{\nabla}$, we have:

Theorem 2.2. *Let $M^n = (M^n, g)$ $(n \geq 2)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$. If $\bar{s} - s \geq 0$, then the connections $\bar{\nabla}$ and ∇ coincide.*

Proof. Taking account of both (7) and the definition of $\bar{s}$, we have

$$\bar{s} = s - \frac{1}{2}n(n-1)g(U,U), \tag{8}$$

since $tr\ du = 0$.

Thus if $\bar{s} - s \geq 0$, the above identity yields $U = 0$, equivalently $u = 0$. Therefore it follows from (1) that $\bar{\nabla} = \nabla$. The proof of Theorem 2.2 is completed. □

Concerning Weyl curvature tensors, we have:

Corollary 2.3. *Let $M^n = (M^n, g)$ $(n \geq 2)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$. If $\bar{s} - s \geq 0$, then the Weyl curvature tensors $\bar{W}$ and W coincide.*

Proof. It is an immediate consequence of Theorem 2.2. □

Let σ_p be a two-dimensional plane in the tangent space at a point p spanned by vectors X, Y. Then the sectional curvature $k(\sigma_p)$ is defined by

$$k(\sigma_p) = -\frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

Sectional curvature $k(\sigma_p)$ is uniquely determined by the plane σ_p and is independent of the vectors X, Y in the plane σ_p. If the sectional curvature $k(\sigma_p)$ is a constant for all planes σ_p and each point p on M^n, then M^n is said to be a space of constant curvature. Concerning sectional curvature $k(\sigma_p)$, the following fact [5] (namely, Schur's theorem) is well known: If the sectional curvature $k(\sigma_p)$ is independent of the plane σ_p chosen at each point p on M^n, then M^n is a space of constant curvature. Analogous to the definition of the sectional curvature $k(\sigma_p)$, we define the sectional curvature $\bar{k}(\sigma_p)$ with respect to $\bar{\nabla}$ by

$$\bar{k}(\sigma_p) = -\frac{\bar{R}(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

Concerning the sectional curvature $\bar{k}(\sigma_p)$ with respect to $\bar{\nabla}$, we obtain a generalized Schur's theorem as follows:

Theorem 2.4. *Let $M^n = (M^n, g)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$ whose 1-form u is closed. If the sectional curvature $\bar{k}(\sigma_p)$ with respect to $\bar{\nabla}$ is independent of the plane σ_p chosen at each point p on M^n, then M^n is a space of constant curvature with respect to $\bar{\nabla}$ if and only if the length of the associated vector field U is constant.*

Proof. As a well known fact [7], the curvature tensor is completely determined by the sectional curvature. Therefore taking the curvature symmetries (i), (ii), (iii) and (iv) in Theorem 2.1 into consideration, we have [7]

$$\bar{R}(X,Y,Z,W) = \bar{k}(p)(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)),$$

which yields from (5)

$$\bar{r}(Y,Z) = (n-1)\bar{k}(p)g(Y,Z).$$

Taking account of $du = 0$, (7) and the above identity, we get

$$r(Y,Z) = \frac{n-1}{2}(g(U,U) + 2\bar{k}(p))g(Y,Z),$$

which implies that M^n is Einstein and so [5]

$$g(U,U) + 2\bar{k}(p) = constant.$$

Therefore it is clear that $g(U,U) = constant$ implies $\bar{k}(p) = constant$, and vice versa. This completes the proof of Theorem 2.4. □

In [13], Yano showed that a Riemannian manifold with semi-symmetric metric connection whose curvature tensor vanishes is conformally flat. In case of SSRM connection, we shall prove the following:

Theorem 2.5. *Let $M^n = (M^n, g)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$. If the curvature tensor $\bar{R}$ with respect to $\bar{\nabla}$ vanishes, then M^n is a space of constant curvature, and in addition the length of the associated vector field U is constant.*

Proof. Taking account of (6) and $\bar{R} = 0$ together, we get

$$R(X,Y,Z,W) = 2du(X,Y)g(Z,W) + \frac{1}{2}g(U,U)(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)).$$

Since the curvature tensor R possesses the skew symmetric property with respect to Z, W, we obtain $du = 0$, which makes the last relation reduce to

$$R(X,Y,Z,W) = \frac{1}{2}g(U,U)(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)).$$

Therefore it follows from the last relation and the classical Schur's theorem that M^n is a space of constant curvature, and hence we have $g(U,U) = constant$. The proof of Theorem 2.5 is completed. □

3. Weakly symmetric manifold and SSRM connection

A non-flat Riemannian manifold $M^n = (M^n, g)$ is called a weakly symmetric manifold [11] if its curvature tensor R satisfies

$$\begin{aligned}(\nabla_X R)(Y,Z,V,W) = {} & A(X)R(Y,Z,V,W) + B(Y)R(X,Z,V,W) \\ & + C(Z)R(Y,X,V,W) + D(V)R(Y,Z,X,W) \\ & + E(W)R(Y,Z,V,X),\end{aligned} \tag{9}$$

where the 1-forms A, B, C, D, E are not simultaneously zero. In [4], De and Bandyopadhyay prove that the 1-form B, C, D and E in (9) are related by $B = C$, $D = E$. Thus the equation (9) reduces to

$$\begin{aligned}(\nabla_X R)(Y,Z,V,W) = {} & A(X)R(Y,Z,V,W) + B(Y)R(X,Z,V,W) \\ & + B(Z)R(Y,X,V,W) + D(V)R(Y,Z,X,W) \\ & + D(W)R(Y,Z,V,X).\end{aligned} \tag{10}$$

Now we investigate a weakly symmetric manifold equipped with SSRM connection whose Ricci tensor vanishes. In this case, we have:

Lemma 3.1. *Let $M^n = (M^n, g)$ $(n > 2)$ be a weakly symmetric manifold with SSRM connection $\bar{\nabla}$ whose Ricci tensor $\bar{r}$ vanishes. Then the scalar curvature s of M^n is zero.*

Proof. Let ξ_1, ξ_2, ξ_3 be the associated vector fields corresponding to the 1-forms A, B, D, respectively, that is,

$$g(X,\xi_1) = A(X),\ g(X,\xi_2) = B(X),\ g(X,\xi_3) = D(X).$$

Contracting (10) over Y and W yields

$$\begin{aligned}(\nabla_X r)(Z,V) = {} & A(X)r(Z,V) + R(X,Z,V,\xi_2) + B(Z)r(X,V) \\ & + D(V)r(Z,X) + R(\xi_3,Z,V,X).\end{aligned} \tag{11}$$

Further contracting (11) over Z and V, we get

$$\nabla_X s = A(X)s + 2r(X,\xi_2) + 2r(\xi_3,X). \tag{12}$$

On the other hand, contracting (10) over Y, W and then over X, Z, we obtain from the second Bianchi identity

$$\frac{1}{2}\nabla_V s = D(V)s + r(\xi_1, V) + r(\xi_2, V) - r(\xi_3, V). \tag{13}$$

In a similar manner, contracting (10) with respect to Y, W and then with respect to X, V, we have from the second Bianchi identity

$$\frac{1}{2}\nabla_Z s = B(Z)s + r(Z, \xi_1) - r(Z, \xi_2) + r(Z, \xi_3). \tag{14}$$

It follows from (12), (13) and (14) that the following relations are valid:

$$(A(X) - 2D(X))s = 2(r(\xi_1, X) - 2r(\xi_3, X)), \tag{15}$$

$$(A(X) - 2B(X))s = 2(r(\xi_1, X) - 2r(\xi_2, X)). \tag{16}$$

Suppose that a weakly symmetric manifold $M^n = (M^n, g)$ allows a SSRM connection $\bar{\nabla}$ whose Ricci tensor $\bar{r}$ vanishes. Then taking account of (7), we get

$$r(Y, Z) = \frac{1}{2}(n-1)g(U, U)g(Y, Z) - 2du(Y, Z),$$

which implies

$$r(Y, Z) = \frac{1}{2}(n-1)g(U, U)g(Y, Z)$$

because the Ricci tensor r of M^n is symmetric. Therefore M^n is Einstein and so its scalar curvature s is constant [5]. If we assume that the scalar curvature s of M^n is non-zero, then it follows from (15), (16) and $r = \frac{s}{n}g$ $(n > 2)$ that we have $A = 2B = 2D$. And so (10) reduces to the following form

$$\begin{aligned}(\nabla_X R)(Y, Z, V, W) = 2D(X)R(Y, Z, V, W) &+ D(Y)R(X, Z, V, W) \\ &+ D(Z)R(Y, X, V, W) + D(V)R(Y, Z, X, W) \\ &+ D(W)R(Y, Z, V, X).\end{aligned} \tag{17}$$

Since M^n is Einstein, contracting (17) over Y, W yields

$$\begin{aligned}0 = (\nabla_X r)(Z, V) = 2D(X)r(Z, V) &+ R(X, Z, V, \xi_3) + D(Z)r(X, V) \\ &+ D(V)r(Z, X) + R(\xi_3, Z, V, X).\end{aligned} \tag{18}$$

Permutting cyclically (18) twice over X, Z, V and adding these permutted equations with (18), we get from the first Bianchi identity

$$D(X)r(Z, V) + D(Z)r(V, X) + D(V)r(X, Z) = 0. \tag{19}$$

Further contracting (19) over Z, V yields

$$D(X)s + 2r(\xi_3, X) = 0. \tag{20}$$

Taking account of $r = \frac{s}{n}g$, (20) and $s \neq 0$, we get

$$D = 0,$$

which is a contradiction by the definition of weakly symmetric manifold. Therefore we conclude that a weakly symmetric manifold M^n $(n > 2)$ equipped with SSRM connection $\bar{\nabla}$ whose Ricci tensor $\bar{r}$ vanishes has $s = 0$. This completes the proof of Lemma 3.1. □

As an immediate consequence, Lemma 3.1 leads to the following:

Theorem 3.2. *Let $M^n = (M^n, g)$ be a weakly symmetric manifold M^n $(n > 2)$ with SSRM connection $\bar{\nabla}$ whose Ricci tensor $\bar{r}$ vanishes. Then the connections $\bar{\nabla}$ and ∇ coincide.*

Proof. Since the vanishing of $\bar{r}$ yields $\bar{s} = 0$, it follows from (8) and Lemma 3.1 that $U = 0$, equivalently $u = 0$. And hence we have $\bar{\nabla} = \nabla$. The proof of Theorem 3.2 is completed □

According to Theorem 3.2, we immediately obtain the following:

Corollary 3.3. *There does not exist a weakly symmetric manifold M^n $(n > 2)$ equipped with SSRM connection $\bar{\nabla}$ whose Ricci tensor $\bar{r}$ vanishes unless its Ricci tensor r is zero.*

4. Some semisymmetry conditions

For a $(0, k)$-tensor field A on M^n, we define the tensor $R \cdot A$ by

$$\begin{aligned} &(R(X,Y)\cdot A)(X_1,\ldots,X_k) \\ (21) \quad &= -A(R(X,Y)X_1,X_2,\ldots,X_k) - \cdots - A(X_1,\ldots,X_{k-1},R(X,Y)X_k). \end{aligned}$$

Moreover, if B is a symmetric $(0, 2)$-tensor field, then we define the $(0, k+2)$-tensor $Q(B, A)$ by

$$\begin{aligned} &Q(B,A)(X_1,\ldots,X_k;X,Y) \\ (22) \quad &= -A((X \wedge_B Y)X_1,\ldots,X_k) - \cdots - A(X_1,\ldots,X_{k-1},(X \wedge_B Y)X_k), \end{aligned}$$

where $X \wedge_B Y$ is defined by

$$(X \wedge_B Y)X_i = B(Y, X_i)X - B(X, X_i)Y.$$

If a Riemannian manifold M^n satisfies the condition $R \cdot R = 0$ (resp. $R \cdot r = 0$), then M^n is said to be semisymmetry (resp. Ricci-semisymmetry) [9, 10]. It is easy to see that every locally symmetric manifold is semisymmetry and that every semisymmetry manifold is Ricci-semisymmetry. A vector field V on M^n is said to be concurrent if it satisfies

$$\nabla_X V = fX,$$

where f is a function on M^n. Note that if a vector field V on M^n is concurrent, then its associated 1-form v given by $g(V, X) = v(X)$ is closed. From now on we deal with a Riemannian manifold $M^n = (M^n, g)$ with SSRM connection $\bar{\nabla}$

whose associated vector field U is concurrent. Therefore it follows from (6) and (7) that on such a manifold

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2}g(U,U)(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)), \tag{23}$$

and

$$\bar{r}(Y,Z) = r(Y,Z) - \frac{1}{2}(n-1)g(U,U)g(Y,Z). \tag{24}$$

In this section, we investigate such a manifold satisfying some semisymmetry conditions. First of all, we prove the following statement:

Lemma 4.1. *Let $M^n = (M^n, g)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$ whose associated vector field U is concurrent. Then we have*

$$R \cdot \bar{R} = R \cdot R \tag{25}$$

and

$$\bar{R} \cdot R = R \cdot R - \frac{1}{2}g(U,U)Q(g,R). \tag{26}$$

Proof. Taking account of (21) and (23), we easily obtain

$$R \cdot \bar{R} = R \cdot R.$$

Similarly from (21), (22) and (23), we have

$$\bar{R} \cdot R = R \cdot R - \frac{1}{2}g(U,U)Q(g,R).$$

This completes the proof of Lemma 4.1. □

Consequently, this leads to the following results:

Theorem 4.2. *Let $M^n = (M^n, g)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$ whose associated vector field U is concurrent. Then $R \cdot \bar{R} = 0$ if and only if M^n is semisymmetry.*

Proof. It is obvious from (25) that the statement holds true. □

Theorem 4.3. *Let $M^n = (M^n, g)$ be a semisymmetry Riemannian manifold with SSRM connection $\bar{\nabla}$ whose associated vector field U is concurrent. If the vector field U is nowhere vanishing, and in addition $\bar{R} \cdot R = 0$, then M^n is Einstein.*

Proof. Since M^n is semisymmetry and satisfies the condition $\bar{R} \cdot R = 0$, we get from (26) and $U \neq 0$

$$Q(g,R)(X_1, X_2, X_3, X_4; X, Y) = 0.$$

Contracting the above identity with respect to X_2 and X_3, we have

$$Q(g,r)(X_1, X_4; X, Y) = 0,$$

which implies from (22) that M^n is Einstein. Thus the proof of Theorem 4.3 is completed. □

Theorem 4.4. *Let $M^n = (M^n, g)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$ whose associated vector field U is concurrent. If the vector field U is nowhere vanishing, and in addition $R \cdot \bar{R} - \bar{R} \cdot R = 0$, then M^n is Einstein.*

Proof. Taking account of (25) and (26), we have from the given conditions $R \cdot \bar{R} - \bar{R} \cdot R = 0$ and $U \neq 0$

$$Q(g, R) = 0,$$

which yields by using the same manner as in the proof of Theorem 4.3 that M^n is Einstein. This completes the proof of Theorem 4.4. □

On the other hand, concerning certain Ricci-semisymmetry conditions, we have the following result:

Lemma 4.5. *Let $M^n = (M^n, g)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$ whose associated vector field U is concurrent. Then we have*

$$R \cdot \bar{r} = R \cdot r \tag{27}$$

and

$$\bar{R} \cdot r = R \cdot r - \frac{1}{2} g(U, U) Q(g, r). \tag{28}$$

Proof. Taking account of (21) and (24), we easily obtain

$$R \cdot \bar{r} = R \cdot r.$$

Similarly from (21), (22) and (23), we have

$$\bar{R} \cdot r = R \cdot r - \frac{1}{2} g(U, U) Q(g, r).$$

Hence we get the result as required. □

Consequently, this leads to the following results:

Theorem 4.6. *Let $M^n = (M^n, g)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$ whose associated vector field U is concurrent. Then $R \cdot \bar{r} = 0$ if and only if M^n is Ricci-semisymmetry.*

Proof. It is obvious from (27) that the statement is valid. □

Theorem 4.7. *Let $M^n = (M^n, g)$ be a Ricci-semisymmetry Riemannian manifold with SSRM connection $\bar{\nabla}$ whose associated vector field U is concurrent. If the vector field U is nowhere vanishing, and in addition $\bar{R} \cdot r = 0$, then M^n is Einstein.*

Proof. Since M^n is Ricci-semisymmetry and satisfies the condition $\bar{R} \cdot r = 0$, we have from (28) and $U \neq 0$

$$Q(g, r) = 0,$$

which implies from (22) that M^n is Einstein. Thus the proof of Theorem 4.7 is completed. □

Theorem 4.8. *Let $M^n = (M^n, g)$ be a Riemannian manifold with SSRM connection $\bar{\nabla}$ whose associated vector field U is concurrent. If the vector field U is nowhere vanishing, and in addition $R \cdot \bar{r} - \bar{R} \cdot r = 0$, then M^n is Einstein.*

Proof. Taking account of (27) and (28), we have from the given conditions $R \cdot \bar{r} - \bar{R} \cdot r = 0$ and $U \neq 0$

$$Q(g, r) = 0$$

and hence it follows from (22) that M^n is Einstein. This completes the proof of Theorem 4.8. □

Theorem 4.9. *Let $M^n = (M^n, g)$ be a Ricci-semisymmetry Riemannian manifold with SSRM connection $\bar{\nabla}$ whose associated vector field U is concurrent. If the vector field U is nowhere vanishing, and in addition $\bar{R} \cdot \bar{r} = 0$, then M^n is Einstein.*

Proof. From (21), (22), (23) and (24), we have

$$\bar{R} \cdot \bar{r} = R \cdot r - \frac{1}{2} g(U, U) Q(g, r).$$

If we assume $\bar{R} \cdot \bar{r} = 0$ and $R \cdot r = 0$, then it follows from the above identity and $U \neq 0$ that

$$Q(g, r) = 0,$$

which yields from (22) that M^n is Einstein. Thus the proof of Theorem 4.9 is completed. □

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