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SOLITON FUNCTIONS AND RICCI CURVATURES OF *D*-HOMOTHETICALLY DEFORMED *f*-KENMOTSU ALMOST RIEMANN SOLITONS

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Abstract. The present article contains the study of D-homothetically deformed f-Kenmotsu manifolds. Some fundamental results on the deformed spaces have been deduced. Some basic properties of the Riemannian metric as an inner product on both the original and deformed spaces have been established. Finally, applying the obtained results, soliton functions, Ricci curvatures and scalar curvatures of almost Riemann solitons with several kinds of potential vector fields on the deformed spaces have been characterized.

1. Introduction

The theory of Kenmotsu manifolds was developed by K. Kenmotsu [6] in 1972 and it has been generalized to f-Kenmotsu manifolds (in brief, fKM) [8] in the sequel. A Kenmotsu manifold is warped product of the real line and a Kähler manifold. Warped product manifolds have important applications in the theory of relativity and cosmology. fKMs have been studied by several authors in several contexts. For instances we refer [9, 12, 13].

D-homothetic deformations were introduced by Tanno [11] in order to analyze some topological aspects of a Riemannian manifold. In [7], D-homothetic deformations and Ricci solitons have been studied in the perspective of (κ, μ)-contact metric manifolds.

A Ricci soliton is a fixed solution of Hamiltonian's Ricci flow upto diffeomorphisms and scaling [4]. In 2016, I. E. Hirica and C. Udriste [5], coined the idea of Riemann solitons. After that the notion of Riemann solitons has been extended to that of almost Riemann solitons (in brief, ARS) [1, 2, 10]. A Riemann soliton with potential function λ as a smooth function on a manifold, is called an ARS.

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Recently, A. M. Blaga [2] analyzed certain solitons on Kenmotsu manifolds with the help of D-homothetic deformation and deduced some important results regarding the soliton functions and Ricci curvatures. Now, it is an interesting problem to generalize the results of Blaga to fKMs. To this end, we study D-homothetic deformations on fKMs.

The present paper is arranged as follows: In the preliminary section, we give some basic definitions of fKMs and ARSs. D-homothetically deformed f-Kenmotsu manifolds (in brief DHDfKMs) and its relation with fKMs are discussed in Section 3. In Section 4, we establish some basic properties of the Riemannian metric as an inner product on both the original and deformed spaces. In the last two sections, we study ARSs applying the deduced results of earlier sections.

2. Preliminaries

A $(2n+1)$ dimensional almost contact metric manifold is a differentiable manifold N endowed with an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a $(1,1)$-tensor field, ξ is the characteristic vector field, η is a 1-form and g is the Riemannian metric on N satisfying [3]

$$\phi^2(E) = -E + \eta(E)\xi, \quad \eta(\xi) = 1, \quad \eta(E) = g(E,\xi),$$
$$g(\phi E, \phi F) = g(E,F) - \eta(E)\eta(F).$$

By the consequences of the above, we have

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi E, F) = -g(E, \phi F)$$

for all vector fields E, $F \in \chi(N)$, the set of all vector fields on N. The almost contact metric manifold N is called fKM if the covariant differentiation of ϕ satisfies

$$(\nabla_E \phi)F = f\big(g(\phi E, F)\xi - \eta(F)\phi E\big), \tag{2.1}$$

where $f \in C^\infty(N)$ and $df \wedge \eta = 0$ for dim(N)≥ 5. If f=β, a non-zero constant, then the manifold is called β-Kenmotsu and if $f = 1$, then it is called Kenmotu manifold. An fKM is called regular if $f^2 + \xi(f) \neq 0$.

If we replace F by ξ in (2.1), a direct calculation gives

$$\nabla\xi = f\big(I - \eta \otimes \xi\big). \tag{2.2}$$

As an application of (2.2), we have

$$(\nabla_E \eta)F = f\big(g(E,F) - \eta(E)\eta(F)\big) \tag{2.3}$$

for any vector fields E and $F \in \chi(N)$. For a $(2n+1)$-dimensional fKM, we can compute the following:

$$\pounds_\xi g = 2f\big(g - \eta \otimes \eta\big), \tag{2.4}$$

$$div(\xi) = 2nf, \tag{2.5}$$

$$R(E,F)\xi = (Ef)(F - \eta(F)\xi) - (Ff)(E - \eta(E)\xi) + f^2\big(\eta(E)F - \eta(F)E\big), \tag{2.6}$$

$$Ric(F,\xi) = (1-2n)(Ff) - (2nf^2 + \xi(f))\eta(F), \tag{2.7}$$

where $\pounds$, div, R and Ric indicate Lie-derivative operator, divergence, Riemann curvature and Ricci tensor, respectively, on the manifold N.

A vector field X on N is said to be of solenoidal type if $div(X) = 0$.

A vector field X on N is called conformal-Killing if $\pounds_X g = \rho g$ for some $\rho \in C^\infty(N)$ and if $\rho = 0$, then the conformal-Killing vector field is known as a Killing vector field.

Again, a concircular vector field X on N satisfies the condition $\nabla_E X = \rho E$ for any $E \in \chi(N)$, $\rho \in C^\infty(N)$ and vector field X is parallel if $\rho = 0$.

For a differentiable manifold N of dimension $(2n+1)$ with Riemannian metric g and Riemann curvature R, equation of the Riemann soliton is given by

$$2R + \lambda g \odot g + g \odot \pounds_X g = 0, \tag{2.8}$$

where λ is a scalar named as potential function, X indicates potential vector field and $\pounds_X$ denotes the Lie-derivative along the smooth vector field X on N. If λ is a smooth function on N, then the Riemann soliton is called an ARS. The ARS (g, X, λ) is expanding, steady or shrinking according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

For two $(0,2)$-tensor fields P_1 and P_2, the Kulkarni-Nomizu product is defined by

$$\begin{aligned}(P_1 \odot P_2)(E,F,W,U) = {} & P_1(E,U)P_2(F,W) + P_1(F,W)P_2(E,U) \\ & - P_1(E,W)P_2(F,U) - P_1(F,U)P_2(E,W).\end{aligned} \tag{2.9}$$

By virtue of (2.8) and (2.9), we have

$$\pounds_X g + \frac{2}{2n-1} Ric + \frac{2(2n\lambda + divX)}{2n-1} g = 0. \tag{2.10}$$

By tracing (2.10) we have the scalar curvature

$$scal = -2n(2n+1)\lambda - 4n\, divX. \tag{2.11}$$

If the potential vector field X is a gradient of a smooth function, then the ARS is called gradient ARS. On a $(2n+1)$-dimensional smooth manifold N, the gradient ARS is given by

$$Hess(\psi) + \frac{1}{2n-1} Ric + \frac{(2n\lambda + \triangle(\psi))}{2n-1} g = 0, \tag{2.12}$$

where $\psi \in C^\infty(N)$, $grad(\psi)$ is considered as a potential vector field and $div(grad(\psi)) = \triangle(\psi)$, known as the Laplacian operator of ψ. From (2.11), we get

$$scal = -2n(2n+1)\lambda - 4n \triangle (\psi). \tag{2.13}$$

3. D-homothetically deformed f-Kenmotsu manifolds

For a $(2n+1)$-dimensional almost contact metric manifold (N, ϕ, ξ, η, g) and the contact distribution $D := ker(\eta)$, we can define the D-homothetic deformation [3]

$$\overline{g} = ag + a(a-1)\eta \otimes \eta, \quad \overline{\eta} = a\eta, \quad \overline{\xi} = \frac{1}{a}\xi, \quad \overline{\phi} = \phi, \tag{3.1}$$

for a positive constant $a(\neq 1)$. Then $(N, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is also a $(2n+1)$-dimensional almost contact metric manifold.

Let $A := \{e_1, e_2, \ldots, e_{2n}, e_{2n+1} = \xi\}$ and $B := \{\overline{e_1}, \overline{e_2}, \ldots, \overline{e}_{2n}, \overline{e}_{2n+1} = \overline{\xi}\}$ be two orthonormal bases of $\chi(N)$ with respect to g and $\overline{g}$, respectively, where $\overline{e_i} = \frac{1}{\sqrt{a}}e_i$, $i = 1, 2, \ldots, 2n$ and $\overline{\xi} = \frac{1}{a}\xi$.

Proposition 3.1. *If $\overline{\nabla}$ and ∇ are the Levi-Civita connections with respect to the metric of a DHDfKM and a fKM, respectively, then*

$$\overline{\nabla} = \nabla + \frac{a-1}{a} f(g - \eta \otimes \eta) \otimes \xi. \tag{3.2}$$

Proof. Using (2.3), (3.1) and Koszul's formula on the deformed manifold, we have

$$\begin{aligned} &g(\overline{\nabla}_E F, G) + (a-1)\eta(\overline{\nabla}_E F)\eta(G) \\ &= g(\nabla_E F, G) + (a-1)\big(f(g(E,F) - \eta(E)\eta(F)) + \eta(\nabla_E F)\big)\eta(G), \end{aligned} \tag{3.3}$$

where, E, F and G are vector fields of $\chi(N)$.

For any vector field $G \in \chi(N)$, from the above we get

$$\begin{aligned} &\overline{\nabla}_E F + (a-1)\eta(\overline{\nabla}_E F)\xi \\ &= \nabla_E F + (a-1)\big(f(g(E,F) - \eta(E)\eta(F)) + \eta(\nabla_E F)\big)\xi. \end{aligned} \tag{3.4}$$

Taking inner product in (3.4) with ξ, we have

$$\eta(\overline{\nabla}_E F) = \eta(\nabla_E F) + \frac{a-1}{a} f\big(g(E,F) - \eta(E)\eta(F)\big). \tag{3.5}$$

Application of (3.5) in (3.4) gives the result. □

By a direct calculation using the previous proposition, we obtain:

Lemma 3.1. *In a DHDfKM of dimension* $(2n+1)$ *the following relations hold*:

$$£_{\overline{\xi}}\overline{g} = £_{\xi} g = 2f(g - \eta \otimes \eta), \tag{3.6}$$

$$\overline{\nabla}\overline{\xi} = \frac{1}{a}\nabla\xi = \frac{1}{a} f(I - \eta \otimes \xi), \tag{3.7}$$

$$\overline{div}(\overline{\xi}) = \frac{1}{a} div(\xi) = \frac{1}{a}(2nf), \tag{3.8}$$

$$(\overline{\nabla}_E \overline{\phi})F = (\nabla_E \phi)F + \frac{a-1}{a} f g(E, \phi F)\xi \tag{3.9}$$

$$= f\big(\frac{1}{a}g(\phi E, F)\xi - \eta(F)\phi E\big)$$

for any vector fields E and F of $\chi(N)$.

Proposition 3.2. *For the Levi-Civita connections $\overline{\nabla}$ with respect to the metric of a DHDfKM and ∇ with respect to the metric of a fKM the following hold:*

$$\overline{R}(E,F)G = R(E,F)G + \frac{a-1}{a}\big(f^2(g(\phi F,\phi G)E - g(\phi E,\phi G)F) + (Ef)g(\phi F,\phi G)\xi - (Ff)g(\phi E,\phi G)\xi\big), \tag{3.10}$$

$$\overline{R}(E,F,G,H) = aR(E,F,G,H) + (a-1)f^2\big(\eta(G)(\eta(E)g(F,H) - \eta(F)g(E,H)) - g(E,G)(g(F,H) - \eta(F)\eta(H)) + g(F,G)(g(E,H) - \eta(E)\eta(H))\big), \tag{3.11}$$

$$\overline{Ric} = Ric + \frac{a-1}{a}(2nf^2 + \xi(f))(g - \eta\otimes\eta), \tag{3.12}$$

$$\overline{scal} = \frac{1}{a}scal + \frac{2n(a-1)}{a^2}((2n+1)f^2 + 2\xi(f)) \tag{3.13}$$

for the vector fields E, F, G and H on $\chi(N)$. Here R, Ric and $scal$ being Riemann curvature, Ricci curvature and scalar curvature respectively on the manifold N.

Proof. The Riemann curvature on a DHDfKM is

$$\overline{R}(E,F)G = \overline{\nabla}_E\overline{\nabla}_F G - \overline{\nabla}_F\overline{\nabla}_E G - \overline{\nabla}_{[E,F]}G.$$

Using (2.2), (2.3) and (3.2) in the above, we obtain (3.10). By the definition of Riemann curvature of type $(0,4)$ on the D-homothetically deformed manifold (in brief DHDM), we can write

$$\overline{R}(E,F,G,H) = \overline{g}(\overline{R}(E,F)G,H).$$

Applying (3.1) and (3.10) in the above equation, we get

$$\begin{aligned}\overline{R}(E,F,G,H) &= aR(E,F,G,H) + (a-1)[f^2\big(g(\phi F,\phi G)g(E,H) - g(\phi E,\phi G)g(F,H)\big)\\ &\quad + (Ef)g(\phi F,\phi G)\eta(H) - (Ff)g(\phi E,\phi G)\eta(H) + a\eta(H)\eta(R(E,F)G)\\ &\quad + (a-1)\eta(H)\big(f^2(g(\phi F,\phi G)\eta(E) - g(\phi E,\phi G)\eta(F))\\ &\quad + (Ef)g(\phi F,\phi G) - (Ff)g(\phi E,\phi G)\big)].\end{aligned} \tag{3.14}$$

Again, from (2.6) we obtain

$$\begin{aligned}\eta(R(E,F)G) &= (Ff)(g(E,G) - \eta(E)\eta(G)) - (Ef)(g(F,G) - \eta(F)\eta(G))\\ &\quad + f^2(\eta(F)g(E,G) - \eta(E)g(F,G)).\end{aligned} \tag{3.15}$$

Use of (3.15) in (3.14) gives our desired expression (3.11). Tracing (3.11) with respect to $\overline{g}$, we have

$$\begin{aligned} \overline{Ric}(F,G) &+ \frac{a-1}{a^2}\overline{g}(\overline{R}(\xi,F)G,\xi) \\ &= Ric(F,G) + \frac{(2n-1)(a-1)}{a}f^2(g(F,G)-\eta(F)\eta(G)). \end{aligned} \tag{3.16}$$

Use of (3.11) and (3.15) gives

$$\overline{g}(\overline{R}(\xi,F)G,\xi) = -a(f^2+\xi(f))(g(F,G)-\eta(F)\eta(G)). \tag{3.17}$$

By (3.16) and (3.17), we obtain the relation (3.12). Again, by tracing (3.12) with respect to $\overline{g}$ and putting $Ric(\xi,\xi) = -2n(f^2+\xi(f))$, we get (3.13). □

Proposition 3.3. *In a DHDfKM of dimension* $(2n+1)$*, the following occurs*:

$$\overline{grad}(\psi) = \frac{1}{a}grad(\psi) - \frac{a-1}{a^2}\xi(\psi)\xi, \tag{3.18}$$

$$\overline{Hess}(\psi) = Hess(\psi) - \frac{a-1}{a}f\xi(\psi)(g-\eta\otimes\eta), \tag{3.19}$$

$$\overline{\triangle}(\psi) = \frac{1}{a}\triangle(\psi) - \frac{2n(a-1)}{a^2}f\xi(\psi) - \frac{a-1}{a^2}\xi(\xi(\psi)), \tag{3.20}$$

$$\overline{div} = div \tag{3.21}$$

for any $\psi \in C^\infty(N)$ *and* $\overline{grad}(\psi)$, $\overline{Hess}(\psi)$, $\overline{\triangle}(\psi)$ *and* $\overline{div}$ *indicate gradient, Hessian, Laplacian operator and divergence with respect to the deformed metric* $\overline{g}$.

Proof. For two orthonormal bases A and B with respect to g and $\overline{g}$, respectively, the gradient of a smooth function is defined by $grad(\psi) = g^{ij}\, e_j(\psi)e_i$, g^{ij} being the inverse of $g = g_{ij}$ $(i,j = 1,2,\ldots,2n+1)$. Using the expression of $grad(\psi)$, by direct computation, we get (3.18). By the definition of Hessian on a DHDM, we obtain

$$\overline{Hess}(\psi)(E,F) = ag(\overline{\nabla}_E\overline{grad}(\psi),F) + a(a-1)\eta(\overline{\nabla}_E\overline{grad}(\psi))\eta(F). \tag{3.22}$$

Again, by Proposition 3.1 and equation (3.18), we can calculate that

$$\begin{aligned} &\overline{\nabla}_E\overline{grad}(\psi) \\ &= \frac{1}{a}\nabla_E grad(\psi) - \frac{a-1}{a^2}\big(\eta(\nabla_E grad(\psi))\xi + f\xi(\psi)(E-\eta(E)\xi\big). \end{aligned} \tag{3.23}$$

Putting the value of the previous equation in (3.22) yields (3.19). For a DHDM in view of (3.18), the Laplacian operators is transferred to

$$\overline{\triangle}(\psi) = \frac{1}{a}\sum_{i=1}^{2n+1} g(\nabla_{e_i}grad(\psi),e_i) - \frac{a-1}{a^2}\sum_{i=1}^{2n+1} g(\nabla_{e_i}(\xi(\psi)\xi),e_i), \tag{3.24}$$

e_i being a orthonormal basis vector field of $\chi(N)$.

By a straight-forward computation we get

$$\nabla_{e_i}(\xi(\psi)\xi) = \nabla_{e_i} g(grad(\psi),\xi)\xi + \xi(\psi)\nabla_{e_i}\xi. \tag{3.25}$$

Combining (3.24) and (3.25), we obtain (3.20).

The definition of divergence in a DHDM and Proposition 3.1 implies (3.21). □

Lemma 3.2. *For a conformal-Killing vector field X on a DHDfKM*

$$divX = \frac{(2n+a)\rho}{2} - (a-1)\xi(\eta(X)), \tag{3.26}$$

where $\rho \in C^\infty(N)$ is given by $\pounds_X \overline{g} = \rho\overline{g}$.

Proof. If a vector field X is conformal-Killing on a DHDfKM, then

$$\overline{g}(\overline{\nabla}_E X, F) + \overline{g}(\overline{\nabla}_F X, E) = \rho\overline{g}(E,F). \tag{3.27}$$

By Proposition 3.1 and the equation (3.27), we get

$$\begin{aligned} &a\big(g(\nabla_E X, F) + g(\nabla_F X, E)\big) + a(a-1)[f\big(g(E,X)\eta(F) + g(F,X)\eta(E) \\ &- 2\eta(E)\eta(F)\eta(X)\big) + \big(\eta(\nabla_E X)\eta(F) + \eta(\nabla_F X)\eta(E)\big)] \\ &= \rho\big(ag(E,F) + a(a-1)\eta(E)\eta(F)\big). \end{aligned}$$

By tracing the above equation we obtain (3.26). □

From the above lemma, we get:

Corollary 3.1. *If $\overline{\xi}$ is conformal-Killing in a DHDfKM, then*

$$\rho = \frac{4nf}{a(2n+a)}.$$

Remark 3.1. By Lemma 3.2, a Killing vector field, which is $\overline{g}$-orthogonal to ξ on a DHDfKM, is solenoidal.

Lemma 3.3. *If a vector field X is concircular in a DHDfKM, then*

$$divX = (2n+a)\rho - (a-1)\xi(\eta(X)) \tag{3.28}$$

for $\rho \in C^\infty(N)$ given by $\overline{\nabla}X = \rho I$, I being identity operator on $\chi(N)$.

Proof. For a concircular vector field X on a DHDfKM, we have

$$\overline{g}(\overline{\nabla}_E X, F) = \overline{g}(\rho E, F). \tag{3.29}$$

Using Proposition 3.1 and tracing we obtain (3.28). □

In view of Lemma 3.3, one obtains the following:

Corollary 3.2. *If $\overline{\xi}$ is concircular in a DHDfKM, then*

$$\rho = \frac{2nf}{a(2n+a)}.$$

Remark 3.2. From Lemma 3.3, a parallel vector field which is $\overline{g}$-orthogonal to ξ on a DHDfKM, is solenoidal.

4. Relation between the inner products of the original space and the deformed space

Let $A := \{e_1, e_2, \ldots, e_{2n}, e_{2n+1} = \xi\}$ and $B := \{\overline{e_1}, \overline{e_2}, \ldots, \overline{e}_{2n}, \overline{e}_{2n+1} = \overline{\xi}\}$ be two orthonormal bases of $\chi(N)$ with respect to g and $\overline{g}$, respectively, where $\overline{e_i} = \frac{1}{\sqrt{a}}e_i$, $i = 1, 2, \ldots, 2n$ and $\overline{\xi} = \frac{1}{a}\xi$. We consider $\{dx^1, dx^2, \ldots, dx^{2n}, dx^{2n+1}\}$ and $\{\overline{dx}^1, \overline{dx}^2, \ldots, \overline{dx}^{2n}, \overline{dx}^{2n+1}\}$ as two orthonormal bases of the dual space of $\chi(N)$ with respect to g and $\overline{g}$, respectively, where $\overline{dx}^i = \sqrt{a}\, dx^i$, $i = 1, 2, \ldots, 2n$ and $\overline{dx}^{2n+1} = a\ dx^{2n+1}$. By definition, $dx^i(e_i) = 1$ for $i = 1, 2, \ldots, (2n + 1)$, $dx^i(e_j) = 0$ for $i \neq j$ and $\overline{dx}^i(\overline{e}_i) = 1$ for $i = 1, 2, \ldots, (2n + 1)$, $\overline{dx}^i(\overline{e}_j) = 0$ for $i \neq j$. Then any symmetric $(0, 2)$ tensor field on a manifold with metric g and on a DHDM with metric $\overline{g}$ is of the form

$$P = P_{ij}dx^i \otimes dx^j \quad \text{and} \quad \overline{P} = \overline{P}_{ij}\overline{dx}^i \otimes \overline{dx}^j,$$

respectively, for $i, j = 1, 2, \ldots, 2n, (2n + 1)$, where $P_{ij} = P(e_i, e_j)$ and $\overline{P}_{ij} = \overline{P}(\overline{e}_i, \overline{e}_j)$. Now, for any two symmetric $(0, 2)$-tensor fields P and Q on a manifold with metric g and on a DHDM with metric $\overline{g}$, we can define the following inner product [2]:

$$\langle P, Q\rangle_g = \sum_{i=1}^{2n+1}\sum_{j=1}^{2n+1} P_{ij}Q_{ij}, \tag{4.1}$$

$$\langle P, Q\rangle_{\overline{g}} = \frac{1}{a^2}\langle P, Q\rangle_g - \frac{a^2-1}{a^4}P(\xi, \xi)Q(\xi, \xi), \tag{4.2}$$

$$\langle \overline{P}, \overline{Q}\rangle_g = a^2\sum_{i=1}^{2n+1}\sum_{j=1}^{2n+1} \overline{P}_{ij}\overline{Q}_{ij} + a^2(a^2-1)\overline{P}(\overline{\xi}, \overline{\xi})\overline{Q}(\overline{\xi}, \overline{\xi}), \tag{4.3}$$

$$\langle \overline{P}, \overline{Q}\rangle_{\overline{g}} = \frac{1}{a^2}\langle \overline{P}, \overline{Q}\rangle_g - \frac{a^2-1}{a^4}\overline{P}(\xi, \xi)\overline{Q}(\xi, \xi). \tag{4.4}$$

The Hilbert-Schmidt norms (HSNs) of symmetric $(0, 2)$-tensor fields P and $\overline{P}$ with respect to g and $\overline{g}$ satisfy the following:

$$|P|_g^2 = \sum_{i=1}^{2n+1}\sum_{j=1}^{2n+1}(P_{ij})^2, \tag{4.5}$$

$$|P|_{\overline{g}}^2 = \frac{1}{a^2}|P|_g^2 - \frac{a^2-1}{a^4}(P(\xi, \xi))^2, \tag{4.6}$$

$$|\overline{P}|_g^2 = a^2\sum_{i=1}^{2n+1}\sum_{j=1}^{2n+1}(\overline{P}_{ij})^2 + a^2(a^2-1)(\overline{P}(\overline{\xi}, \overline{\xi}))^2, \tag{4.7}$$

$$|\overline{P}|^2_{\overline{g}} = \frac{1}{a^2}|\overline{P}|^2_g - \frac{a^2-1}{a^4}(\overline{P}(\xi,\xi))^2. \tag{4.8}$$

Using equation (4.1), we can state the following:

Lemma 4.1. *In a* $(2n+1)$*-dimensional fKM, we get*:

$$\langle g, g\rangle_g = 2n+1, \quad \langle g, \eta\otimes\eta\rangle_g = 1, \quad \langle g, Ric\rangle_g = scal,$$

$$\langle g, Hess(\psi)\rangle_g = \triangle(\psi), \quad \langle Ric, \eta\otimes\eta\rangle_g = -2n(f^2+\xi(f)),$$

$$\langle Hess(\psi), \eta\otimes\eta\rangle_g = \xi(\xi(\psi)), \quad \langle \eta\otimes\eta, \eta\otimes\eta\rangle_g = 1,$$

$$\langle Ric, Hess(\psi)\rangle_g = \sum_{i=1}^{2n} Ric(\nabla_{e_i} grad(\psi), e_i) + Ric(\nabla_\xi grad(\psi), \xi).$$

As a consequences of the above lemma and the Hilbert-Schmidt norm given by (4.6), we have the following:

Proposition 4.1. *If* g *and* $\overline{g}$ *are the metric of a* $(2n+1)$*-dimensional fKM and its D-homothetically deformed manifolds, respectively, then the following relations hold*:

$$\langle g, \eta\otimes\eta\rangle_{\overline{g}} = \frac{1}{a^4},$$

$$\langle g, Ric\rangle_{\overline{g}} = \frac{1}{a^2} scal + \frac{2n(a^2-1)}{a^4}(f^2+\xi(f)),$$

$$\langle g, Hess(\psi)\rangle_{\overline{g}} = \frac{1}{a^2}\triangle(\psi) - \frac{a^2-1}{a^4}\xi(\xi(\psi)),$$

$$\langle Ric, \eta\otimes\eta\rangle_{\overline{g}} = -\frac{2n}{a^4}(f^2+\xi(f)),$$

$$\langle Ric, Hess(\psi)\rangle_{\overline{g}} = \frac{1}{a^2}\langle Ric, Hess(\psi)\rangle_g + \frac{2n(a^2-1)}{a^4}(f^2+\xi(f))\xi(\xi(\psi)),$$

$$\langle Hess(\psi), \eta\otimes\eta\rangle_{\overline{g}} = \frac{1}{a^4}\xi(\xi(\psi)).$$

Moreover, the HSNs with respect to g *and* $\overline{g}$ *agree with*

$$|g|^2_{\overline{g}} = \frac{2na^2+1}{a^4},$$

$$|\eta\otimes\eta|^2_{\overline{g}} = \frac{1}{a^4},$$

$$|Ric|^2_{\overline{g}} = \frac{1}{a^2}|Ric|^2_g - \frac{4n^2(a^2-1)}{a^4}(f^2+\xi(f))^2,$$

$$|Hess(\psi)|^2_{\overline{g}} = \frac{1}{a^2}|Hess(\psi)|^2_g - \frac{a^2-1}{a^4}(\xi(\xi(\psi)))^2.$$

As an application of Lemma 4.1, we get:

Proposition 4.2. *If g and $\overline{g}$ are the metric of a $(2n+1)$-dimensional fKM and its D-homothetically deformed manifolds, respectively, then the following relations hold*:

$$\langle \overline{g}, \overline{\eta}\otimes\overline{\eta}\rangle_g = a^4,$$

$$\langle \overline{g}, \overline{Ric}\rangle_g = ascal + 2n(a-1)(2nf^2+\xi(f)) - 2na(a-1)(f^2+\xi(f)),$$

$$\langle \overline{g}, \overline{Hess}(\psi)\rangle_g = a\triangle(\psi) - 2n(a-1)(\xi(\psi))f + a(a-1)\xi(\xi(\psi)),$$

$$\langle \overline{Ric}, \overline{\eta}\otimes\overline{\eta}\rangle_g = -2na^2(f^2+\xi(f)),$$

$$\begin{aligned}\langle \overline{Ric}, \overline{Hess}(\psi)\rangle_g = {} & \langle Ric, Hess(\psi)\rangle_g - \frac{a-1}{a}f(\xi(\psi))scal \\ & + \frac{a-1}{a}(2nf^2+\xi(f))(\triangle(\psi)-\xi(\xi(\psi))) \\ & - \frac{2n(a-1)}{a}f(\xi(\psi))(f^2+\xi(f)) \\ & - \frac{2n(a-1)^2}{a^2}f(\xi(\psi))(2nf^2+\xi(f)),\end{aligned}$$

$$\langle \overline{Hess}(\psi), \overline{\eta}\otimes\overline{\eta}\rangle_g = a^2\xi(\xi(\psi)).$$

Also, the HSNs with respect to g and $\overline{g}$ are of the form:

$$|\overline{g}|_g^2 = (2n+a^2)a^2,$$

$$|\overline{\eta}\otimes\overline{\eta}|_g^2 = a^4,$$

$$\begin{aligned}|\overline{Ric}|_g^2 = {} & |Ric|_g^2 + \frac{2(a-1)}{a}(2nf^2+\xi(f))(scal+2n(f^2+\xi(f))) \\ & + \frac{2n(a-1)^2}{a^2}(2nf^2+\xi(f))^2,\end{aligned}$$

$$\begin{aligned}|\overline{Hess}(\psi)|_g^2 = {} & |Hess(\psi)|_g^2 - \frac{2(a-1)}{a}f(\xi(\psi))(\triangle(\psi)-\xi(\xi(\psi))) \\ & + \frac{2n(a-1)^2}{a^2}f^2(\xi(\psi))^2.\end{aligned}$$

By Proposition 4.1, we obtain:

Proposition 4.3. *In a $(2n+1)$-dimensional DHDfKM, the following relations hold*:

$$\langle \overline{g}, \overline{\eta}\otimes\overline{\eta}\rangle_{\overline{g}} = 1,$$

$$\langle \overline{g}, \overline{Ric}\rangle_{\overline{g}} = \frac{1}{a}scal + \frac{2n(a-1)}{a^2}((2n+1)f^2+2\xi(f)),$$

$$\langle \overline{g}, \overline{Hess}(\psi)\rangle_{\overline{g}} = \frac{1}{a}\triangle(\psi) - \frac{2n(a-1)}{a^2}(\xi(\psi))f - \frac{a-1}{a^2}\xi(\xi(\psi)),$$

$$\langle \overline{Ric}, \overline{\eta}\otimes\overline{\eta}\rangle_{\overline{g}} = -\frac{2n}{a^2}(f^2+\xi(f)),$$

$$\begin{aligned}\langle\overline{Ric},\overline{Hess}(\psi)\rangle_{\overline{g}} = &\frac{1}{a^2}\langle Ric, Hess(\psi)\rangle_g - \frac{a-1}{a^3}f(\xi(\psi))scal \\ &+ \frac{2n(a^2-1)}{a^4}(f^2+\xi(f))\xi(\xi(\psi)) \\ &+ \frac{a-1}{a^3}(2nf^2+\xi(f))(\triangle(\psi)-\xi(\xi(\psi))) \\ &- \frac{2n(a-1)}{a^3}f(\xi(\psi))(f^2+\xi(f)) \\ &- \frac{2n(a-1)^2}{a^4}f(\xi(\psi))(2nf^2+\xi(f)),\end{aligned}$$

$$\langle\overline{Hess}(\psi),\overline{\eta}\otimes\overline{\eta}\rangle_{\overline{g}} = \frac{1}{a^2}\xi(\xi(\psi)).$$

Moreover, the HSNs with respect to g and $\overline{g}$ agree with

$$|\overline{g}|^2_{\overline{g}} = 2n+1, \quad |\overline{\eta}\otimes\overline{\eta}|^2_{\overline{g}} = 1,$$

$$\begin{aligned}|\overline{Ric}|^2_{\overline{g}} = &\frac{1}{a^2}|Ric|^2_g + \frac{2(a-1)}{a^3}(2nf^2+\xi(f))(scal+2n(f^2+\xi(f))) \\ &+ \frac{2n(a-1)^2}{a^4}(2nf^2+\xi(f))^2 - \frac{4n^2(a^2-1)}{a^4}(f^2+\xi(f))^2,\end{aligned}$$

$$\begin{aligned}|\overline{Hess}(\psi)|^2_{\overline{g}} = &\frac{1}{a^2}|Hess(\psi)|^2_g - \frac{a^2-1}{a^4}(\xi(\xi(\psi)))^2 \\ &- \frac{2(a-1)}{a^3}f(\xi(\psi))(\triangle(\psi)-\xi(\xi(\psi))) + \frac{2n(a-1)^2}{a^4}f^2(\xi(\psi))^2.\end{aligned}$$

5. Soliton function and Ricci curvature of a *D*-homothetically deformed fKM with almost Riemann soliton

The ARS $(\overline{g}, X, \overline{\lambda})$ on a DHDM is

$$\pounds_X\overline{g} + \frac{2}{2n-1}\overline{Ric} + \frac{2(2n\overline{\lambda}+\overline{div}X)}{2n-1}\overline{g} = 0. \tag{5.1}$$

The contraction of the above equation with respect to $\overline{g}$ gives

$$\overline{scal} = -2n(2n+1)\overline{\lambda} - 4n\overline{div}X. \tag{5.2}$$

Theorem 5.1. *If the ARS on a DHDfKM is defined by $(\overline{g}, \overline{\xi}, \overline{\lambda})$, then*

$$\overline{\lambda} = \frac{1}{a^2}(f^2+\xi(f)) - \frac{1}{a}f, \tag{5.3}$$

$$Ric = [-2n(f^2+f) - \frac{2n+a-1}{a}\xi(f)+f]g + (2n-1)[f - \frac{a-1}{a}\xi(f)]\eta\otimes\eta, \tag{5.4}$$

$$scal = 2n(2n+1)(f-f^2) - 8n^2f - \frac{2n(2n+2a-1)}{a}\xi(f). \tag{5.5}$$

Proof. The ARS $(\overline{g}, \overline{\xi}, \overline{\lambda})$ on a DHDM is

$$\pounds_{\overline{\xi}}\overline{g} + \frac{2}{2n-1}\overline{Ric} + \frac{2(2n\overline{\lambda} + \overline{div\xi})}{2n-1}\overline{g} = 0. \tag{5.6}$$

Replacing (3.6) and (3.8) in (5.6), we get

$$\overline{Ric} = -[(4n-1)f + 2na\overline{\lambda}]g + [(4n-2na-1)f - 2na(a-1)\overline{\lambda}]\eta \otimes \eta. \tag{5.7}$$

Equating (3.12) and (5.7),

$$\begin{aligned} Ric = & -[(4n-1)f + 2na\overline{\lambda} + \frac{a-1}{a}(2nf^2 + \xi(f))]g \\ & + [(4n-2na-1)f - 2na(a-1)\overline{\lambda} + \frac{a-1}{a}(2nf^2 + \xi(f))]\eta \otimes \eta. \end{aligned} \tag{5.8}$$

Then by tracing, we obtain

$$scal = -2na(2n+a)\overline{\lambda} - 2n(4n+a-1)f - \frac{2n(a-1)}{a}(2nf^2 + \xi(f)). \tag{5.9}$$

Again, by using (3.13) and (5.2),

$$scal = -2n(2n+1)a\overline{\lambda} - 8n^2 f - \frac{2n(2n+1)(a-1)}{a}f^2 - \frac{4n(a-1)}{a}\xi(f). \tag{5.10}$$

Comparing (5.9) and (5.10), we have

$$\overline{\lambda} = \frac{1}{a^2}(f^2 + \xi(f)) - \frac{1}{a}f.$$

By replacing the value of $\overline{\lambda}$ in (5.8) and (5.10), we get (5.4) and (5.5), respectively. □

By virtue of the above theorem, we have:

Corollary 5.1. *The ARS $(\overline{g}, \overline{\xi}, \overline{\lambda})$ on a D-homothetically deformed non-regular fKM is expanding, steady and shrinking according as $f < 0$, $f = 0$ and $f > 0$, respectively.*

Remark 5.1. Under the assumption of Theorem 5.1, if $grad(f)$ is $\overline{g}$-orthogonal to ξ, then

$$|Ric|^2 = 4n^2(2n+1)f^4 + 8n^2(2n-1)f^3 + 2n(4n^2 - 4n + 1)f^2.$$

Theorem 5.2. *In an ARS $(\overline{g}, X, \overline{\lambda})$ with solenoidal type potential vector field X on a $(2n+1)$-dimensional DHDfKM the following relations hold:*

$$\overline{\lambda} = \frac{1}{a^2}(f^2 + \xi(f)) - \frac{2n-1}{2n}\xi(\eta(X)), \tag{5.11}$$

$$\begin{aligned} & Ric(E,F) \\ & = [-\frac{(2n+a-1)}{a}\xi(f) - 2nf^2 + (2n-1)a\xi(\eta(X))]g(E,F) \\ & \quad - \frac{(2n-1)(a-1)}{a}\big(\xi(f) - a^2\xi(\eta(X))\big)\eta(E)\eta(F) \end{aligned} \tag{5.12}$$

$$- \frac{(2n-1)a}{2}[g(\nabla_E X, F) + g(\nabla_F X, E) + (a-1)f\big(g(E,X)\eta(F) + g(F,X)\eta(E) - 2\eta(E)\eta(F)\eta(X)\big) + (a-1)\big(\eta(\nabla_E X)\eta(F) + \eta(\nabla_F X)\eta(E)\big)],$$

$$scal = -2n(2n+1)f^2 - \frac{2n(2n+2a-1)}{a}\xi(f) + (4n^2-1)a\xi(\eta(X)) \tag{5.13}$$

for any E, F on $\chi(N)$.

Proof. If the potential vector field X is solenoidal in an ARS on a DHDfKM, then from (5.1),

$$\begin{aligned} \overline{Ric}(E,F) = & -2n\overline{\lambda}\big(ag(E,F) + a(a-1)\eta(E)\eta(F)\big) \\ & - \frac{(2n-1)}{2}[a\big(g(\nabla_E X, F) + g(\nabla_F X, E)\big) \\ & + a(a-1)f\big(g(E,X)\eta(F) + g(F,X)\eta(E) - 2\eta(E)\eta(F)\eta(X)\big) \\ & + a(a-1)\big(\eta(\nabla_E X)\eta(F) + \eta(\nabla_F X)\eta(E)\big)]. \end{aligned} \tag{5.14}$$

Equation (3.12) and the above (5.14) give that

$$\begin{aligned} Ric(E,F) = & -\Big(2na\overline{\lambda} + \frac{(a-1)}{a}(2nf^2 + \xi(f))\Big)g(E,F) \\ & - \Big(2na(a-1)\overline{\lambda} - \frac{(a-1)}{a}(2nf^2 + \xi(f))\Big)\eta(E)\eta(F) \\ & - \frac{(2n-1)}{2}[a\big(g(\nabla_E X, F) + g(\nabla_F X, E)\big) \\ & + a(a-1)f\big(g(E,X)\eta(F) + g(F,X)\eta(E) - 2\eta(E)\eta(F)\eta(X)\big) \\ & + a(a-1)\big(\eta(\nabla_E X)\eta(F) + \eta(\nabla_F X)\eta(E)\big)]. \end{aligned} \tag{5.15}$$

By tracing (5.15), we have

$$scal = -2na(2n+a)\overline{\lambda} - (2n-1)a(a-1)\xi(\eta(X)) - \frac{2n(a-1)}{a}(2nf^2 + \xi(f)). \tag{5.16}$$

Again, the equations (3.13) and (5.2) with $divX = 0$ imply that

$$scal = -2n(2n+1)a\overline{\lambda} - \frac{2n(a-1)}{a}\big((2n+1)f^2 + 2\xi(f)\big). \tag{5.17}$$

By virtue of (5.16) and (5.17), we infer that

$$\overline{\lambda} = \frac{1}{a^2}(f^2 + \xi(f)) - \frac{2n-1}{2n}\xi(\eta(X)).$$

Hence, by replacing $\overline{\lambda}$ in (5.15) and (5.17), we get (5.12) and (5.13), respectively. □

Theorem 5.3. *If the potential vector field X of an ARS $(\overline{g}, X, \overline{\lambda})$ is conformal-Killing on a $(2n+1)$-dimensional DHDfKM, then the following relations hold*:

$$\overline{\lambda} = \frac{1}{a^2}(f^2 + \xi(f)) - \frac{(2n-a)}{2n}\xi(\eta(X)) - \frac{(2n+a)}{4n}\rho, \tag{5.18}$$

$$scal = -2n(2n+1)f^2 - \frac{2n(2n+2a-1)}{a}\xi(f) - \frac{(2n-1)(2n+a)a}{2}\big(\rho - 2\xi(\eta(X))\big), \tag{5.19}$$

where ρ is given by

$$\rho = -\frac{1}{2n(2n+a)}[scal + 2n(2n+1)\lambda - 4n(a-1)\xi(\eta(X))]. \tag{5.20}$$

Proof. For an ARS $(\overline{g}, X, \overline{\lambda})$ on a $(2n+1)$-dimensional DHDfKM, using the equation (3.12), we have

$$\begin{aligned} Ric = & -[2na\overline{\lambda} + adivX + \frac{(2n-1)a}{2}\rho + \frac{(a-1)}{a}(2nf^2 + \xi(f))]g \\ & - \frac{(a-1)}{2a}[4na^2\overline{\lambda} + 2a^2 divX + (2n-1)a^2\rho - 2(2nf^2 + \xi(f))]\eta \otimes \eta. \end{aligned} \tag{5.21}$$

By tracing (5.21),

$$\begin{aligned} scal = & -2na(2n+a)\overline{\lambda} - a(2n+a)divX \\ & - \frac{(2n-1)(2n+a)a}{2}\rho - \frac{2n(a-1)}{a}(2nf^2 + \xi(f)). \end{aligned} \tag{5.22}$$

Again, from (3.13) and (5.2), we get

$$scal = -2n(2n+1)a\overline{\lambda} - 4nadivX - \frac{2n(a-1)}{a}\big((2n+1)f^2 + 2\xi(f)\big). \tag{5.23}$$

Equating (5.22) and (5.23) and using (3.26) we obtain

$$\overline{\lambda} = \frac{1}{a^2}(f^2 + \xi(f)) - \frac{(2n-a)}{2n}\xi(\eta(X)) - \frac{(2n+a)}{4n}\rho.$$

Use of $\overline{\lambda}$ in (5.23) gives (5.19). Moreover, the equations (3.21), (3.26) and (5.2) imply that

$$\rho = -\frac{1}{2n(2n+a)}[scal + 2n(2n+1)\lambda - 4n(a-1)\xi(\eta(X))].$$

□

Theorem 5.4. *For an ARS $(\overline{g}, X, \overline{\lambda})$ with concircular potential vector field X on a $(2n+1)$-dimensional DHDfKM the following relations hold*:

$$\overline{\lambda} = \frac{1}{a^2}(f^2 + \xi(f)) - \frac{(2n-a)}{2n}\xi(\eta(X)) - \frac{(2n+a)}{2n}\rho, \tag{5.24}$$

$$\begin{aligned} scal = & -2n(2n+1)f^2 - \frac{2n(2n+2a-1)}{a}\xi(f) \\ & - (2n-1)(2n+a)a\big(\rho - \xi(\eta(X))\big), \end{aligned} \tag{5.25}$$

where ρ is of the form

$$\rho = -\frac{1}{4n(2n+a)}[scal + 2n(2n+1)\lambda - 4n(a-1)\xi(\eta(X))]. \tag{5.26}$$

Proof. For an ARS $(\overline{g}, X, \overline{\lambda})$ on a $(2n+1)$-dimensional DHDfKM, considering the equation (3.12) and by tracing, we obtain

$$\begin{aligned} scal = & -2n(2n+a)a\overline{\lambda} - (2n+a)adivX \\ & -(2n-1)(2n+a)a\rho - \frac{2n(a-1)}{a}(2nf^2 + \xi(f)). \end{aligned} \tag{5.27}$$

From (3.13) and (5.2),

$$scal = -2n(2n+1)a\overline{\lambda} - 4nadivX - \frac{2n(a-1)}{a}\big((2n+1)f^2 + 2\xi(f)\big). \tag{5.28}$$

Considering (5.27), (5.28) and using (3.28), we have

$$\overline{\lambda} = \frac{1}{a^2}(f^2 + \xi(f)) - \frac{(2n-a)}{2n}\xi(\eta(X)) - \frac{(2n+a)}{2n}\rho.$$

Replacing the above in (5.28), we get (5.25) and equations (3.21), (3.26) and (5.2) give

$$\rho = -\frac{1}{4n(2n+a)}[scal + 2n(2n+1)\lambda - 4n(a-1)\xi(\eta(X))].$$

□

6. Soliton function and Ricci curvature of a *D*-homothetically deformed fKM with gradient almost Riemann soliton

The ARS $(\overline{g}, X = \overline{grad}(\psi), \overline{\lambda})$ on a DHDM is

$$\overline{Hess}(\psi) + \frac{1}{2n-1}\overline{Ric} + \frac{(2n\overline{\lambda} + \overline{\triangle}(\psi))}{2n-1}\overline{g} = 0, \tag{6.1}$$

where $\frac{1}{2}\pounds_{\overline{grad}(\psi)}\overline{g} = \overline{Hess}(\psi)$ and $\overline{div}(\overline{grad})(\psi) = \overline{\triangle}(\psi)$. The contraction of the above equation gives

$$\overline{scal} = -2n(2n+1)\overline{\lambda} - 4n\overline{\triangle}(\psi). \tag{6.2}$$

Theorem 6.1. *Let $(\overline{g}, X = \overline{grad}(\psi), \overline{\lambda})$ be a gradient ARS on a DHDfKM. Then*

$$\overline{\lambda} = \frac{1}{a^2}(f^2 + \xi(f)) - \frac{1}{2na}\triangle(\psi) + \frac{(a-1)}{a^2}f\xi(\psi) - \frac{(2n-a)}{2na^2}\xi(\xi(\psi)) \tag{6.3}$$

for some smooth function $\psi \in C^\infty(N)$.

Proof. For a gradient ARS $(\overline{g}, X = \overline{grad}(\psi), \overline{\lambda})$ on a DHDfKM, using (6.1), (3.12), (3.19) and (3.20) and by tracing, we have

$$\begin{aligned} scal = & -2n(2n+a)a\overline{\lambda} - (4n+a-1)\triangle(\psi) \\ & - \frac{2n(a-1)}{a}[2nf^2 + \xi(f) - (4n+a-1)\xi(\psi)f - \frac{(2n+a)}{2n}\xi(\xi(\psi))]. \end{aligned} \tag{6.4}$$

Again, (6.2), (3.13) and (3.20) give that

$$scal = -2n(2n+1)a\overline{\lambda} - 4n \triangle(\psi) + \frac{8n^2(a-1)}{a}\xi(\psi)f + \frac{4n(a-1)}{a}\xi(\xi(\psi)) - \frac{2n(a-1)}{a}\big((2n+1)f^2 + 2\xi(f)\big). \tag{6.5}$$

By considering (6.4) and (6.5), we obtain

$$\overline{\lambda} = \frac{1}{a^2}(f^2 + \xi(f)) - \frac{1}{2na} \triangle(\psi) + \frac{(a-1)}{a^2} f\xi(\psi) - \frac{(2n-a)}{2na^2}\xi(\xi(\psi)). \qquad \square$$

Corollary 6.1. *For a gradient ARS* $(\overline{g}, X = \overline{grad}(\psi), \overline{\lambda})$ *on a DHDfKM, if* $grad(\psi)$ *is* $\overline{g}$*-orthogonal to* ξ*, then*

$$\overline{\lambda} = \frac{1}{a^2}(f^2 + \xi(f)) - \frac{1}{2na} \triangle(\psi),$$

$$scal = -(2n-1) \triangle(\psi) - 2n(2n+1)f^2 - \frac{2n(2n+2a-1)}{a}\xi(f).$$

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