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TRANSVERSAL LIGHTLIKE SUBMERSIONS FROM INDEFINITE SASAKIAN MANIFOLDS ONTO LIGHTLIKE MANIFOLDS

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Abstract. In this paper, we introduce and study two new classes of lightlike submersions, called radical transversal and transversal lightlike submersions between an indefinite Sasakian manifold and a lightlike manifold. We give examples and investigate the geometry of distributions involved in the definitions of these lightlike submersions. We also study radical transversal and transversal lightlike submersions from an indefinite Sasakian manifold onto a lightlike manifold with totally contact umbilical fibers.

1. Introduction

In 1966, O'Neill [14] initiated the study of Riemannian submersions and Gray [8] further continued it. Let $\pi : (M_1, g_1) \to (M_2, g_2)$ be a smooth map, where (M_1, g_1) and (M_2, g_2) are Riemannian manifolds. Then π is called a Riemannian submersion if π has maximal rank and $\pi_\star$ preserves the length of horizontal vectors. In [2], Chinea studied almost contact metric submersions between manifolds equipped with different structures. Most of the research on Riemannian submersions can be found in the book [7]. In [20], Sahin introduced slant submersions from almost Hermitian manifolds onto Riemannian manifolds as a generalization of almost Hermitian and anti-invariant submersions. Following this research, Küpeli Erken and Murathan [13] studied slant Riemannian submersions from Sasakian manifolds. In [18], Sahin introduced screen conformal lightlike submersions from lightlike manifolds onto semi-Riemannian manifolds.

On the other hand, it is known that when M_1 and M_2 are Riemannian manifolds, then fibers of π are Riemannian manifolds. But when M_1 and M_2 are semi-Riemannian manifolds, then the fibers of π may not be semi-Riemannian. In view of this fact, O'Neill [15] introduced the notion of semi-Riemannian submersions between semi-Riemannian manifolds, and Sahin [19] introduced

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screen lightlike submersions from lightlike manifolds onto semi-Riemannian manifolds. Also, Sahin and Gündüzalp [21] studied lightlike submersions from semi-Riemannian manifolds onto lightlike manifolds. Some recent studies on the geometry of lightlike submersions can be seen in ([10–12, 16, 17, 22]). The geometry of totally umbilical lightlike submanifolds of semi-Riemannian manifolds was studied by Duggal and Jin [4]. Radical transversal and transversal lightlike submanifolds of indefinite Sasakian manifolds were defined and studied by Yildirim and Sahin [25]. They also studied totally contact umbilical radical transversal and transversal lightlike submanifolds of indefinite Sasakian manifolds. Later, Wang and Liu [24] introduced generalized transversal lightlike submanifolds of indefinite Sasakian manifolds. The above theories motivated us to study some new classes of lightlike submersions. In the present paper, we introduce the notions of transversal and radical transversal lightlike submersions from indefinite Sasakian manifolds onto lightlike manifolds. We also study radical transversal and transversal lightlike submersions between an indefinite Sasakian manifold and a lightlike manifold with totally contact umbilical fibers. The paper is organized as follows. In Section 2, we collect basic definitions and formulae as needed for this paper. In Section 3, we define radical transversal lightlike submersions, provide two examples and discuss the integrability and geodesic foliations of distributions on a fiber of such lightlike submersions. We also prove a necessary condition for the induced connection to be a metric connection. In Section 4, we study the geometry of radical transversal lightlike submersions with totally contact umbilical fibers. We also obtain an existence (non-existence) theorem for radical transversal lightlike submersions from indefinite Sasakian space forms with totally contact umbilical fibers. In Section 5, we introduce transversal lightlike submersions, give two examples and study the geometry of distributions.

2. Preliminaries

In this section, we recall several definitions and results which will be required throughout the paper.

A smooth semi-Riemannian manifold (M, g) of dimension $2m+1$ is said to have an almost contact structure (ϕ, ξ, η) if it carries a (1, 1) tensor field ϕ, a vector field ξ called characteristic vector field and a 1-form η on M, satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{1}$$

where I denotes the identity tensor.

If a semi-Riemannian manifold (M, g) has an almost contact structure satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(TM), \tag{2}$$

then (ϕ, ξ, η, g) is called an (ϵ)-almost contact metric structure on M [6, 23], where $\epsilon = -1$ or 1 according as ξ is timelike or spacelike. From (1) and (2), we get

$$g(\xi,\xi)=\epsilon, \quad \eta(X)=\epsilon g(X,\xi), \quad g(X,\phi Y)+g(\phi X,Y)=0. \tag{3}$$

An (ϵ)-almost contact metric structure (ϕ,ξ,η,g) on M is an indefinite Sasakian structure if and only if

$$(\nabla_X\phi)Y=g(X,Y)\xi-\epsilon\eta(Y)X \tag{4}$$

for all $X,Y\in\Gamma(TM)$, where ∇ denotes the Riemannian connection for g [6, Theorem 7.1.6].

A semi-Riemannian manifold M equipped with an indefinite Sasakian structure (ϕ,ξ,η,g) is called an indefinite Sasakian manifold and it is denoted by (M,ϕ,ξ,η,g). Setting $Y=\xi$ in (4), we get

$$\nabla_X\xi=-\epsilon\phi X, \qquad \forall X\in\Gamma(TM). \tag{5}$$

In this paper, we assume that the characteristic vector field ξ is spacelike.

Example 2.1 ([5])**.** Let $(\mathbb{R}^{2n+1}_{2q},g)$ be a semi-Riemannian manifold with its usual contact form

$$\eta=\frac{1}{2}\left(dz-\sum_{i=1}^{n}y_idx_i\right).$$

The characteristics vector field ξ is given by $2\frac{\partial}{\partial z}$ and its semi-Riemannian metric g and tensor field ϕ are given by

$$g=\eta\otimes\eta+\frac{1}{4}\left(-\sum_{i=1}^{q}dx_i\otimes dx_i+dy_i\otimes dy_i+\sum_{i=q+1}^{n}dx_i\otimes dx_i+dy_i\otimes dy_i\right),$$

$$\phi\left(\sum_{i=1}^{n}\left(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i}\right)+Z\frac{\partial}{\partial z}\right)=\sum_{i=1}^{n}\left(Y_i\frac{\partial}{\partial x_i}-X_i\frac{\partial}{\partial y_i}\right)+\sum_{i=1}^{n}Y_iy_i\frac{\partial}{\partial z},$$

where (x_i,y_i,z) $(i=1,2,\dots,n)$ are the Cartesian coordinates on $\mathbb{R}^{2n+1}_{2q}$. This gives a contact metric structure on $\mathbb{R}^{2n+1}$.

Now, it can be proved that $(\mathbb{R}^{2n+1}_{2q},\phi,\xi,\eta,g)$ is an indefinite Sasakian manifold. The vector fields $E_i=2\frac{\partial}{\partial y_i}$, $E_{n+i}=2\left(\frac{\partial}{\partial x_i}+y_i\frac{\partial}{\partial z}\right)$ and ξ form a ϕ-basis for the contact metric structure.

Let (M,g) be a real m-dimensional smooth semi-Riemannian manifold. Then $Rad\ T_pM=\{V\in T_pM: g(V,X)=0, X\in T_pM\}$ is a subspace of T_pM called the radical subspace with respect to g. Suppose $\dim(RadT_pM)=r$. Then the mapping Rad $TM: p\in M\to Rad\ T_pM$ is said to be the radical distribution of rank r on M. The manifold M is said to be an r-lightlike manifold [3] if $r>0$.

Let $f:(M_1,g_1)\to(M_2,g_2)$ be a smooth submersion from a semi-Riemannian manifold M_1 onto an r-lightlike manifold M_2. Then kernel of f_* at $p\in M_1$ and its orthogonal complement are given by $Kerf_{*p}=\{X\in T_pM_1: f_{*p}X=0\}$, and $(Kerf_{*p})^{\perp}=\{Y\in T_pM_1: g_1(Y,X)=0, X\in Kerf_{*p}\}$, respectively. As

T_pM_1 is a semi-Riemannian vector space, $Kerf_*$ may not be complementary to $(Kerf_*)^\perp$. We now consider the case when $\Delta_p = Kerf_{*p} \cap (Kerf_{*p})^\perp \neq \{0\}$ with $0 < \dim\Delta < \min\{\dim(Kerf_*), \dim(Kerf_*)^\perp\}$, then Δ and $Kerf_*$ are radical and lightlike distributions on $f^{-1}(x)$, respectively. Thus, there exists an orthogonal complementary distribution to Δ in $Kerf_*$ which is non-degenerate and we denote it by $S(Kerf_*)$. Therefore we have $Kerf_* = \Delta \perp S(Kerf_*)$. Using the last reasoning again for $(Kerf_*)^\perp$, we get $(Kerf_*)^\perp = \Delta \perp S(Kerf_*)^\perp$, where $S(Kerf_*)^\perp$ is a complementary distribution to Δ in $(Kerf_*)^\perp$.

Let $\{V_i\}$ be any local basis of Δ. Then there exists a local null frame $\{N_i\}$ of smooth sections with values in the orthogonal complement of $S(Kerf_*)^\perp$ in $(S(Kerf_*))^\perp$ satisfying $g_1(V_i, N_j) = \delta_{ij}$ and $g_1(N_i, N_j) = 0$. The vector bundle locally spanned by N_1, N_2,…, N_r is called a lightlike transversal vector bundle and it is denoted by $ltr(Kerf_*)$ ([3, page 144]). Consider the vector bundle $tr(Kerf_*) = ltr(Kerf_*) \perp S(Kerf_*)^\perp$, which is complementary (but not orthogonal) vector bundle to $Kerf_*$ in $TM_1|_{f^{-1}(x)}$. Then we get

$$TM_1|_{f^{-1}(x)} = Kerf_* \oplus tr(Kerf_*),$$

$$TM_1|_{f^{-1}(x)} = S(Kerf_*) \perp [\Delta \oplus ltr(Kerf_*)] \perp S(Kerf_*)^\perp.$$

It should be noted that $ltr(Kerf_*)$ and $Kerf_*$ are not orthogonal to each other. Next, we will denote $\mathcal{V} = Kerf_*$, the vertical space of T_pM_1 and $\mathcal{H} = tr(Kerf_*)$, the horizontal space. Therefore we get

$$TM_1 = \mathcal{H} \oplus \mathcal{V}.$$

Also, we have $\mathcal{V}_p = T_pf^{-1}(x)$, where $p \in f^{-1}(x)$.

Definition ([21]). A submersion $f : M_1 \to M_2$ from a semi-Riemannian manifold (M_1, g_1) onto an r-lightlike manifold (M_2, g_2) is called an r-lightlike submersion if

(a) $\dim\Delta = \dim\{(Kerf_*) \cap (Kerf_*)^\perp\} = r$, $0 < r < \min\{\dim(Kerf_*), \dim(Kerf_*)^\perp\}$.

(b) f_* preserves the length of horizontal vectors, i.e., $g_1(X, Y) = g_2(f_*X, f_*Y)$ for $X, Y \in \Gamma\mathcal{H}$.

We now have the following particular cases:

(i) If $\dim\Delta = \dim(Kerf_*) < \dim(Kerf_*)^\perp$, then we get $\mathcal{V} = \Delta$ and $\mathcal{H} = S(Kerf_*)^\perp \perp ltr(Kerf_*)$ and f is called an isotropic submersion.

(ii) If $\dim\Delta = \dim(Kerf_*)^\perp < \dim(Kerf_*)$, then we have $\mathcal{V} = S(Kerf_*) \perp \Delta$ and $\mathcal{H} = ltr(Kerf_*)$ and f is called a co-isotropic submersion.

(iii) If $\dim\Delta = \dim(Kerf_*)^\perp = \dim(Kerf_*)$, then we get $\mathcal{V} = \Delta$ and $\mathcal{H} = ltr(Kerf_*)$ and f is called a totally lightlike submersion.

As we know, the geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$. Therefore, Sahin and Gündüzalp [21] defined these tensors for a lightlike submersion as

$$\mathcal{T}_XY = h\nabla_{\nu X}\nu Y + \nu\nabla_{\nu X}hY, \qquad \mathcal{A}_XY = \nu\nabla_{hX}hY + h\nabla_{hX}\nu Y, \tag{6}$$

where $h : TM_1 \to \mathcal{H}$ and $\nu : TM_1 \to \mathcal{V}$ denote the natural projections and ∇ be the Levi-Civita connection of g_1.

We now study the induced geometric objects on a fiber of lightlike submersions. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a lightlike submersion from an $(m+n)$-dimensional semi-Riemannian manifold M_1 onto an n-dimensional lightlike manifold M_2. Then by definition, $Kerf_*$ is an m-dimensional lightlike distribution on $f^{-1}(x)$. Also, we denote the induced metric on $f^{-1}(x)$ by $\hat{g}$. Then for any $U, V \in \Gamma(Kerf_*)$ and $X \in \Gamma(tr(Kerf_*))$, using (6) we have

$$\nabla_U V = \hat{\nabla}_U V + \mathcal{T}_U V, \tag{7}$$

$$\nabla_U X = \mathcal{T}_U X + \nabla^t_U X, \tag{8}$$

where $\hat{\nabla}_U V = \nu \nabla_U V$ and $\nabla^t_U X = h \nabla_U X$. Further we note that $\{\hat{\nabla}_U V, \mathcal{T}_U X\}$ and $\{\mathcal{T}_U V, \nabla^t_U X\}$ belongs to $\Gamma(Kerf_*)$ and $\Gamma(tr(Kerf_*))$, respectively. Here $\hat{\nabla}$ and ∇^t are linear connections on $f^{-1}(x)$ and $tr(Kerf_*)$, respectively.

Let $S(Kerf_*)^\perp \neq 0$, that is, f is either an r-lightlike submersion or isotropic submersion. Next, we denote the projection of $tr(Kerf_*)$ on $ltr(Kerf_*)$ and $S(Kerf_*)^\perp$ by L and S, respectively. Then (7) and (8) take the following form

$$\nabla_U V = \hat{\nabla}_U V + \mathcal{T}^l_U V + \mathcal{T}^s_U V, \tag{9}$$

$$\nabla_U X = \mathcal{D}^l_U X + \mathcal{D}^s_U X + \mathcal{T}_U X, \tag{10}$$

where $\mathcal{T}^l_U V = L(\mathcal{T}_U V)$, $\mathcal{T}^s_U V = S(\mathcal{T}_U V)$ and $\mathcal{D}^l_U X = L(\nabla^t_U X)$, $\mathcal{D}^s_U X = S(\nabla^t_U X)$. $\mathcal{T}^l$ and $\mathcal{T}^s$ are called the lightlike second fundamental form and the screen second fundamental form of a fiber of f, respectively. We also note that the differential operators $\mathcal{D}^l$ and $\mathcal{D}^s$ define two Otsuki connections on $tr(Kerf_*)$ with respect to the vector bundle morphism L and S, respectively. Now, for any $U \in \Gamma(Kerf_*)$ we define the following differential operators

$$\nabla^l_U : \Gamma(ltr(Kerf_*)) \to \Gamma(ltr(Kerf_*)); \nabla^l_U(LX) = \mathcal{D}^l_U(LX), \tag{11}$$

and

$$\nabla^s_U : \Gamma(S(Kerf_*)^\perp) \to \Gamma(S(Kerf_*)^\perp); \nabla^s_U(SX) = \mathcal{D}^s_U(SX), \tag{12}$$

where $X \in \Gamma(tr(Kerf_*))$. By a simple calculation, it follows that both ∇^l and ∇^s are linear connections on $ltr(Kerf_*)$ and $S(Kerf_*^\perp)$, respectively. These connections are called the lightlike and the screen transversal connection on $f^{-1}(x)$.

Further, we define mappings

$$\mathcal{D}^l : \Gamma(Kerf_*) \times \Gamma(S(Kerf_*)^\perp) \to \Gamma(ltr(Kerf_*)); \mathcal{D}^l(U, SX) = \mathcal{D}^l_U(SX) \tag{13}$$

and

$$\mathcal{D}^s : \Gamma(Kerf_*) \times \Gamma(ltr(Kerf_*)) \to \Gamma(S(Kerf_*)^\perp) : \mathcal{D}^s(U, LX) = \mathcal{D}^s_U(LX), \tag{14}$$

where $U \in \Gamma(Kerf_*)$ and $X \in \Gamma(tr(Kerf_*))$. Now using (10)-(14), we get

$$\nabla_U X = \mathcal{T}_U X + \nabla^l_U LX + \nabla^s_U SX + \mathcal{D}^l(U, SX) + \mathcal{D}^s(U, LX). \tag{15}$$

In particular, when $X = N \in \Gamma(ltr(Kerf_*))$ and $X = W \in \Gamma(S(Kerf_*)^{\perp})$ then from (15), we obtain

$$\nabla_U N = \mathcal{T}_U N + \nabla^l_U N + \mathcal{D}^s(U, N), \tag{16}$$

and

$$\nabla_U W = \mathcal{T}_U W + \nabla^s_U W + \mathcal{D}^l(U, W). \tag{17}$$

Now using (9), (17), (16) and metric connection ∇, we get

$$g_1(\mathcal{T}^s_U V, W) + g_1(V, \mathcal{D}^l(U, W) + \hat{g}(\mathcal{T}_U W, V) = 0, \tag{18}$$

$$g_1(\mathcal{D}^s(U, N), W) + g_1(N, \mathcal{T}_U W) = 0. \tag{19}$$

Suppose $S(Kerf_*) \neq 0$ and σ denotes the projection of $Kerf_*$ on $S(Kerf_*)$. Then for $U, V \in \Gamma(Kerf_*)$ and $Z \in \Gamma(\Delta)$ we have

$$\hat{\nabla}_U \sigma V = \nabla^*_U \sigma V + \mathcal{T}^*_U \sigma V, \tag{20}$$

$$\hat{\nabla}_U Z = \mathcal{T}^*_U Z + \nabla^{*t}_U Z, \tag{21}$$

where $\{\nabla^*_U \sigma V, \mathcal{T}^*_U Z\}$ and $\{\mathcal{T}^*_U \sigma V, \nabla^{*t}_U Z\}$ belongs to $\Gamma(S(Kerf_*))$ and $\Gamma(\Delta)$, respectively. Here ∇^* and ∇^{*t} are induced metric linear connections on $S(Kerf_*)$ and Δ, respectively. From (9), (21), (16) and (20) we obtain

$$g_1(\mathcal{T}^l_U \sigma V, Z) + \hat{g}(\sigma V, \mathcal{T}^*_U Z) = 0, \tag{22}$$

$$g_1(\mathcal{T}^*_U \sigma V, N) + \hat{g}(\mathcal{T}_U N, \sigma V) = 0, \tag{23}$$

$$g_1(\mathcal{T}^l_U Z, Z) = 0, \qquad \mathcal{T}^*_Z Z = 0, \tag{24}$$

where $U, V \in \Gamma(Kerf_*)$, $Z \in \Gamma(\Delta)$ and $N \in \Gamma(ltr(Kerf_*))$.

As ∇ is a metric connection on M_1, using (9) we get

$$(\hat{\nabla}_U \hat{g})(V, W) = g_1(\mathcal{T}^l_U V, W) + g_1(\mathcal{T}^l_U W, V). \tag{25}$$

Finally, we obtain the Gauss equation for fibers of an r-lightlike submersion. By using (11) and (12), we define the following covariant derivatives

$$(\nabla_U \mathcal{T}^l)(V, W) = \nabla^l_U \mathcal{T}^l_V W - \mathcal{T}^l_{\hat{\nabla}_U V} W - \mathcal{T}^l_V \hat{\nabla}_U W, \tag{26}$$

$$(\nabla_U \mathcal{T}^s)(V, W) = \nabla^s_U \mathcal{T}^s_V W - \mathcal{T}^s_{\hat{\nabla}_U V} W - \mathcal{T}^s_V \hat{\nabla}_U W, \tag{27}$$

for any $U, V, W \in \Gamma(Kerf_*)$. Let R and $\hat{R}$ denote the curvature tensors of ∇ and $\hat{\nabla}$, respectively. Then by using (9), (16), (17), (25) and (26), we derive

$$\begin{aligned} R(U, V)W = {} & \hat{R}(U, V)W + \mathcal{T}_U \mathcal{T}^l_V W - \mathcal{T}_V \mathcal{T}^l_U W + \mathcal{T}_U \mathcal{T}^s_V W \\ & - \mathcal{T}_V \mathcal{T}^s_U W + (\nabla_U \mathcal{T}^l)(V, W) - (\nabla_V \mathcal{T}^l)(U, W) \\ & + \mathcal{D}^l(U, \mathcal{T}^s_V W) - \mathcal{D}^l(V, \mathcal{T}^s_U W) + (\nabla_U \mathcal{T}^s)(V, W) \\ & - (\nabla_V \mathcal{T}^s)(U, W) + \mathcal{D}^s(U, \mathcal{T}^l_V W) - \mathcal{D}^s(V, \mathcal{T}^l_U W) \end{aligned} \tag{28}$$

for $U, V, W \in \Gamma(Kerf_*)$.

3. Radical transversal lightlike submersions

In this section, we introduce radical transversal lightlike submersions from indefinite Sasakian manifolds onto lightlike manifolds such that the structure vector field ξ is tangent to fiber. Also, we provide examples and study the geometry of such lightlike submersions.

Definition. Let $(M_1, \phi, \xi, \eta, g_1)$ be an indefinite Sasakian manifold and (M_2, g_2) be a lightlike manifold. Suppose that $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ is a lightlike submersion with the characteristic vector field ξ tangent to $f^{-1}(x)$, i.e., ξ belongs to $S(Kerf_*)$. Then, f is called a radical transversal lightlike submersion if

(i) $\phi(\Delta) = ltr(Kerf_*)$,

(ii) there exists a non-degenerate subbundle $\mathcal{D}$ of $S(Kerf_*)$ such that $\phi(\mathcal{D}) = \mathcal{D}$, where $S(Kerf_*) = \mathcal{D} \perp \langle\xi\rangle$.

A radical transversal lightlike submersion is said to be proper if $\mathcal{D} \neq 0$. Now, we construct some examples of proper radical transversal lightlike submersions.

Example 3.1. Consider an indefinite Sasakian manifold as given in Example 2.1 for $m = 4$ and $q = 1$, i.e., $(\mathbb{R}^9_2, \phi, \xi, \eta, g_1)$. Let $(\mathbb{R}^4, g_2)$ be a lightlike manifold, where $g_2 = \frac{1}{8}\{(da_2)^2 + (da_4)^2\}$ and a_1, a_2, a_3, a_4 are the usual coordinates on $\mathbb{R}^4$. Define a map $f : \mathbb{R}^9_2 \to \mathbb{R}^4$ by

$$f(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) = (x_1 - x_2, x_3 - x_4, y_1 + y_2, y_3 - y_4).$$

After some computations, we have $Kerf_* = Span\{V_1 = E_5 + E_6, V_2 = E_7 + E_8, V_3 = E_1 - E_2, V_4 = E_3 + E_4, V_5 = E_9 = \xi\}$, $(Kerf_*)^\perp = Span\{V_1, V_3, W_1 = E_7 - E_8, W_2 = E_3 - E_4\}$ with $\Delta = Kerf_* \cap (Kerf_*)^\perp = Span\{V_1, V_3\}$ which implies $S(Kerf_*) = \mathcal{D} \perp \langle\xi\rangle$, where $\mathcal{D} = Span\{V_2, V_4\}$ and $S(Kerf_*)^\perp = Span\{W_1, W_2\}$. Now, we obtain $ltr(Kerf_*) = Span\{N_1 = -\frac{1}{2}(E_5 - E_6), N_2 = -\frac{1}{2}(E_1 + E_2)\}$. Then it is easy to see that f is a 2-lightlike submersion. Moreover, we have $\phi(V_1) = 2N_2$, $\phi(V_3) = -2N_1$, $\phi V_2 = -V_4$, $\phi(V_4) = V_2$ which implies $\phi(\Delta) = ltr(Kerf_*)$ and $\phi(\mathcal{D}) = \mathcal{D}$. Thus f is a proper radical transversal 2-lightlike submersion.

Example 3.2. Consider an indefinite Sasakian manifold as given in Example 2.1 for $m = 5$ and $q = 1$, i.e., $(\mathbb{R}^{11}_2, \phi, \xi, \eta, g_1)$. Let $(\mathbb{R}^6, g_2)$ be a lightlike manifold, where $g_2 = \frac{1}{8}\{(da_2)^2 + 2(da_3)^2 + (da_5)^2 + 2(da_6)^2\}$ and $a_1, a_2, a_3, a_4, a_5, a_6$ are the usual coordinates on $\mathbb{R}^6$. Define a map $f : \mathbb{R}^{11}_2 \to \mathbb{R}^6$ by

$$f(x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5, z) = (x_1 + x_4, x_2 + x_5, x_3, y_1 - y_4, y_2 + y_5, y_3).$$

Then by direct calculations, we get $Kerf_* = Span\{V_1 = E_6 - E_9, V_2 = E_7 - E_{10}, V_3 = E_1 + E_4, V_4 = E_2 - E_5, V_5 = E_{11} = \xi\}$, $(Kerf_*)^\perp = Span\{V_1, V_3, W_1 = E_7 + E_{10}, W_2 = E_8, W_3 = E_2 + E_5, W_4 = E_3\}$ with $\Delta = Kerf_* \cap (Kerf_*)^\perp = Span\{V_1, V_3\}$ which implies $S(Kerf_*) = \mathcal{D} \perp \langle\xi\rangle$, where $\mathcal{D} =$

$Span\{V_2, V_4\}$ and $S(Kerf_*)^\perp = Span\{W_1, W_2, W_3, W_4\}$. Next, we obtain $ltr(Kerf_*) = Span\{N_1 = -\frac{1}{2}(E_6 + E_9), N_2 = -\frac{1}{2}(E_1 - E_4)\}$. Now it is easy to see that f is a 2-lightlike submersion. Further we have $\phi(V_1) = 2N_2$, $\phi(V_3) = -2N_1$, $\phi V_2 = -V_4$, $\phi(V_4) = V_2$ which implies $\phi(\Delta) = ltr(Kerf_*)$ and $\phi(\mathcal{D}) = \mathcal{D}$. Therefore f is a proper radical transversal 2-lightlike submersion.

Theorem 3.3. *There does not exist radical transversal* 1*-lightlike submersions between indefinite Sasakian manifolds and lightlike manifolds.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal 1-lightlike submersion from an indefinite Sasakian manifold M_1 onto lightlike manifold M_2. Then we have $\Delta = span\{V\}$, which implies $ltr(Kerf_*) = span\{N\}$. Now using (1)-(3), we derive $g_1(\phi V, V) = g_1(\phi^2 V, \phi V) = -g_1(V, \phi V) + \eta(V) g_1(\xi, \phi V)$ which gives $g_1(\phi V, V) = 0$.

Also, from the definition we have $\phi V = N$. Therefore, we get $g_1(\phi V, V) = g_1(N, V) = 1$, which is a contradiction. Thus, we deduce that f can not be a radical transversal 1-lightlike submersion. □

Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold M_1 onto a lightlike manifold M_2. Then, we have the following remarks: (i) $\dim(\Delta) \geq 2$, (ii) $\dim(S(Kerf_*)) \neq 2m$, $m \geq 1$, (iii) Any proper radical transversal lightlike submersion from an 11-dimensional indefinite Sasakian manifold onto a 6-dimensional lightlike manifold must be 2-lightlike.

Theorem 3.4. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, the screen transversal distribution $S(Kerf_*)^\perp$ is invariant with respect to ϕ.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion. Then, for any $U \in \Gamma(S(Kerf_*)), V \in \Gamma(\Delta)$ and $W \in \Gamma(S(Kerf_*)^\perp)$, using (3) we get $g_1(\phi W, V) = -g_1(W, \phi V) = 0$ and $g_1(\phi W, U) = -g_1(W, \phi U) = 0$. This imply that $\phi(S(Kerf_*)^\perp) \cap \Delta = \{0\}$ and $\phi(S(Kerf_*)^\perp) \cap S(Kerf_*) = \{0\}$. Similarly, for $N \in \Gamma(ltr(Kerf_*))$, we obtain $g_1(\phi W, N) = -g_1(W, \phi N) = 0$ which implies that $\phi(S(Kerf_*)^\perp) \cap ltr(Kerf_*) = \{0\}$. Thus the proof is completed. □

Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold M_1 onto lightlike manifold M_2. Suppose that Q and P denote the projections of $Kerf_*$ on Δ and $\mathcal{D}$, respectively. Then for $U \in \Gamma(Kerf_*)$, we write

$$U = QU + PU + \eta(U)\xi, \tag{29}$$

where $QU \in \Gamma(\Delta)$ and $PU \in \Gamma(\mathcal{D})$. On applying ϕ to (29), we get

$$\phi U = \phi QU + \phi PU. \tag{30}$$

If we set $\phi QU = \omega U$ and $\phi PU = \tau U$, then (30) becomes

$$\phi U = \omega U + \tau U, \tag{31}$$

where $\tau U \in \Gamma(\mathcal{D})$ and $\omega U \in \Gamma(ltr(Kerf_*))$.

From (4), we have

$$\hat{g}(U, V)\xi - \eta(V)U = \nabla_U \phi V - \phi(\nabla_U V), \tag{32}$$

where $U, V \in \Gamma(Kerf_*)$. Now using (32), (31), (9) and (16), we obtain

$$\begin{aligned}\hat{g}(U, V)\xi - \eta(V)U = \hat{\nabla}_U \tau V + \mathcal{T}^l_U \tau V + \mathcal{T}^s_U \tau V + \mathcal{T}_U \omega V + \nabla^l_U \omega V + \mathcal{D}^s(U, \omega V)\\ - \tau(\hat{\nabla}_U V) - \omega(\hat{\nabla}_U V) - \phi(\mathcal{T}^l_U V) - \phi(\mathcal{T}^s_U V).\end{aligned}$$

Then, equating the tangential, screen transversal and lightlike transversal parts of the above equation, we get

$$(\hat{\nabla}_U \tau)V = \phi(\mathcal{T}^l_U V) - \mathcal{T}_U \omega V + \hat{g}(U, V)\xi - \eta(V)U, \tag{33}$$

$$\mathcal{T}^s_U \tau V + \mathcal{D}^s(U, \omega V) - \phi(\mathcal{T}^s_U V) = 0, \tag{34}$$

$$\mathcal{T}^l_U \tau V + \nabla^l_U \omega V - \omega(\hat{\nabla}_U V) = 0. \tag{35}$$

Lemma 3.5. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then for $U, V \in \Gamma(Kerf_* - \langle\xi\rangle)$, we have*

(i) $\hat{g}(\hat{\nabla}_U V, \xi) = g_1(V, \phi U)$,
(ii) $\hat{g}([U, V], \xi) = 2g_1(V, \phi U)$.

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion. As ∇ is a metric connection, for any $U, V \in \Gamma(Kerf_* - \langle\xi\rangle)$, using (9) and (5), we get

$$\hat{g}(\hat{\nabla}_U V, \xi) = g_1(V, \phi U). \tag{36}$$

Since $\hat{\nabla}$ is a symmetric connection, from (36) and (3) we have (ii). □

As the induced connection on a fiber of a lightlike submersion is not a metric connection, we now find a necessary condition for $\hat{\nabla}$ to be a metric connection.

Theorem 3.6. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). If the induced connection $\hat{\nabla}$ on $f^{-1}(x)$ is a metric connection, then $\mathcal{T}_U \phi V$ has no components in $\mathcal{D}$ for any $U \in \Gamma(Kerf_*)$ and $V \in \Gamma(\Delta)$.*

Proof. We know that the induced connection $\hat{\nabla}$ on $f^{-1}(x)$ is a metric connection if and only if $\hat{\nabla}_U V \in \Gamma(\Delta)$ for $U \in \Gamma(Kerf_*)$ and $V \in \Gamma(\Delta)$ [1, Theorem 4]. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion and $\hat{\nabla}$ be a metric connection. Then, for any $W \in \Gamma(\mathcal{D})$ using (9), we get $g_1(\nabla_U V, W) = 0$. From the last equation and (2), we obtain $g_1(\phi\nabla_U V, \phi W) + \eta(\nabla_U V)\eta(W) = 0$, which implies that $g_1(\phi\nabla_U V, \phi W)$=0.

Next, using (4) and (16), we derive $g_1(\mathcal{T}_U\phi V, \phi W) = 0$. Therefore $\mathcal{T}_U\phi V$ has no components in $\mathcal{D}$. □

Theorem 3.7. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, $\mathcal{D} \perp \langle\xi\rangle$ is integrable if and only if $\mathcal{T}^l_U \tau V = \mathcal{T}^l_V \tau U$ for any $U, V \in \Gamma(\mathcal{D} \perp \langle\xi\rangle)$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion. Suppose that $U, V \in \Gamma(\mathcal{D} \perp \langle\xi\rangle)$. Then (35) becomes

$$\mathcal{T}^l_U \tau V - \omega(\hat{\nabla}_U V) = 0. \tag{37}$$

On interchanging the role of U and V in (37), we get

$$\mathcal{T}^l_V \tau U - \omega(\hat{\nabla}_V U) = 0. \tag{38}$$

Now from (37) and (38), we derive

$$\mathcal{T}^l_U \tau V - \mathcal{T}^l_V \tau U - \omega[U, V] = 0. \tag{39}$$

Then the proof follows from (39). □

Corollary 3.8. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then the distribution $\mathcal{D}$ is not integrable.*

Proof. Suppose that $\mathcal{D}$ is integrable. Then for any $U, V \in \Gamma(D)$, using Lemma 3.5 we have, $2g_1(V, \phi U) = \hat{g}([U, V], \xi) = 0$. This is a contradiction to the fact that $\mathcal{D}$ is non-degenerate distribution of $f^{-1}(x)$. □

Theorem 3.9. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, $\Delta \perp \langle\xi\rangle$ is integrable if and only if $\mathcal{T}_U \omega V - \mathcal{T}_V \omega U = \eta(U)V - \eta(V)U$ for any $U, V \in \Gamma(\Delta \perp \langle\xi\rangle)$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion. Suppose that $U, V \in \Gamma(\Delta \perp \langle\xi\rangle)$. Then (33) becomes

$$\tau(\hat{\nabla}_U V) = \mathcal{T}_U \omega V + \eta(V)U - \phi(\mathcal{T}^l_U V) - \hat{g}(U, V). \tag{40}$$

Interchanging the role of U and V in (40), we obtain

$$\tau(\hat{\nabla}_V U) = \mathcal{T}_V \omega U + \eta(U)V - \phi(\mathcal{T}^l_V U) - \hat{g}(V, U). \tag{41}$$

Sine $\hat{\nabla}$ is a symmetric connection, using (40) and (41), we get

$$\tau([U, V]) = \mathcal{T}_U \omega V - \mathcal{T}_V \omega V + \eta(V)U - \eta(U)V. \tag{42}$$

Then the proof follows from (42). □

Corollary 3.10. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then Δ is not integrable.*

Proof. Suppose that Δ is integrable. Then for any $U, V \in \Gamma(\Delta)$, from Lemma 3.5 we have, $2g_1(V, \phi U) = \hat{g}([U, V], \xi) = 0$. Since we know that for any $U \in \Gamma(\Delta)$, there exists $V \in \Gamma(\Delta)$ such that $g_1(U, \phi V) \neq 0$ as $\phi(\Delta) = ltr(Kerf_*)$. Thus we derive a contradiction. □

Theorem 3.11. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then $\Delta \perp \langle\xi\rangle$ defines a totally geodesic foliation if and only if $\hat{g}(\mathcal{T}_U \phi QV, \phi W) = \eta(V)\eta(\hat{\nabla}_U W)$ for any $U, V \in \Gamma(\Delta \perp \langle\xi\rangle)$ and $W \in \Gamma(\mathcal{D})$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion. Then, $\Delta \perp \langle\xi\rangle$ defines a totally geodesic foliation if and only if $\hat{\nabla}_U V \in \Gamma(\Delta \perp \langle\xi\rangle)$ for $U, V \in \Gamma(\Delta \perp \langle\xi\rangle)$. Since ∇ is a metric connection, using (9) for any $U, V \in \Gamma(\Delta \perp \langle\xi\rangle)$ and $W \in \Gamma(\mathcal{D})$, we get $\hat{g}(\hat{\nabla}_U V, W) = -g_1(V, \nabla_U W)$. Next, from (2), (4), (9) and (29) we derive $\hat{g}(\hat{\nabla}_U V, W) = -g_1(\phi QV, \hat{\nabla}_U \phi W) - \eta(V)\eta(\hat{\nabla}_U W)$. Then from (20) and (23), we obtain

$$\hat{g}(\hat{\nabla}_U V, W) = \hat{g}(\mathcal{T}_U \phi QV, \phi W) - \eta(V)\eta(\hat{\nabla}_U W). \tag{43}$$

Thus, our assertion follows from (43). □

Theorem 3.12. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, $S(Kerf_*)$ defines a totally geodesic foliation if and only if $\mathcal{T}^*_U \phi N$ has no components in $\mathcal{D}$ for any $U \in \Gamma(S(Kerf_*))$ and $N \in \Gamma(ltr(Kerf_*))$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion. Then, $S(Kerf_*)$ defines a totally geodesic foliation if and only if $\hat{\nabla}_U V \in S(Kerf_*)$ for $U, V \in \Gamma(S(Kerf_*))$. Using (9) and (2) for any $U, V \in \Gamma(S(Kerf_*))$ and $N \in \Gamma(ltr(Kerf_*))$, we get $g_1(\hat{\nabla}_U V, N) = g_1(\phi\nabla_U V, \phi N)$. Now from (4), (9) and (22), we obtain

$$g_1(\hat{\nabla}_U V, N) = -\hat{g}(\phi PV, \mathcal{T}^*_U \phi N). \tag{44}$$

Then the proof follows from (44). □

4. Radical transversal lightlike submersions with totally contact umbilical fibers

In this section, we introduce radical transversal lightlike submersions from indefinite Sasakian manifolds onto lightlike manifolds with totally contact umbilical fibers such that the structure vector field ξ is tangent to fiber. We also study the geometry of such lightlike submersions.

Definition. Let $(M_1, \phi, \xi, \eta, g_1)$ be an indefinite Sasakian manifold and (M_2, g_2) be a lightlike manifold. Suppose that $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ is a lightlike submersion with the characteristic vector field ξ tangent to $f^{-1}(x)$, i.e., ξ

belongs to $S(Kerf_*)$. Then, f is called with totally contact umbilical fibers if for any $U, V \in \Gamma(Kerf_*)$, we have

$$\mathcal{T}^l_U V = [\hat{g}(U,V) - \eta(U)\eta(V)]\beta_l + \eta(U)\mathcal{T}^l_V \xi + \eta(V)\mathcal{T}^l_U \xi, \tag{45}$$

$$\mathcal{T}^s_U V = [\hat{g}(U,V) - \eta(U)\eta(V)]\beta_s + \eta(U)\mathcal{T}^s_V \xi + \eta(V)\mathcal{T}^s_U \xi, \tag{46}$$

where $\beta_l \in \Gamma(ltr(Kerf_*))$ and $\beta_s \in \Gamma(S(Kerf_*^{\perp}))$.

Theorem 4.1. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2) with totally contact umbilical fibers. Then, $\beta_l = 0$ if and only if $S(kerf_*)$ is integrable.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion with totally contact umbilical fibers. Then, using (9), (4) and (2) for any $U, V \in \Gamma(\mathcal{D})$ and $N \in \Gamma(ltr(Kerf_*))$, we obtain

$$g_1(\mathcal{T}^l_U \phi V, \phi N) - g_1(\mathcal{T}^l_V \phi U, \phi N) = g_1([U,V], N). \tag{47}$$

From (45), we have

$$\mathcal{T}^l_U \phi V - \mathcal{T}^l_V \phi U = \hat{g}(U, \phi V)\beta_l - \hat{g}(V, \phi U)\beta_l. \tag{48}$$

Now using (47), (48) and (3), we derive

$$g_1([U,V], N) = 2\hat{g}(U, \phi V) g_1(\beta_l, \phi N), \tag{49}$$

which completes the proof. □

Theorem 4.2. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2) with totally contact umbilical fibers. Then, $\beta_l = 0$ if and only if $\mathcal{T}^*_U \phi V = 0$ for any $U, V \in \Gamma(\mathcal{D})$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion with totally contact umbilical fibers. Now from (4), (9) and (31) for any $U, V \in \Gamma(\mathcal{D})$, we get

$$\hat{\nabla}_U \phi V - \phi(\mathcal{T}^l_U V) = g(U,V)\xi - \mathcal{T}^l_U \phi V - \mathcal{T}^s_U \phi V + \tau \hat{\nabla}_U V + \omega \hat{\nabla}_U V + \phi(\mathcal{T}^s_U V).$$

Then for any $Z \in \Gamma(\Delta)$, we have

$$g_1(\hat{\nabla}_U V, \phi Z) = g_1(\phi(\mathcal{T}^l_U V), \phi Z). \tag{50}$$

Using (50), (20), (2) and (45), we obtain

$$g_1(\mathcal{T}^*_U \phi V, \phi Z) = \hat{g}(U,V) g_1(\beta_l, Z). \tag{51}$$

Then, our assertion follows from (51). □

Theorem 4.3. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2) with totally contact umbilical fibers. If the induced connection $\hat{\nabla}$ on $f^{-1}(x)$ is a metric connection, then $\mathcal{T}_U \phi Z = \eta(U) Z$ for any $U \in \Gamma(Kerf_*)$ and $Z \in \Gamma(\Delta)$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion with totally contact umbilical fibers. Also, suppose that $\hat{\nabla}$ is a metric connection. Then, from (4), (9), (16), (45), (46) and (31) we get

$$\mathcal{T}_U \phi Z + \nabla^l_U \phi Z + \mathcal{D}^s(U, \phi Z) = \tau \hat{\nabla}_U Z + \omega \hat{\nabla}_U Z + \eta(U)\phi(\mathcal{T}^l_Z \xi) + \eta(U)\phi(\mathcal{T}^s_Z \xi). \tag{52}$$

Equating tangential components of (52), we get

$$\mathcal{T}_U \phi Z = \tau \hat{\nabla}_U Z + \phi(\mathcal{T}^l_Z \xi)\eta(U). \tag{53}$$

Also, using (5) and (9) we have

$$\mathcal{T}^l_Z \xi = -\phi Z. \tag{54}$$

Now, from (53) and (54) we obtain

$$\mathcal{T}_U \phi Z = \tau \hat{\nabla}_U Z - \phi^2 Z\, \eta(U) = \tau \hat{\nabla}_U Z + \eta(U) Z$$

which imply that $\mathcal{T}_U \phi Z = \eta(U) Z$. □

Theorem 4.4. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2) with totally contact umbilical fibers. If Δ is parallel, then $\mathcal{T}_{Z_1} \phi Z_2 = \phi \mathcal{T}^l_{Z_1} Z_2$ for any $Z_1, Z_2 \in \Gamma(\Delta)$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion with totally contact umbilical fibers. Suppose that Δ is parallel distribution. Then, for any $Z_1, Z_2 \in (\Delta)$, using (4), (9), (16) and (31) we get

$$\mathcal{T}_{Z_1} \phi Z_2 + \nabla^l_{Z_1} \phi Z_2 + \mathcal{D}^s(Z_1, \phi Z_2) = \tau \hat{\nabla}_{Z_1} Z_2 + \omega \hat{\nabla}_{Z_1} Z_2 + \phi(\mathcal{T}^l_{Z_1} Z_2) + \phi(\mathcal{T}^s_{Z_1} Z_2).$$

On equating tangential parts of the above equation, we obtain

$$\mathcal{T}_{Z_1} \phi Z_2 = \tau \hat{\nabla}_{Z_1} Z_2 + \phi(\mathcal{T}^l_{Z_1} Z_2).$$

As Δ is a parallel distribution, we have $\tau \hat{\nabla}_{Z_1} Z_2 = 0$. This completes the proof. □

Let (M, ϕ, ξ, η, g) be an indefinite Sasakian manifold. Then a plane section in T_pM is called a ϕ-section if it is span by a unit vector U orthogonal to ξ and ϕU, where $U \in T_pM$. A ϕ-sectional curvature of M at p is defined as the sectional curvature of M at p with respect to a ϕ-section. If the ϕ-sectional curvature on M is constant for every ϕ-section, then M is called an indefinite Sasakian space form, denoted by $M(c)$, where c is the ϕ-sectional curvature. In [9], the curvature tensor R of an indefinite Sasakian space form $M(c)$ is given as follows:

$$\begin{aligned} R(U,V)W = {} & \frac{(c+3)}{4}\{g(V,W)U - g(U,W)V\} + \frac{(c-1)}{4}\{\epsilon\eta(U)\eta(W)V \\ & - \epsilon\eta(V)\eta(W)U + g(U,W)\eta(V)\xi - g(V,W)\eta(U)\xi \\ & + g(\phi V, W)\phi U + g(\phi W, U)\phi V - 2g(\phi U, V)\phi W\}, \end{aligned} \tag{55}$$

where $U, V, W \in \Gamma(TM)$.

At the last of this section, we investigate the existence (non-existence) of radical transversal lightlike submersion from an indefinite Sasakian space form onto a lightlike manifold with totally contact umbilical fibers. For this purpose, we first prove some lemmas.

Lemma 4.5. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2) with totally contact umbilical fibers. Then, $\beta_s = 0$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion with totally contact umbilical fibers. Then for $U \in \Gamma(\mathcal{D})$, using (4), (9) and (31), we get

$$\hat{\nabla}_U \phi U + \mathcal{T}^l_U \phi U + \mathcal{T}^s_U \phi U - \tau \hat{\nabla}_U V - \omega \hat{\nabla}_U V - \phi(\mathcal{T}^l_U V) - \phi(\mathcal{T}^s_U U) = \hat{g}(U, U)\xi.$$

Equating the components on $S(Kerf_*)^\perp$ in the above equation, we get

$$\mathcal{T}^s_U \phi U = \phi(\mathcal{T}^s_U U). \tag{56}$$

Now, using (56) and (46) for $W \in \Gamma(S(Kerf_*)^\perp)$, we obtain

$$\hat{g}(U, U) g_1(\beta_s, \phi W) = -\hat{g}(U, \phi U) g_1(\beta_s, W),$$

which imply that $\hat{g}(U, U) g_1(\beta_s, W) = 0$. As $S(Kerf_*)$ and $S(Kerf_*)^\perp$ are non-degenerate, we derive $\beta_s = 0$. □

Lemma 4.6. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2) with totally contact umbilical fibers. Then, for any $U \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\Delta)$, we have*

$$\mathcal{T}^l_{\hat{\nabla}_U \phi U} Z = -\hat{g}(\hat{\nabla}_U \phi U, \xi)\phi Z, \tag{57}$$

$$\mathcal{T}^l_{\phi U} \xi = 0, \tag{58}$$

$$\hat{g}(U, \hat{\nabla}_{\phi U} Z) = -g_1(\mathcal{T}^l_{\phi U} U, Z), \tag{59}$$

$$\hat{g}(\phi U, \hat{\nabla}_U Z) = -g_1(\mathcal{T}^l_U \phi U, Z). \tag{60}$$

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion with totally contact umbilical fibers. Then, using (5) and (9), we obtain

$$\hat{\nabla}_Z \xi + \mathcal{T}^l_Z \xi + \mathcal{T}^s_Z \xi = -\phi Z.$$

Considering the components on $ltr(Kerf_*)$ in the above equation, we get

$$\mathcal{T}^l_Z \xi = -\phi Z. \tag{61}$$

Also, from (45) we derive

$$\mathcal{T}^l_{\hat{\nabla}_U \phi U} Z = \eta(\hat{\nabla}_U \phi U) \mathcal{T}^l_Z \xi. \tag{62}$$

Now, using (62) and (61), we get $\mathcal{T}^l_{\hat{\nabla}_U \phi U} Z = -\eta(\hat{\nabla}_U \phi U)\phi Z$. Thus we have (57).

From (9), (5) and (1), we get

$$U = \hat{\nabla}_{\phi U}\xi + \mathcal{T}^l_{\phi U}\xi + \mathcal{T}^s_{\phi U}\xi,$$

which proves (58).

As ∇ is a metric connection, we get

$$g_1(U, \nabla_{\phi U} Z) = -g_1(\nabla_{\phi U} U, Z).$$

Then, by using (9) we derive (59).

By a simple calculation, we obtain

$$g_1(\phi U, \nabla_U Z) = -g_1(\nabla_U \phi U, Z).$$

Thus, from (9) we have (60). □

Lemma 4.7. *Let f be a radical transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, for any $U \in \Gamma(\mathcal{D})$ we have*

$$\hat{g}(\hat{\nabla}_U \phi U, \xi) = \hat{g}(\phi U, \phi U), \tag{63}$$

$$\hat{g}(\hat{\nabla}_{\phi U} U, \xi) = -\hat{g}(U, U). \tag{64}$$

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a radical transversal lightlike submersion. Since ∇ is a metric connection, we obtain

$$g_1(\nabla_U \phi U, \xi) = -g_1(\phi U, \nabla_U \xi). \tag{65}$$

Now, from (65), (9) and (5) we derive (63). Following similar steps as above, we have (64). □

Theorem 4.8. *There exists no proper radical transversal lightlike submersion from an indefinite Sasakian space form $(M_1(c), \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2) with totally contact umbilical fibers and $c \neq -3$.*

Proof. Let $f : M_1(c) \to M_2$ be a proper radical transversal lightlike submersion with totally contact umbilical fibers and $c \neq -3$. Then for any $U \in \Gamma(\mathcal{D})$ and $Z_1, Z_2 \in \Gamma(\Delta)$, using (28), (55), (46) and Lemma 4.5, we obtain

$$\frac{1-c}{2}\hat{g}(\phi U, \phi U) g_1(\phi Z_1, Z_2) = g_1\big((\nabla_U \mathcal{T}^l)(\phi U, Z_1) - (\nabla_{\phi U}\mathcal{T}^l)(U, Z_1), Z_2\big), \tag{66}$$

where

$$(\nabla_U \mathcal{T}^l)(\phi U, Z_1) = \nabla^l_U \mathcal{T}^l_{\phi U} Z_1 - \mathcal{T}^l_{\hat{\nabla}_U \phi U} Z_1 - \mathcal{T}^l_{\phi U}\hat{\nabla}_U Z_1, \tag{67}$$

$$(\nabla_{\phi U} \mathcal{T}^l)(U, Z_1) = \nabla^l_{\phi U} \mathcal{T}^l_U Z_1 - \mathcal{T}^l_{\hat{\nabla}_{\phi U} U} Z_1 - \mathcal{T}^l_U \hat{\nabla}_{\phi U} Z_1. \tag{68}$$

Using (45), (63), (61) and (3), we get

$$\mathcal{T}^l_{\hat{\nabla}_U \phi U} Z_1 = -\hat{g}(\phi U, \phi U)\phi Z_1. \tag{69}$$

From (45) and (58), we have

$$\mathcal{T}^l_{\phi U}\hat{\nabla}_U Z_1 = \hat{g}(\phi U, \hat{\nabla}_U Z_1)\beta_l. \tag{70}$$

By using (67), (69), (70) and (45), we obtain

$$(\nabla_U \mathcal{T}^l)(\phi U, Z_1) = \hat{g}(U,U)\phi Z_1 - \hat{g}(\phi U, \hat{\nabla}_U Z_1)\beta_l. \tag{71}$$

From (45), (61), (3) and (64), we get

$$\mathcal{T}^l_{\hat{\nabla}_{\phi U} U} Z_1 = \hat{g}(U,U)\phi Z_1. \tag{72}$$

Using (45) and (58), we have

$$\mathcal{T}^l_U \hat{\nabla}_{\phi U} Z_1 = \hat{g}(U, \hat{\nabla}_{\phi U} Z_1)\beta_l. \tag{73}$$

Next, from (68), (72), (73) and (45), we derive

$$(\nabla_{\phi U} \mathcal{T}^l)(U, Z_1) = -\hat{g}(U,U)\phi Z_1 - \hat{g}(U, \hat{\nabla}_{\phi U} Z_1)\beta_l. \tag{74}$$

Thus, from (66), (71) and (74), we get

$$\begin{aligned}&\frac{1-c}{2}\hat{g}(U,U)g_1(\phi Z_1, Z_2)\\ &= 2\hat{g}(U,U)g_1(\phi Z_1, Z_2) + \hat{g}(U, \hat{\nabla}_{\phi U} Z_1)g_1(\beta_l, Z_2) - \hat{g}(\phi U, \hat{\nabla}_U Z_1)g_1(\beta_l, Z_2).\end{aligned}$$

Now, using (59), (60) and the above equation, we have

$$\begin{aligned}&\frac{1-c}{2}\hat{g}(U,U)g_1(\phi Z_1, Z_2)\\ &= 2\hat{g}(U,U)g_1(\phi Z_1, Z_2) - g_1(\mathcal{T}^l_{\phi U} U, Z_1)g_1(\beta_l, Z_2) + g_1(\mathcal{T}^l_U \phi U, Z_1)g_1(\beta_l, Z_2),\end{aligned}$$

which imply that $(3+c)\hat{g}(U,U)g_1(\phi Z_1, Z_2) = 0$. As $\Delta \oplus ltr(Kerf_*)$ and $S(Kerf_*)$ are non-degenerate, we can choose Z_1, Z_2 and U such that $\hat{g}(U,U) \neq 0$ and $g_1(\phi Z_1, Z_2) \neq 0$. Thus, we have $c = -3$, which is a contradiction. □

5. Transversal lightlike submersions

In this section, we study transversal lightlike submersions from indefinite Sasakian manifolds onto lightlike manifolds such that the structure vector field ξ is tangent to fiber.

Definition. Let $(M_1, \phi, \xi, \eta, g_1)$ be an indefinite Sasakian manifold and (M_2, g_2) be a lightlike manifold. Suppose that $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ is a lightlike submersion with the characteristic vector field ξ tangent to $f^{-1}(x)$, i.e., ξ belongs to $S(Kerf_*)$. Then f is called a transversal lightlike submersion if

(i) $\phi(\Delta) = ltr(Kerf_*)$,
(ii) $\phi(\mathcal{D}) \subseteq S(Kerf_*)^{\perp}$, where $\mathcal{D}$ is a non-degenerate subbundle of $S(Kerf_*)$ such that $S(Kerf_*) = \mathcal{D} \perp \langle \xi \rangle$.

Suppose that μ is the orthogonal complementary subbundle to $\phi(\mathcal{D})$ in $S(Kerf_*)^{\perp}$, that is,

$$S(Kerf_*)^{\perp} = \phi(\mathcal{D}) \perp \mu. \tag{75}$$

Then it is easy to see that μ is invariant with respect to ϕ. In view of the above definition, we have the following result.

Theorem 5.1. *There does not exist transversal* 1*-lightlike submersion from an indefinite Sasakian manifold* $(M_1, \phi, \xi, \eta, g_1)$ *onto a lightlike manifold* (M_2, g_2).

A transversal lightlike submersion is said to be proper if $S(Kerf_*)^{\perp} \neq 0$ and $\mathcal{D} \neq 0$. Let f be a transversal lightlike submersion from an indefinite Sasakian manifold onto a lightlike manifold. Then we have: (i) $\dim(\Delta) \geq 2$, (ii) Any proper transversal lightlike submersion from a 7-dimensional indefinite Sasakian manifold onto a 2-dimensional lightlike manifold must be 2-lightlike.

Now, we give two examples of proper transversal lightlike submersions.

Example 5.2. Consider an indefinite Sasakian manifold as given in Example 2.1 for $m = 6$ and $q = 1$, i.e., $(\mathbb{R}^{13}_2, \phi, \xi, \eta, g_1)$. Let $(\mathbb{R}^6, g_2)$ be a lightlike manifold, where $g_2 = \frac{1}{8}\big\{(da_2)^2 + (da_3)^2 + (da_5)^2 + (da_6)^2\big\}$ and $a_1, a_2, a_3, a_4, a_5, a_6$ are the usual coordinates on $\mathbb{R}^6$. Define a map $f : \mathbb{R}^{13}_2 \to \mathbb{R}^6$ by

$$f(x_1, \ldots, x_6, y_1, \ldots, y_6, z) = (x_1 + x_3, x_2 - x_4, x_5 + x_6, y_1 - y_3, y_2 + y_4, y_5 - y_6).$$

After some computations, we have $Kerf_* = Span\big\{V_1 = E_7 - E_9, V_2 = E_8 + E_{10}, V_3 = E_{11} - E_{12}, V_4 = E_1 + E_3, V_5 = E_2 - E_4, V_6 = E_5 + E_6, V_7 = E_{13} = \xi\big\}$, $(Kerf_*)^{\perp} = Span\big\{V_1, V_4, W_1 = E_8 - E_{10}, W_2 = E_{11} + E_{12}, W_3 = E_2 + E_4, W_4 = E_5 - E_6\big\}$ with $\Delta = Kerf_* \cap (Kerf_*)^{\perp} = Span\big\{V_1, V_4\big\}$, which implies $S(Kerf_*) = \mathcal{D} \perp \langle\xi\rangle$, where $\mathcal{D} = Span\big\{V_2, V_3, V_5, V_6\big\}$ and $S(Kerf_*)^{\perp} = Span\big\{W_1, W_2, W_3, W_4\big\}$. Now, we get $ltr(Kerf_*) = Span\big\{N_1 = -\frac{1}{2}(E_7 + E_9), N_2 = -\frac{1}{2}(E_1 - E_3)\big\}$. Then it is easy to see that f is a 2-lightlike submersion. Also, we have $\phi(V_1) = 2N_2$, $\phi(V_4) = -2N_1$, $\phi V_2 = -W_3$, $\phi(V_3) = -W_4, \phi(V_5) = W_1$ and $\phi(V_6) = W_2$, which implies $\phi(\Delta) = ltr(Kerf_*)$ and $\phi(\mathcal{D}) \subseteq S(Kerf_*)^{\perp}$. Hence f is a proper transversal 2-lightlike submersion.

Example 5.3. Consider an indefinite Sasakian manifold as given in Example 2.1 for $m = 7$ and $q = 1$, i.e., $(\mathbb{R}^{15}_2, \phi, \xi, \eta, g_1)$. Let $(\mathbb{R}^8, g_2)$ be a lightlike manifold, where $g_2 = \frac{1}{8}\big\{(da_2)^2 + (da_3)^2 + 2(da_4)^2 + (da_6)^2 + (da_7)^2 + 2(da_8)^2\big\}$ and $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ are the usual coordinates on $\mathbb{R}^8$. Define a map $f : \mathbb{R}^{15}_2 \to \mathbb{R}^8$ by

$$f(x_1, \ldots, x_7, y_1, \ldots, y_7, z) = (x_1 + x_5, x_2 + x_6, x_3 + x_4, x_7, y_1 - y_5, y_2 - y_6, y_3 - y_4, y_7).$$

Then by direct calculations, we get $Kerf_* = Span\big\{V_1 = E_8 - E_{12}, V_2 = E_9 - E_{13}, V_3 = E_{10} - E_{11}, V_4 = E_1 + E_5, V_5 = E_2 + E_6, V_6 = E_3 + E_4, V_7 = E_{15} = \xi\big\}$, $(Kerf_*)^{\perp} = Span\big\{V_1, V_4, W_1 = E_9 + E_{13}, W_2 = E_{10} + E_{11}, W_3 = E_{14}, W_4 = E_2 - E_6, W_5 = E_3 - E_4, W_6 = E_7\big\}$ with $\Delta = Kerf_* \cap (Kerf_*)^{\perp} = Span\big\{V_1, V_4\big\}$ which implies $S(Kerf_*) = \mathcal{D} \perp \langle\xi\rangle$, where $\mathcal{D} = Span\big\{V_2, V_3, V_5, V_6\big\}$ and $S(Kerf_*)^{\perp} = Span\big\{W_1, W_2, W_3, W_4, W_5, W_6\big\}$. Thus, we obtain $ltr(Kerf_*) = Span\big\{N_1 = -\frac{1}{2}(E_8 + E_{12}), N_2 = -\frac{1}{2}(E_1 - E_5)\big\}$. Now it is easy to see that f is a 2-lightlike submersion. Further we have $\phi(V_1) = 2N_2$, $\phi(V_4) = -2N_1$, $\phi(V_2) = -W_4$, $\phi(V_3) = -W_5, \phi(V_5) = W_1$ and $\phi(V_6) = W_2$ which implies $\phi(\Delta) = ltr(Kerf_*)$ and $\phi(\mathcal{D}) \subset S(Kerf_*)^{\perp}$. Therefore f is a proper transversal 2-lightlike submersion.

Let f be a transversal lightlike submersion from an indefinite Sasakian manifold M_1 onto a lightlike manifold M_2. Also, suppose that Q and $\mathcal{P}$ denote the projections of $Kerf_*$ on Δ and $\mathcal{D}$, respectively. Then, for $U \in \Gamma(Kerf_*)$, we write

$$U = QU + \mathcal{P}U + \eta(U)\xi, \tag{76}$$

where $QU \in \Gamma(\Delta)$ and $\mathcal{P}U \in \Gamma(\mathcal{D})$. On applying ϕ to (76), we have

$$\phi U = \phi QU + \phi \mathcal{P}U. \tag{77}$$

If we set $\phi QU = LU$ and $\phi\mathcal{P}U = SU$, then (77) becomes

$$\phi U = LU + SU, \tag{78}$$

where $LU \in \Gamma(ltr(Kerf_*))$ and $SU \in \Gamma(S(Kerf_*)^{\perp})$. Using (75), for $W \in \Gamma(S(Kerf_*)^{\perp})$, we have

$$\phi W = \mathcal{B}W + \mathcal{C}W, \tag{79}$$

where $\mathcal{B}W \in \Gamma(\mathcal{D})$ and $\mathcal{C}W \in \Gamma(\mu)$.

Now, using (4), (78), (9), (16), (17) and (79), for $U, V \in \Gamma(Kerf_*)$, we obtain

$$\mathcal{T}_U LV + \mathcal{T}_U SV - \phi\mathcal{T}^l_U V - \mathcal{B}\mathcal{T}^s_U V = \hat{g}(U,V)\xi - \eta(V)U, \tag{80}$$

$$\mathcal{D}^s(U, LV) + \nabla^s_U SV - S\hat{\nabla}_U V = \mathcal{C}\mathcal{T}^s_U V, \tag{81}$$

$$\nabla^l_U LV + \mathcal{D}^l(U, SV) = L(\hat{\nabla}_U V). \tag{82}$$

Now, we discuss the integrability of distributions on a fiber of transversal lightlike submersions.

Theorem 5.4. *Let f be a transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, $\Delta \perp \langle\xi\rangle$ is integrable if and only if $\mathcal{D}^s(U, LV) = \mathcal{D}^s(V, LU)$ for $U, V \in \Gamma(\Delta \perp \langle\xi\rangle)$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a transversal lightlike submersion. Suppose that $U, V \in \Gamma(\Delta \perp \langle\xi\rangle)$. Then, (81) becomes

$$\mathcal{D}^s(U, LV) - S\hat{\nabla}_U V - \mathcal{C}\mathcal{T}^s_U V = 0. \tag{83}$$

Interchanging the role of U and V in (83), we get

$$\mathcal{D}^s(V, LU) - S\hat{\nabla}_V U - \mathcal{C}\mathcal{T}^s_V U = 0. \tag{84}$$

As $\hat{\nabla}$ is symmetric connection, using (83) and (84), we obtain

$$\mathcal{D}^s(U, LV) - \mathcal{D}^s(V, LU) = S[U, V]. \tag{85}$$

Then the proof follows from (85). □

Corollary 5.5. *Let f be a transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifolds (M_2, g_2). Then, Δ is not integrable.*

The proof of the above corollary is similar as that of Corollary 3.10, so we omit it.

Theorem 5.6. *Let f be a transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, $\mathcal{D} \perp \langle\xi\rangle$ is integrable if and only if $\mathcal{D}^l(U, SV) = \mathcal{D}^l(V, SU)$ for $U, V \in \Gamma(\mathcal{D} \perp \langle\xi\rangle)$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a transversal lightlike submersion. Suppose that $U, V \in \Gamma(\mathcal{D} \perp \langle\xi\rangle)$. Then (82) becomes

$$\mathcal{D}^l(U, SV) = L\hat{\nabla}_U V. \tag{86}$$

On interchanging the role of U and V in (86), we get

$$\mathcal{D}^l(V, SU) = L\hat{\nabla}_V U. \tag{87}$$

Now, from (86) and (87), we obtain

$$\mathcal{D}^l(U, SV) - \mathcal{D}^l(V, SU) = L[U, V].$$

Thus, the proof follows from the above equation. □

Theorem 5.7. *Let f be a transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, $\Delta \perp \langle\xi\rangle$ defines a totally geodesic foliation if and only if $\mathcal{D}^s(U, LV) = \mathcal{C}\mathcal{T}^s_U V$ for $U, V \in \Gamma(\Delta \perp \langle\xi\rangle)$.*

Proof. Since we have, $\Delta \perp \langle\xi\rangle$ defines a totally geodesic foliation if and only if $\hat{\nabla}_U V \in \Gamma(\Delta \perp \langle\xi\rangle)$ for $U, V \in \Gamma(\Delta \perp \langle\xi\rangle)$. Using (81), for $U, V \in \Gamma(\Delta \perp \langle\xi\rangle)$, we obtain $\mathcal{D}^s(U, LV) - S\hat{\nabla}_U V - \mathcal{C}\mathcal{T}^s_U V = 0$. Then, the proof follows from the last equation. □

Theorem 5.8. *Let f be a transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, $\mathcal{D} \perp \langle\xi\rangle$ defines a totally geodesic foliation if and only if $\mathcal{D}^l(V, SU) = 0$ for $U, V \in \Gamma(\mathcal{D} \perp \langle\xi\rangle)$.*

Proof. As we have, $\mathcal{D} \perp \langle\xi\rangle$ defines a totally geodesic foliation if and only if $\hat{\nabla}_U V \in \Gamma(\mathcal{D} \perp \langle\xi\rangle)$ for $U, V \in \Gamma(\mathcal{D} \perp \langle\xi\rangle)$. By using (82), for $U, V \in \Gamma(\mathcal{D} \perp \langle\xi\rangle)$, we get $\mathcal{D}^l(U, SV) = L\hat{\nabla}_U V$. Thus the proof is completed. □

Theorem 5.9. *Let f be a transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, the induced connection $\hat{\nabla}$ on $f^{-1}(x)$ is a metric connection if and only if $\mathcal{B}\mathcal{D}^s(U, \phi V) = \eta(\hat{\nabla}_U V)\xi$ for $U \in \Gamma(Kerf_*)$ and $V \in \Gamma(\Delta)$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a transversal lightlike submersion. Using (4), (9), (16), (78) and (79), for $U \in \Gamma(Kerf_*)$ and $V \in \Gamma(\Delta)$, we get

$$-\hat{\nabla}_U V = L\mathcal{T}_U \phi V + S\mathcal{T}_U \phi V + \phi\nabla^l_U \phi V + \mathcal{B}\mathcal{D}^s(U, \phi V)$$

$$+\mathcal{C}\mathcal{D}^s(U,\phi V)-\eta(\hat{\nabla}_U V)\xi+\mathcal{T}^l_U V+\mathcal{T}^s_U V.$$

Equating the tangential components of the above equation, we obtain

$$-\hat{\nabla}_U V=\mathcal{B}\mathcal{D}^s(U,\phi V)+\phi\nabla^l_U\phi V-\eta(\hat{\nabla}_U V)\xi. \tag{88}$$

Since we have, the induced connection $\hat{\nabla}$ on $f^{-1}(x)$ is a metric connection if and only if $\hat{\nabla}_U V\in\Gamma(\Delta)$ for $U\in\Gamma(Kerf_*)$ and $V\in\Gamma(\Delta)$. Thus, the proof follows from (88). □

Theorem 5.10. *Let f be a transversal lightlike submersion from an indefinite Sasakian manifold (M_1,ϕ,ξ,η,g_1) onto a lightlike manifold (M_2,g_2) with totally contact umbilical fibers. If the induced connection $\hat{\nabla}$ on $f^{-1}(x)$ is a metric, then we have $\mathcal{D}^s(U,\phi Z)=\eta(U)\mathcal{C}\mathcal{T}^s_Z\xi$ for $U\in\Gamma(Kerf_*)$ and $Z\in\Gamma(\Delta)$.*

Proof. Let $f:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a transversal lightlike submersion with totally contact umbilical fibers. Also, suppose that $\hat{\nabla}$ is a metric connection. Then, using (4), (9), (16) and (78), for $U\in\Gamma(Kerf_*)$ and $Z\in\Gamma(\Delta)$, we get

$$\mathcal{T}_U\phi Z+\nabla^l_U\phi Z+\mathcal{D}^s(U,\phi Z)-L\hat{\nabla}_U Z-S\hat{\nabla}_U Z-\phi\mathcal{T}^l_U Z-\phi\mathcal{T}^s_U Z=0. \tag{89}$$

From (89), (45), (46) and (79), we have

$$\begin{aligned}&\mathcal{T}_U\phi Z+\nabla^l_U\phi Z+\mathcal{D}^s(U,\phi Z)-L\hat{\nabla}_U Z-S\hat{\nabla}_U Z\\&-\eta(U)\phi\mathcal{T}^l_Z\xi-\eta(U)\mathcal{B}\mathcal{T}^s_Z\xi-\eta(U)\mathcal{C}\mathcal{T}^s_Z\xi=0.\end{aligned}$$

Considering the components on $S(Kerf_*)^\perp$ in the above equation, we obtain

$$\mathcal{D}^s(U,\phi Z)-\eta(U)\mathcal{C}\mathcal{T}^s_Z\xi=S\hat{\nabla}_U Z. \tag{90}$$

Since $\hat{\nabla}_U Z\in\Gamma(\Delta)$, from (90) we have $\mathcal{D}^s(U,\phi Z)=\eta(U)\mathcal{C}\mathcal{T}^s_Z\xi$. □

Now, we obtain a classification theorem for transversal lightlike submersions between indefinite Sasakian manifolds and lightlike manifolds with totally contact umbilical fibers.

Lemma 5.11. *Let f be a transversal lightlike submersion from an indefinite Sasakian manifold (M_1,ϕ,ξ,η,g_1) onto a lightlike manifold (M_2,g_2) with totally contact umbilical fibers. Then, $\beta_l=0$ if and only if $\mathcal{D}^s(U,\phi Z)$ has no components in $\phi(\mathcal{D})$ for $U\in\Gamma(D)$ and $Z\in\Gamma(\Delta)$.*

Proof. Let $f:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a transversal lightlike submersion with totally contact umbilical fibers. Then, using (4) for $U\in\Gamma(\mathcal{D})$ and $Z\in\Gamma(\Delta)$, we get

$$\nabla_U\phi U-\phi(\nabla_U U)=\hat{g}(U,U)\xi. \tag{91}$$

From (91), (9), (17), (78) and (79), we obtain

$$\begin{aligned}\hat{g}(U,U)\xi=&\,\mathcal{T}_U\phi U+\mathcal{D}^l(U,\phi U)+\nabla^s_U\phi U-L\hat{\nabla}_U U-S\hat{\nabla}_U U-\phi\mathcal{T}^l_U U\\&-\mathcal{B}\mathcal{T}^s_U U-\mathcal{C}\mathcal{T}^s_U U.\end{aligned}$$

Equating tangential parts in the above equation, we have

$$\hat{g}(U,U)\xi = \mathcal{T}_U \phi U - \mathcal{B}\mathcal{T}^s_U U - \phi \mathcal{T}^l_U U. \tag{92}$$

Thus, we get

$$g_1(\mathcal{T}_U \phi U, \phi Z) - g_1(\phi \mathcal{T}^l_U U, \phi Z) = 0. \tag{93}$$

Now, using (93), (2), (45) and (19), we derive

$$g_1(\mathcal{D}^s(U, \phi Z), \phi U) + \hat{g}(U,U) g_1(\beta_l, Z) = 0. \tag{94}$$

Since $\mathcal{D}$ is non-degenerate, our assertion follows from (94). □

Theorem 5.12. *Let f be a transversal lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2) with totally contact umbilical fibers and satisfying $\phi(\mathcal{D}) = S(Kerf_*)^\perp$. Then, $\beta_s = 0$ or* $\dim(\mathcal{D}) = 1$.

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a transversal lightlike submersion with totally contact umbilical fibers. Then, for $V \in \Gamma(D)$, using (92), (79) and (3), we get

$$\hat{g}(\mathcal{T}_U \phi U, V) = -g_1(\mathcal{T}^s_U U, \phi V). \tag{95}$$

Also from (18), we have

$$\hat{g}(\mathcal{T}_U \phi U, V) = -g_1(\mathcal{T}^s_U V, \phi U). \tag{96}$$

From (95), (96) and (46), we obtain

$$\hat{g}(U,U) g_1(\beta_s, \phi V) = \hat{g}(U,V) g_1(\beta_s, \phi U). \tag{97}$$

On interchanging the role of U and V in (97), we obtain

$$\hat{g}(V,V) g_1(\beta_s, \phi U) = \hat{g}(V,U) g_1(\beta_s, \phi V). \tag{98}$$

Now, using (97) and (98), we derive

$$g_1(\beta_s, \phi U) = \frac{\hat{g}(U,V)^2}{\hat{g}(U,U)\hat{g}(V,V)} g_1(\beta_s, \phi U).$$

Since $S(Kerf_*)^\perp$ is non-degenerate, we have either $\beta_s = 0$ or $\mathcal{D}$ is one dimensional. □

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