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FORMULAS AND RELATIONS FOR BERNOULLI-TYPE NUMBERS AND POLYNOMIALS DERIVE FROM BESSEL FUNCTION

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Abstract. The main purpose of this paper is to give some new identities and properties related to Bernoulli type numbers and polynomials associated with the Bessel function of the first kind. We give symmetric properties of the Bernoulli type numbers and polynomials. Moreover, using generating functions and the Faà di Bruno's formula, we derive some new formulas and relations related to not only these polynomials, but also the Bernoulli numbers and polynomials and the Euler numbers and polynomials.

1. Introduction

Special numbers and polynomials with their generating functions have numerous applications in almost all applied sciences, including all subjects of mathematics. Thanks to these important applications, many researchers still continue to work and research intensively, including the generating functions of these numbers and polynomials.

The motivation of this study is to give new relations and formulas on Bernoulli-type numbers and polynomials with the help of Bessel functions and Euer gamma functions, which are in their very active and well-known classes in the theory of special functions.

Before we start to give these results, we will briefly give some definitions and relations below.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, and $\mathbb{Z}^- = \{\ldots, -2, -1\}$. Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively, denote the sets of integers, real numbers, and complex numbers.

The Bernoulli numbers and polynomials are, respectively, defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \ |t| < 2\pi, \tag{1}$$

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and

$$\frac{te^{tz}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\ \ |t|<2\pi, \tag{2}$$

(*cf.* [1–21]).

The Euler polynomials and numbers are, respectively, defined by

$$\frac{2e^{tx}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\ \ |t|<\pi, \tag{3}$$

and

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\ \ |t|<\pi \tag{4}$$

(*cf.* [1–21]).

Recently, Frappier ([2,3]) gave new type generalized Bernoulli numbers and polynomials, which are denoted by $B_{n,\alpha}$, $B_{n,\alpha}(x)$. In order to give generating functions for these numbers and polynomials, we need the following special functions:

$$g_{\alpha}(z)=\frac{\Gamma(\alpha+1)\,J_{\alpha}(z)\,2^{\alpha}}{z^{\alpha}}, \tag{5}$$

where

$$J_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+\alpha}}{\Gamma(k+1)\,\Gamma(\alpha+k+1)\,2^{2k+\alpha}}$$

is the Bessel function of the first kind of order α and $\Gamma(z)$ is the Euler gamma function (*cf.* [2,15]). The function $\frac{J_{\alpha}(z)}{z^{\alpha}}$ is an even entire function of exponential type one. Here $\alpha\notin\mathbb{Z}^{-}$. The zeros $j_k=j_k(\alpha)$ of $\frac{J_{\alpha}(z)}{z^{\alpha}}$ may be ordered in such a way that $j_{-k}=-j_k$ and $0\le|j_1|\le|j_2|\le\cdots$. Frappier [2] gave the following generating functions for the Bernoulli-type polynomials $B_{n,\alpha}(x)$ as follows:

$$\frac{e^{\left(x-\frac{1}{2}\right)z}}{g_{\alpha}\left(\frac{iz}{2}\right)}=\sum_{n=0}^{\infty}B_{n,\alpha}(x)\frac{z^n}{n!},\ \ |z|<2|j_1|. \tag{6}$$

Note that $B_{n,\alpha}(0)=B_{n,\alpha}$ denotes the Bernoulli-type numbers.

Setting $\alpha=\frac{1}{2}$ and $\alpha=-\frac{1}{2}$, (6) reduces to (2) and (3), respectively.

In [2], Frappier gave fundamental properties of the α-Bernoulli type polynomials and numbers. By using different values of α, Frappier found the values of the α-Bernoulli type polynomials and numbers. In [3], Frappier gave not only the relation between the sums $\sum_{k=1}^{\infty}j_k^{-2r}$, $r\in\mathbb{Z}^{+}$, where $j_k=j_k(\alpha)$ denotes the zeros of the Bessel function of the first kind of order α, and α-Bernoulli type polynomials but also many applications related to Euler-MacLaurin summation formula and infinite series. In [4], Frappier constructed the α-calculus with the help of the α-Bernoulli type polynomials. Frappier gave many applications of the α-calculus. We can give many new relations and identities related to the

α-Bernoulli type polynomials. We also study on relations between logarithmic function which involve the function $g_{\alpha}(z)$ trigonometric function and classical Bernoulli numbers and Euler numbers as well.

Yang [21] studied on symmetry for the classical Bernoulli polynomials of higher order. Yang gave a relation of symmetry between power sum polynomials and the classical Bernoulli numbers. This property has also been studied by Tuenter [19]. By using the same motivation of the above studies, we can prove an identity of symmetry for the α-Bernoulli type polynomials.

We summarize our results as follows: In Section 2, we give some formulas and relations of the Bernoulli-type numbers and polynomials. We prove Raabe type multiplication formula for these polynomials. In Section 3, using generating functions, which related to logarithm of functions involving trigonometric functions and the function $g_{\alpha}(z)$, we give some new identities for the Bernoulli-type numbers and polynomials. We also establish some formulas with the aid of the function $\frac{1}{g_{\alpha}(z)}$ and Faà di Bruno's formula gives an explicit formula.

2. Some identities related to the Bernoulli-type numbers and polynomials

In this section, by the help of generating functions, we give many relations and identities associated with the Bernoulli-type numbers and polynomials.

Theorem 2.1. *Let $n \in \mathbb{N}$. For an arbitrary α (not a negative integer), we have*

$$\sum_{k=0}^{n} \binom{n}{k} B_{n-k,\alpha}(x) \frac{\Gamma(\alpha+1)}{4^k \Gamma(\alpha+k+1)} = \left(x - \frac{1}{2}\right)^n. \tag{7}$$

Proof. Using (6), we get

$$\sum_{n=0}^{\infty} \frac{\left(x-\frac{1}{2}\right)^n z^n}{n!} = g_{\alpha}\left(\frac{iz}{2}\right) \sum_{n=0}^{\infty} B_{n,\alpha}(x) \frac{z^n}{n!}.$$

Combining the above equation with (5), we obtain

$$\sum_{n=0}^{\infty} \frac{\left(x-\frac{1}{2}\right)^n z^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} B_{n-k,\alpha}(x) \frac{\Gamma(\alpha+1)}{4^k \Gamma(\alpha+k+1)} \frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$ both sides of the above, we arrive at the desire result. □

Substituting $x = \frac{1}{2}$ into (7), we arrive at the following result:

Corollary 2.2. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{B_{n-k,\alpha}\left(\frac{1}{2}\right) \Gamma(\alpha+1)}{4^k \Gamma(\alpha+k+1)} = 0.$$

Integrating both sides of the equation (7) from a to b, we obtain

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}B_{j,\alpha}\frac{\Gamma(\alpha+1)}{4^{k}\Gamma(\alpha+k+1)}\int_{a}^{b}x^{n-k-j}dx=\int_{a}^{b}\left(x-\frac{1}{2}\right)^{n}dx.$$

Therefore, we get

$$\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}\frac{(n+1)\,\Gamma(\alpha+1)\left(b^{n-k-j+1}-a^{n-k-j+1}\right)}{4^{k}\,(n-k-j+1)\,\Gamma(\alpha+k+1)}B_{j,\alpha}$$
$$=\left(b-\frac{1}{2}\right)^{n+1}-\left(a-\frac{1}{2}\right)^{n+1}.$$

Combining the above equation with the following well-known identities for the Euler gamma function:

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$$

and

$$\Gamma(\alpha+k+1)=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k)\Gamma(\alpha),$$

we get the following theorem:

Theorem 2.3. *Let $n\in\mathbb{N}$. For an arbitrary α (not a negative integer), we have*

$$\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}\frac{4^{-k}\,(n+1)\left(b^{n-k-j+1}-a^{n-k-j+1}\right)}{(n-k-j+1)\,(\alpha+1)(\alpha+2)\cdots(\alpha+k)}B_{j,\alpha}\tag{8}$$
$$=\left(b-\frac{1}{2}\right)^{n+1}-\left(a-\frac{1}{2}\right)^{n+1}.$$

Putting $b=1$ and $a=0$ in (8), we have the following formula for the finite sums:

Corollary 2.4. *Let $n\in\mathbb{N}$. For an arbitrary α (not a negative integer), we have*

$$\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}\frac{4^{-k}B_{j,\alpha}}{(n-k-j+1)\,(\alpha+1)(\alpha+2)\cdots(\alpha+k)}=\frac{1+(-1)^{n}}{(n+1)\,2^{n+1}}.$$

Some formulas including the Raabe type multiplication formula of the Bernoulli-type polynomials were given by the first author [14]. Now we briefly summarize this formula by following theorem.

Theorem 2.5. *Let $n\in\mathbb{N}$. For an arbitrary α (not a negative integer), we have*

$$\sum_{k=0}^{m-1}B_{n,\alpha}\left(x+\frac{k}{m}\right)=\sum_{j=0}^{n}\sum_{v=1}^{m-1}\binom{n}{j}v^{n-j}m^{j}B_{j,\alpha}\left(x\right).\tag{9}$$

Proof. Multiply $\sum_{k=0}^{m-1} B_{n,\alpha}\left(x+\frac{k}{m}\right)$ by $\frac{(mt)^n}{n!}$ and sum over all $n \geq 0$, we get

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m-1} B_{n,\alpha}\left(x+\frac{k}{m}\right)\right)\frac{(mt)^n}{n!} = \sum_{k=o}^{m-1}\left(\sum_{n=0}^{\infty} B_{n,\alpha}\left(x+\frac{k}{m}\right)\frac{(mt)^n}{n!}\right).$$

By using (6), we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m-1} B_{n,\alpha}\left(x+\frac{k}{m}\right)\right)\frac{(mt)^n}{n!} = \sum_{k=0}^{m-1}\frac{e^{\left(x+\frac{k}{m}-\frac{1}{2}\right)mt}}{g_\alpha\left(\frac{imt}{2}\right)}.$$

After some elementary calculations in the above, we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m-1} B_{n,\alpha}\left(x+\frac{k}{m}\right)\right)\frac{(mt)^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\frac{B_{j,\alpha}(x)\left(B_{n+1-j}(m)-B_{n+1-j}\right)m^j}{n+1-j}\right)\frac{t^n}{n!}.$$

Combining the above equation with the following well-known formula for

$$\frac{B_{n+1-j}(m)-B_{n+1-j}}{n+1-j} = \sum_{v=1}^{m-1} v^{n-j}$$

we get

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m-1} B_{n,\alpha}\left(x+\frac{k}{m}\right)\right)\frac{(mt)^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{v=1}^{m-1}\binom{n}{j} v^{n-j} m^j B_{j,\alpha}(x)\right)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ both sides of the above, we arrive at the desired result. □

Theorem 2.6. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_{n,\alpha}(x+1) - B_{n,\alpha}(x-1) = \sum_{k=0}^{n}\binom{n}{k}\left(1-(-1)^{n-k}\right)B_{k,\alpha}(x).$$

Proof. We set

$$f(z) = \frac{e^{\left(x-\frac{1}{2}\right)z}\sinh z}{g_\alpha\left(\frac{iz}{2}\right)}. \tag{10}$$

Therefore

$$f(z) = \frac{1}{2}\left(\frac{e^{\left(x+1-\frac{1}{2}\right)z}}{g_\alpha\left(\frac{iz}{2}\right)} - \frac{e^{\left(x-1-\frac{1}{2}\right)z}}{g_\alpha\left(\frac{iz}{2}\right)}\right).$$

Combining (6) with the above equation, we get the following series for the function $f(z)$:

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{B_{n,\alpha}(x+1) - B_{n,\alpha}(x-1)}{2}\right)\frac{z^n}{n!}. \tag{11}$$

Combining (10) with (11), we obtain

$$f(z) = \left(\sum_{n=0}^{\infty} B_{n,\alpha}(x) \frac{z^n}{n!}\right)\left(\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{n!} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{\left(1-(-1)^{n-k}\right)B_{k,\alpha}(x)}{2}\right)\frac{z^n}{n!}. \tag{12}$$

By (11) and (12), we get

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{\left(1-(-1)^{n-k}\right)B_{k,\alpha}(x)}{2}\right)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{B_{n,\alpha}(x+1)-B_{n,\alpha}(x-1)}{2}\right)\frac{z^n}{n!}.$$

By comparing the coefficient $\frac{z^n}{n!}$ in the both sides of the above equation, we arrive at the desired result. $\square$

Note that proofs of the following identities and relations can be given along the same lines as the proof of Theorem 2.6. We now give just sketch of these proofs as follows.

Corollary 2.7. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_{n,\alpha}(x+1) + B_{n,\alpha}(x-1) = \sum_{k=0}^{n}\binom{n}{k}\left(1+(-1)^{n-k}\right)B_{k,\alpha}(x).$$

Proof. We define

$$f(z) = \frac{e^{\left(x-\frac{1}{2}\right)z}\cosh z}{g_\alpha\left(\frac{iz}{2}\right)}.$$

By the same calculation of Theorem 2.6, we easily arrive at the desire result. $\square$

Corollary 2.8. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\left(B_{k,\alpha}(x+1)+B_{k,\alpha}(x-1)\right)B_{n-k}\left(\frac{1}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}\left(1+(-1)^{n-k}\right)B_{k,\alpha}(x)B_{n-k}.$$

Proof. We define

$$f(z) = \frac{2ze^{\left(x-\frac{1}{2}\right)z}\coth z}{g_\alpha\left(\frac{iz}{2}\right)}.$$

By the same calculation of Theorem 2.6, we easily arrive at the desired result. $\square$

Corollary 2.9. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}2^{n-k}\left(1+(-1)^{k-m}\right)B_{m,\alpha}(x)\,B_{n-k}\left(\frac{1}{2}\right)$$
$$=\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\left(1+(-1)^{n-k}\right)B_{k,\alpha}(x)\,B_{n-k}.$$

Proof. By combining Corollary 2.7 with Corollary 2.8, we obtain the desired result. □

Corollary 2.10. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\binom{n}{k}\left(B_{k,\alpha}(x+1)-B_{k,\alpha}(x-1)\right)E_{n-k}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\frac{2^{k+1}\left(2^{k+1}-1\right)\left((-1)^{k+1}+1\right)B_{n-k,\alpha}(x)\,B_{k+1}}{k+1}.$$

Proof. We define

$$f(z)=\frac{2e^{\left(x-\frac{1}{2}\right)z}\tanh z}{g_{\alpha}\left(\frac{iz}{2}\right)}.$$

By the same calculation of Theorem 2.6, we easily arrive at the desire result. □

Corollary 2.11. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\left(1-(-1)^{k-m}\right)B_{m,\alpha}(x)\,E_{n-k}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\frac{2^{k+1}\left(2^{k+1}-1\right)\left((-1)^{k+1}+1\right)B_{n-k,\alpha}(x)\,B_{k+1}}{k+1}.$$

Proof. By combining Theorem 2.6 with Corollary 2.10, we obtain the desired result. □

We now give symmetry relation for the Bernoulli-type numbers and polynomials by the next theorem.

Theorem 2.12. *Let $n \in \mathbb{N}_0$. For each pair of integer a and b, we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}B_{n-k,\alpha}(ay)\sum_{j=0}^{ab-1}B_{k,\alpha}\left(bx+\frac{j}{a}\right)$$
$$=\sum_{k=0}^{n}\binom{n}{k}b^{k}a^{n-k}B_{n-k,\alpha}(by)\sum_{j=0}^{ab-1}B_{k,\alpha}\left(ax+\frac{j}{b}\right).$$

Proof. Setting

$$f(z;a,b)=\frac{e^{\left(ab(x+y)-\frac{a+b}{2}\right)z}\left(e^{abz}-1\right)}{g_{\alpha}\left(\frac{iaz}{2}\right)g_{\alpha}\left(\frac{ibz}{2}\right)(e^{z}-1)}.$$

This function is symmetric with respect to a and b. From this function, we obtain the following relation

$$\begin{aligned}
f(z;a,b) &= \left(\frac{e^{\left(abxz-\frac{az}{2}\right)}}{g_{\alpha}\left(\frac{iaz}{2}\right)}\frac{\left(e^{abz}-1\right)}{(e^{z}-1)}\right)\frac{e^{\left(abyz-\frac{bz}{2}\right)}}{g_{\alpha}\left(\frac{ibz}{2}\right)} \\
&= \left(\sum_{j=0}^{ab-1}\frac{e^{\left(bx+\frac{j}{a}-\frac{1}{2}\right)az}}{g_{\alpha}\left(\frac{iaz}{2}\right)}\right)\left(\sum_{n=0}^{\infty}B_{n,\alpha}(ay)\frac{(bz)^{n}}{n!}\right) \\
&= \left(\sum_{j=0}^{ab-1}\left(\sum_{n=0}^{\infty}B_{n,\alpha}\left(bx+\frac{j}{a}\right)\frac{(az)^{n}}{n!}\right)\right)\left(\sum_{n=0}^{\infty}B_{n,\alpha}(ay)\frac{(bz)^{n}}{n!}\right).
\end{aligned}$$

Thus

$$f(z;a,b)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}B_{n-k,\alpha}(ay)\sum_{j=0}^{ab-1}B_{k,\alpha}\left(bx+\frac{j}{a}\right)\right)\frac{z^{n}}{n!}. \tag{13}$$

Therefore

$$f(z;a,b)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}b^{k}a^{n-k}B_{n-k,\alpha}(by)\sum_{j=0}^{ab-1}B_{k,\alpha}\left(ax+\frac{j}{b}\right)\right)\frac{z^{n}}{n!}. \tag{14}$$

Combining (13) with (14), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}B_{n-k,\alpha}(ay)\sum_{j=0}^{ab-1}B_{k,\alpha}\left(bx+\frac{j}{a}\right)\right)\frac{z^{n}}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}b^{k}a^{n-k}B_{n-k,\alpha}(by)\sum_{j=0}^{ab-1}B_{k,\alpha}\left(ax+\frac{j}{b}\right)\right)\frac{z^{n}}{n!}.
\end{aligned}$$

By comparing the coefficients of $\frac{z^{n}}{n!}$ both sides of the above, we arrive at the desired result. □

By substituting $\alpha=\frac{1}{2}$ into Theorem 2.12, we have the following results:

Corollary 2.13.

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}B_{n-k}(ay)\sum_{j=0}^{ab-1}B_{k}\left(bx+\frac{j}{a}\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}b^{k}a^{n-k}B_{n-k}(by)\sum_{j=0}^{ab-1}B_{k}\left(ax+\frac{j}{b}\right).
\end{aligned} \tag{15}$$

By substituting $\alpha = -\frac{1}{2}$ into Theorem 2.12, we have the following symmetry property for the Euler polynomials:

Corollary 2.14.

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} E_{n-k}(ay) \sum_{j=0}^{ab-1} E_k\left(bx + \frac{j}{a}\right) = \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} E_{n-k}(by) \sum_{j=0}^{ab-1} E_k\left(ax + \frac{j}{b}\right). \tag{16}$$

There are different proofs of the equations (15) and (16) (*cf.* [8, 9, 17]). The above relations given in (15) and (16) are symmetric with respect to a and b. These relations are also related to summation of powers of integers, respectively:

$$\sigma_m(k) = 1^m + 2^m + \cdots + a^m$$

and

$$\sigma_m^*(k) = 1^m - 2^m + - \cdots + a^m,$$

where $m \in \mathbb{N}_0$.

The summation $\sigma_m(k)$ has been studied by many authors. For example, Johann Faulhaber (1580-1635) and Jacob Bernoulli (1654-1705). After Bernoulli, there are many papers and books related to the sums $\sigma_m(k)$ and $\sigma_m^*(k)$ (*cf.* [1–21]).

Theorem 2.15. *Let $n \in \mathbb{N}$. Then we have*

$$6\sum_{k=0}^{n-1}\left(n\binom{n-1}{k} - 2\binom{n}{k}\right) B_{k,\frac{3}{2}}(x) = n(n-1)(n-2)x^{n-3} - 6nB_{n-1,\frac{3}{2}}(x).$$

Proof. Substituting $\alpha = \frac{3}{2}$ into (6), we have

$$(6(z-2)e^z + 6z + 12)\sum_{n=0}^{\infty} B_{n,\frac{3}{2}}(x)\frac{z^n}{n!} = \frac{d^3}{dx^3}\{e^{zx}\}$$

see also [2]. After some calculations in the above equation, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} n(n-1)(n-2)x^{n-3}\frac{z^n}{n!} \\
=& -12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B_{k,\frac{3}{2}}(x)\frac{z^n}{n!} + 6\sum_{n=0}^{\infty} n\sum_{k=0}^{n-1}\binom{n-1}{k}B_{k,\frac{3}{2}}(x)\frac{z^n}{n!} \\
&+ 12\sum_{n=0}^{\infty} B_{n,\frac{3}{2}}(x)\frac{z^n}{n!} + 6\sum_{n=0}^{\infty} nB_{n-1,\frac{3}{2}}(x)\frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ both sides of the above, we arrive at the desire result. $\square$

3. Formulas derive from logarithmic function and the function $g_\alpha(z)$

Relations between Bernoulli numbers, Euler numbers, logarithmic function, trigonometric functions are given in Berndt [1] in detailed.

In this section, we give relations between logarithms of functions involving cosine function and cosecant function and Euler and Bernoulli numbers. Our method is similar to that of in Berndt [1, Entry 12 and Entry 16 in Chapter 5].

Theorem 3.1. *Let $n \in \mathbb{N}$, and $|z| < \frac{\pi}{2}$. Then we have*

$$\ln\left(\frac{g_{-\frac{1}{2}}(1)}{g_{-\frac{1}{2}}(z)}\right) = \sum_{n=0}^{\infty} \frac{2^{2n+1}(-1)^{n+1}E_{2n+1}}{(2n+2)!}(z^{2n+2}-1).$$

Proof. Our proof is same as that of Berndt [1]. Let $|z| < \frac{\pi}{2}$, we define

$$\ln\left(\frac{z}{\cos z}\right) - \ln\left(\frac{1}{\cos 1}\right) = \int_1^z \frac{\left(\frac{t}{\cos t}\right)'}{\frac{t}{\cos t}} dt = \int_1^z \frac{dt}{t} + \int_1^z (\tan t)\, dt,$$

where

$$(f(t))' = \frac{d}{dt}\{f(t)\}.$$

By using the series expansion of the function $\tan t$ in the above equation, after some elementary calculations, we obtain

$$\ln\left(\frac{\cos 1}{\cos z}\right) = \sum_{n=0}^{\infty} \frac{2^{2n+1}(-1)^{n+1}E_{2n+1}}{(2n+1)!}\left(\frac{z^{2n+2}-1}{2n+2}\right). \tag{17}$$

Besides that, when we put $\alpha = -\frac{1}{2}$ into (5), we obtain

$$g_{-\frac{1}{2}}(z) = \cos z. \tag{18}$$

Combining (17) with (18), we arrive at the desired result. $\square$

Theorem 3.2. *Let $n \in \mathbb{N}$, and $0 < |z| < \pi$. Then we have*

$$\ln\left(\frac{g_{\frac{1}{2}}(1)}{z g_{\frac{1}{2}}(z)}\right) = \sum_{n=0}^{\infty} \frac{2^{2n}(-1)^{n+1}B_{2n}}{2n\,(2n)!}(z^{2n}-1).$$

Proof. Our proof is same as that of Berndt [1]. Let $0 < |z| < \pi$, we define

$$\ln(\csc z) - \ln(\csc 1) = \int_1^z \frac{\left(\frac{1}{\sin t}\right)'}{\frac{1}{\sin t}} dt = -\int_1^z (\cot t)\, dt.$$

By using Taylor series of $\cot t$ in the above, after some elementary calculations, we obtain

$$\ln\left(\frac{\csc z}{\csc 1}\right) = \sum_{n=0}^{\infty} \frac{2^{2n}(-1)^{n+1}B_{2n}}{(2n)!}\left(\frac{z^{2n}-1}{2n}\right). \tag{19}$$

On the other hand, when we put $\alpha = \frac{1}{2}$ into (5), we obtain

$$g_{\frac{1}{2}}(z) = \frac{\sin z}{z}. \tag{20}$$

□

Theorem 3.3. *Let $n \in \mathbb{N}$, and $|z| < \frac{\pi}{2}$. Then we have*

$$\ln\left(g_{-\frac{1}{2}}\left(\frac{iz}{2}\right)\right) = \sum_{n=1}^{\infty} \frac{\left(2^{2n}-1\right) B_{2n}}{2n\,(2n)!} z^{2n}.$$

Proof. Our proof is same as that of Berndt [1]. Let $|z| < \frac{\pi}{2}$, we define

$$\ln(\cosh z) = \int_0^z \frac{(\cosh t)'}{\cosh t} dt = \int_0^z (\tanh t)\, dt.$$

By using Taylor series of $\tanh t$ in the above, after some elementary calculations, we obtain

$$\ln(\cosh z) = \sum_{n=0}^{\infty} \frac{\left(2^{2n}-1\right) 2^{2n} B_{2n}}{2n\,(2n)!} z^{2n}.$$

When we put (18) into the above, we arrive at the desire result. □

Theorem 3.4. *Let $n \in \mathbb{N}$, and $|z| < \frac{\pi}{2}$. Then we have*

$$\ln\left(\frac{z}{2} g_{\frac{1}{2}}\left(\frac{iz}{2}\right)\right) = \sum_{n=0}^{\infty} \frac{B_{2n}}{2n\,(2n)!} z^{2n}.$$

Proof. Our proof is same as that of Berndt [1]. Let $|z| < \frac{\pi}{2}$, we define

$$\ln(\sinh z) = \int_0^z \frac{(\sinh t)'}{\sinh t} dt = \int_0^z (\coth t)\, dt.$$

By using Taylor series of $\coth t$ in the above, after some elementary calculations, we obtain

$$\ln(\sinh z) = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{2n\,(2n)!} z^{2n}.$$

Substituting (20) into the above equation, after some elementary calculations, we arrive at the desire result. □

3.1. Formulas derive from the function $\frac{1}{g_\alpha(z)}$

Faà di Bruno's formula gives an explicit formula for the mth derivative of the composition $g(f(t))$. This formula is given by the following theorem.

Theorem 3.5 ([5], Faà di Bruno's formula)**.** *If f and g are functions with a sufficient number of derivatives, then*

$$\frac{d^m}{dt^m} g(f(t)) = \sum \frac{m!}{b_1! \cdots b_m!} g^{(k)}(f(t)) \prod_{j=1}^{m} \left(\frac{f^{(j)}(t)}{j!}\right)^{b_j}, \tag{21}$$

where the sum is over all different solutions in nonnegative integers $b_1, \dots, b_m$ of $b_1 + 2b_2 + \cdots + mb_m = m$, and $k = b_1 + \cdots + b_m$.

By using (6) and (21), Frappier [3] proved the following expression for the α-Bernoulli type polynomials:

$$B_{n,\alpha}(x) = \left(x-\frac{1}{2}\right)^n + \sum_{s=1}^{\left[\frac{n}{2}\right]} \frac{(-1)^s \Gamma(2s+1)\binom{n}{2s} P_{N(s)}(\alpha)}{2^{4s}\Gamma(s+1)\prod_{v=1}^{s}(\alpha+v)^{\left[\frac{s}{v}\right]}}, \tag{22}$$

where

$$P_{N(s)}(\alpha) = \prod_{v=1}^{s}(\alpha+v)^{\left[\frac{s}{v}\right]} \sum_{r=1}^{s}\sum_{\pi(s,r)} (-1)^{r+s}(r!)\, c(k_1,\ldots,k_s) \prod_{v=1}^{s}\left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+v+1)}\right)^{k_v}$$

is a polynomial in α of degree $N(1)=0$ and

$$N(s) = \sum_{j=1}^{s}\left[\frac{s}{j}\right],$$

$s = 2, 3, \ldots$. The notation $\pi(s,r)$ means that the summation is extended over all nonnegative integers $k_1, k_2, \ldots, k_s$ such that $k_1 + 2k_2 + \cdots + sk_s = s$, $k_1 + k_2 + \cdots + k_s = r$, and

$$c(k_1,k_2,\ldots,k_s) := \left(\frac{s!}{k_1!k_2!\cdots k_s!\,(1!)^{k_1}\cdots(s!)^{k_s}}\right).$$

The polynomials $P_{N(s)}(\alpha)$ appear in the MacLaurin expansion of the function $\frac{1}{g_\alpha(z)}$, that is,

$$\frac{1}{g_\alpha(z)} = 1 + \sum_{s=1}^{\infty} \frac{P_{N(s)}(\alpha) z^{2s}}{2^{2s}\Gamma(s+1)\prod_{v=1}^{s}(\alpha+v)^{\left[\frac{s}{v}\right]}}. \tag{23}$$

For instances, $P_0(\alpha)=1$, $P_1(\alpha)=\alpha+3$, $P_2(\alpha)=\alpha^2+8\alpha+19$ and $P_4(\alpha) = \alpha^4+17\alpha^3+117\alpha^2+379\alpha+422$.

By using (23) and (6), in the next theorem, we give an application which is related to the polynomial $P_{N(s)}(\alpha)$ and the α-Bernoulli type polynomials.

Theorem 3.6. *Let s be a positive integer. Then we have*

$$\begin{aligned} B_{s,\alpha}(x) = {} & \frac{P_{N\left(\left[\frac{s}{2}\right]\right)}(\alpha)\left(1+(-1)^s\right)\Gamma(s+1)}{2^{\left(2\left[\frac{s}{2}\right]+1\right)}\Gamma\left(\left[\frac{s}{2}\right]+1\right)\prod_{v=1}^{\left[\frac{s}{2}\right]}(\alpha+v)^{\left[\frac{s}{2v}\right]}} + \left(x-\frac{1}{2}\right)^s \\ & + \sum_{k=1}^{s} \frac{P_{N\left(\left[\frac{k}{2}\right]\right)}(\alpha)\Gamma(s+1)\left(1+(-1)^k\right)\left(x-\frac{1}{2}\right)^{s-k}}{2^{\left(2\left[\frac{k}{2}\right]+1\right)}\Gamma\left(\left[\frac{k}{2}\right]+1\right)\Gamma(s-k+1)\prod_{v=1}^{\left[\frac{k}{2}\right]}(\alpha+v)^{\left[\frac{k}{2v}\right]}}. \end{aligned} \tag{24}$$

Proof. We modified (23) as follows:

$$\frac{1}{g_{\alpha}(z)}-1=\sum_{s=1}^{\infty}\frac{P_{N\left(\left[\frac{s}{2}\right]\right)}(\alpha)\left(1+(-1)^{s}\right)z^{s}}{2^{\left(2\left[\frac{s}{2}\right]+1\right)}\Gamma\left(\left[\frac{s}{2}\right]+1\right)\prod\limits_{v=1}^{\left[\frac{s}{2}\right]}(\alpha+v)^{\left[\frac{s}{2v}\right]}}. \tag{25}$$

By (25) and the function $e^{\left(x-\frac{1}{2}\right)z}-1$, we define the following relation

$$\left(\frac{1}{g_{\alpha}(z)}-1\right)\left(e^{\left(x-\frac{1}{2}\right)z}-1\right)=\frac{e^{\left(x-\frac{1}{2}\right)z}}{g_{\alpha}(z)}-e^{\left(x-\frac{1}{2}\right)z}-\frac{1}{g_{\alpha}(z)}+1.$$

After some elementary calculations and using (6) in the above equation, we obtain

$$\left(\sum_{s=1}^{\infty}\frac{P_{N\left(\left[\frac{s}{2}\right]\right)}(\alpha)\left(1+(-1)^{s}\right)z^{s}}{2^{\left(2\left[\frac{s}{2}\right]+1\right)}\Gamma\left(\left[\frac{s}{2}\right]+1\right)\prod\limits_{v=1}^{\left[\frac{s}{2}\right]}(\alpha+v)^{\left[\frac{s}{2v}\right]}}\right)\left(\sum_{s=1}^{\infty}\left(x-\frac{1}{2}\right)^{s}\frac{z^{s}}{s!}\right)$$
$$=\sum_{s=1}^{\infty}\frac{B_{s,\alpha}(x)z^{s}}{s!}-\sum_{s=1}^{\infty}\left(x-\frac{1}{2}\right)^{s}\frac{z^{s}}{s!}-\sum_{s=1}^{\infty}\frac{P_{N\left(\left[\frac{s}{2}\right]\right)}(\alpha)\left(1+(-1)^{s}\right)z^{s}}{2^{2\left[\frac{s}{2}\right]+1}\left[\frac{s}{2}\right]!\prod\limits_{v=1}^{\left[\frac{s}{2}\right]}(\alpha+v)^{\left[\frac{s}{2v}\right]}}.$$

By using Cauchy product in the left side of the above equation yields

$$\sum_{s=1}^{\infty}\left(\sum_{k=1}^{s}\frac{P_{N\left(\left[\frac{k}{2}\right]\right)}(\alpha)\left(1+(-1)^{k}\right)\left(x-\frac{1}{2}\right)^{s-k}}{2^{\left(2\left[\frac{k}{2}\right]+1\right)}\Gamma\left(\left[\frac{k}{2}\right]+1\right)\Gamma(s-k+1)\prod\limits_{v=1}^{\left[\frac{k}{2}\right]}(\alpha+v)^{\left[\frac{k}{2v}\right]}}\right)z^{s}$$
$$=\sum_{s=1}^{\infty}\left(\frac{B_{s,\alpha}(x)-\left(x-\frac{1}{2}\right)^{s}}{\Gamma(s+1)}-\frac{P_{N\left(\left[\frac{s}{2}\right]\right)}(\alpha)\left(1+(-1)^{s}\right)z^{s}}{2^{\left(2\left[\frac{s}{2}\right]+1\right)}\Gamma\left(\left[\frac{s}{2}\right]+1\right)\prod\limits_{v=1}^{\left[\frac{s}{2}\right]}(\alpha+v)^{\left[\frac{s}{2v}\right]}}\right)z^{s}.$$

By comparing the coefficients of z^s both sides of the above, we arrive at the desired result. □

Remark 3.7. Proof of (24) is different from that of (22). Proof of (22) is related to the Faa di Bruno formula.

References

[1] B. C. Berndt, *Ramanujan's notebooks. Part I*, Springer, New York, 1989.

[2] C. Frappier, *Representation formulas for entire functions of exponential type and generalized Bernoulli polynomials*, J. Austral. Math. Soc. Ser. A **64** (1998), no. 3, 307–316.

[3] C. Frappier, *Generalised Bernoulli polynomials and series*, Bull. Austral. Math. Soc. **61** (2000), no. 2, 289–304. `https://doi.org/10.1017/S0004972700022292`

[4] C. Frappier, *A unified calculus using the generalized Bernoulli polynomials*, J. Approx. Theory **109** (2001), no. 2, 279–313. `https://doi.org/10.1006/jath.2000.3550`

[5] W. P. Johnson, *The curious history of Faà di Bruno's formula*, Amer. Math. Monthly **109** (2002), no. 3, 217–234. `https://doi.org/10.2307/2695352`

[6] N. Kilar and Y. Simsek, *Formulas involving sums of powers, special numbers and polynomials arising from p-adic integrals, trigonometric and generating functions*, Publ. Inst. Math. (Beograd) (N.S.) **108(122)** (2020), 103–120. `https://doi.org/10.2298/pim2022103k`

[7] T. Kim, *Sums of powers of consecutive q-integers*, Adv. Stud. Contemp. Math. (Kyungshang) **9** (2004), no. 1, 15–18.

[8] T. Kim, *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Difference Equ. Appl. **14** (2008), no. 12, 1267–1277. `https://doi.org/10.1080/10236190801943220`

[9] T. Kim, *Note on the Euler numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **17** (2008), no. 2, 131–136.

[10] T. Kim, D. S. Kim, and J. Kwon, *Analogues of Faulhaber's formula for poly-Bernoulli and type 2 poly-Bernoulli polynomials*, Montes Taurus J. Pure Appl. Math. **3** (2021), no. 1, 1–6.

[11] T. Kim, S.-H. Rim, and Y. Simsek, *A note on the alternating sums of powers of consecutive q-integers*, Adv. Stud. Contemp. Math. (Kyungshang) **13** (2006), no. 2, 159–164.

[12] I. Küçükoğlu and Y. Simsek, *Identities and relations on the q-Apostol type Frobenius-Euler numbers and polynomials*, J. Korean Math. Soc. **56** (2019), no. 1, 265–284. `https://doi.org/10.4134/JKMS.j180185`

[13] D. J. Maširević, R. K. Parmar, and T. K. Pogány, *(p, q)-extended Bessel and modified Bessel functions of the first kind*, Results Math. **72** (2017), no. 1-2, 617–632. `https://doi.org/10.1007/s00025-016-0649-1`

[14] S. S. Ozbek, *Generating functions for Bernoulli numbers and polynomials*, Master Thesis, Akdeniz University 2009.

[15] T. K. Pogány and E. E. Süli, *Integral representation for Neumann series of Bessel functions*, Proc. Amer. Math. Soc. **137** (2009), no. 7, 2363–2368. `https://doi.org/10.1090/S0002-9939-09-09796-2`

[16] Y. Simsek, *Special functions related to Dedekind-type DC-sums and their applications*, Russ. J. Math. Phys. **17** (2010), no. 4, 495–508. `https://doi.org/10.1134/S1061920810040114`

[17] Y. Simsek, *Complete sum of products of (h, q)-extension of Euler polynomials and numbers*, J. Difference Equ. Appl. **16** (2010), no. 11, 1331–1348. `https://doi.org/10.1080/10236190902813967`

[18] Y. Simsek, *Explicit formulas for p-adic integrals*: *Approach to p-adic distributions and some families of special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **1** (2019), no. 1, 1–76.

[19] H. J. H. Tuenter, *A symmetry of power sum polynomials and Bernoulli numbers*, Amer. Math. Monthly **108** (2001), no. 3, 258–261. `https://doi.org/10.2307/2695389`

[20] T. Usmana, N. Khanb, M. Saifc, and J. Choi, *A unified family of Apostol-Bernoulli based poly-Daehee polynomials*, Montes Taurus J. Pure Appl. Math. **3** (2021), no. (3), 1–11.

[21] S. L. Yang, *An identity of symmetry for the Bernoulli polynomials*, Discrete Math. **308** (2008), no. 4, 550–554. `https://doi.org/10.1016/j.disc.2007.03.030`

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