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CIRCULAR SPECTRUM AND ASYMPTOTIC PERIODIC SOLUTIONS TO A CLASS OF NON-DENSELY DEFINED EVOLUTION EQUATIONS

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Abstract. In this paper, for the bounded solution of the non-densely defined non-autonomous evolution equation, we present the condition for asymptotic periodicity by using the circular spectral theory of functions on the half line and the extrapolation theory of non-densely defined evolution equation.

1. Introduction

Studying the periodicity of solutions is one of the great problems for the qualitative theory of evolution equations. The existence and uniqueness of periodic solutions have been proved for several important classes of densely defined evolution equations by using classical approaches such as the fixed point method [6, 16, 20], the use of ultimate boundedness of solutions and the compactness of Poincare map over compact embedding [1, 11, 18], the spectral theory of functions [14, 15, 17], ergodic approach [10]. As indicated in [3], we sometimes need to deal with non-densely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on $[0, \pi]$ and consider $\Delta = \frac{\partial^2}{\partial x^2}$ in $C([0, \pi], \mathbb{R})$, in order to measure the solutions in the sup-norm, then the domain

$$\mathcal{D}(\Delta) = \left\{u \in C^2([0, \pi], \mathbb{R}) : u(0) = u(\pi) = 0\right\}$$

is not dense in $C([0, \pi], \mathbb{R})$ with the sup-norm since

$$\overline{\mathcal{D}(\Delta)} = \{u \in C([0, \pi], \mathbb{R}) : u(0) = u(\pi) = 0\} \neq C([0, \pi], \mathbb{R}).$$

Many results on the existence and uniqueness of periodic solutions of non-densely defined evolution equations are obtained [2, 4, 6, 7]. Especially, in [5] K. Ezzinbi and M. Jazar gave a new criterion related to Massera's approach which

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is more general than the known exponential dichotomy for the existence of periodic and almost periodic solutions for some evolution equations in a Banach space of the form

$$(1)\qquad \begin{cases} \dfrac{d}{dt}x(t) = (A + B(t))x(t) + f(t) & \text{for } t \geq 0, \\ x(0) = x_0, \end{cases}$$

where $A : \mathcal{D}(A) \subset X \to X$ is a nondensely defined linear operator on a Banach space X which satisfies the Hille-Yosida condition:

($\mathbf{M}_1$): there exist $M_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $(\omega_0, +\infty) \subset \rho(A)$ and

$$|R(\xi, A)^n| \leq \frac{M_0}{(\xi - \omega_0)^n} \quad \text{for } n \in \mathbb{N} \text{ and } \xi > \omega_0,$$

where $\rho(A)$ is the resolvent set of A and $R(\xi, A) = (\xi - A)^{-1}$; the function $f : \mathbb{R}^+ \to X$ is bounded continuous; for every $t \geq 0, B(t)$ is a bounded linear operator on X.

Recently, Luong et al., in [12], studied the densely defined case of Eq. (1) when $A(t) := A + B(t)$ generates a 1-periodic strongly continuous evolutionary process $(U(t, s))_{t \geq s \geq 0}$ defined on the whole space X and f is asymptotic 1-periodic in the sense that f is bounded, continuous and $\lim_{t\to\infty}(f(t + 1) - f(t)) = 0$ (see e.g. [8] and its references). We recall that a function $x(\cdot)$ is an asymptotic solution to Eq. (1) if there is a continuous function $\epsilon(\cdot)$ such that $\lim\limits_{t\to\infty} \epsilon(t) = 0$ and

$$x'(t) = (A + B(t))x(t) + f(t) + \epsilon(t), \quad \forall t \geq 0.$$

By using the spectral theory of functions on the half line and the induced evolution semigroups in various spectral function spaces, Luong et al. [12] introduced a new condition for the unique existence bounded solution to be asymptotic 1-periodic on the half line. More precisely, they showed that a bounded and continuous function $g : \mathbb{R} \to X$ is asymptotic 1-periodic if and only if its *circular spectrum* $\sigma(g)$ (see [15] for more detail of this notion) satisfies $\sigma(g) \subset \{1\}$. Therefore, the existence of asymptotic 1-periodic solutions is reduced to that of solutions $x(\cdot)$ such that $\sigma(x(\cdot)) \subset \{1\}$. The search for asymptotic solutions $x(\cdot)$ with $\sigma(x(\cdot)) \subset \{1\}$ can be done by using the evolution semigroup associated with the homogeneous equations $x'(t) = A(t)x(t)$ in appropriate function spaces. In the case that the operator A is not densely defined, the linear part $A + B(t)$ does not generate a strongly continuous evolutionary process on the whole space X, so the results obtained in [12] are not guaranteed. Moreover, the inhomogeneous part $f(\cdot)$ takes value in the whole space X while the values of mild solution $x(\cdot)$ is exactly in $X_0 = \overline{\mathcal{D}(A)}$. To overcome these difficulties, in this paper we first use the theory of extrapolation spaces to express the mild solution of Eq. (1) in terms of an evolution process $(\mathcal{U}_B(t, s))_{t \geq s \geq 0}$ defined on closed subspace X_0 (see [7] and the references therein for more detail). Then, by using the periodicity and boundedness of $(\mathcal{U}_B(t,s))$ combined with the circular spectrum of functions we state the conditions for the unique bounded solution of (1) to be asymptotic periodic which fit the case of densely defined of non-autonomous linear part.

Before concluding this introduction section we give an outline of the paper. We briefly list the main notations in Section 2. This section also contains the definitions as well as properties of circular spectra of functions on the half line and extrapolation spaces. Section 3 contains the main result of the paper that deals with the asymptotic periodicity of solutions to non-densely defined nonautonomous evolution equations of the form (1).

2. Preliminaries

2.1. Notations

In this paper $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ stand for the real line, its positive half line, and the complex plane. If X denotes a (complex) Banach space, then $\mathcal{L}(X)$ stands for the space of all bounded linear operators in X. The spectrum of a linear operator T in a Banach space is denoted by $\sigma(T)$, and $\rho(T) := \mathbb{C}\backslash\sigma(T)$. We denote by $BC(\mathbb{R}^+, X)$ the space of all bounded continuous functions from $\mathbb{R}^+$ to a Banach space X with supremum norm, and $C_0(\mathbb{R}^+, X)$ is the space $\{g \in BC(\mathbb{R}^+, X) : \lim_{t\to\infty} g(t) = 0\}$. Finally, Γ will stand for the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

2.2. Circular spectra of functions on the half line

Many of the concepts and results in this subsection are discussed and proved in [12,13].

We consider the translation operator S in $BC(\mathbb{R}^+, X_0)$ defined as

$$[Sx](\xi) := x(1+\xi), \quad \xi \geq 0, \ x \in BC(\mathbb{R}^+, X_0).$$

Furthermore, we also consider the quotient spaces

$$Y := BC\left(\mathbb{R}^+, X_0\right)/C_0\left(\mathbb{R}^+, X_0\right).$$

Then, S induces operators in Y that will be denoted by $\bar{S}$. It is well known that $\bar{S}$ is an isometry, so $\sigma(\bar{S}) \subset \Gamma$.

For each $x \in BC(\mathbb{R}^+, X_0)$ let us consider the complex function $[\S x](\lambda)$ in $\lambda \in \mathbb{C}\backslash\Gamma$ defined as

$$[\S x](\lambda) := R(\lambda, \bar{S})\bar{x}, \quad \lambda \in \mathbb{C}\backslash\Gamma.$$

Definition ([13])**.** The circular spectrum of a function $x \in BC(\mathbb{R}^+, X_0)$ is defined to be the set of all $\xi_0 \in \Gamma$ such that $[\S x](\lambda)$ has no analytic extension into any neighborhood of ξ_0 in the complex plane. This spectrum of x is denoted by $\sigma(x)$. We will denote by $\rho(x)$ the set $\Gamma\backslash\sigma(x)$.

The following lemma justifies the introduction of these concepts of spectra.

Lemma 2.1 ([13]). *Let $x \in BC(\mathbb{R}^+, X_0)$. Then, for each $x \in BC(\mathbb{R}^+, X_0)$,*

$$\sigma(Qx) \subset \sigma(x)$$

provided that Q is an operator in $BC(\mathbb{R}^+, X_0)$ that commutes with S and leaves $C_0(\mathbb{R}^+, X_0)$ invariant.

2.3. Mild solutions and extrapolation spaces

It is well known that (see [7] and the references therein) the part A_0 of A in X_0 generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0 satisfying $\|T_0(t)\| \leq Me^{\omega t}$, $\forall t \geq 0$. Moreover, for $\lambda \in \rho(A_0)$ the resolvent $R(\lambda, A_0)$ is the restriction of $R(\lambda, A)$ to X_0. On X_0 we introduce the norm $\|x\|_{-1} = \|R(\lambda_0, A_0)x\|$, where $\lambda_0 \in \rho(A)$ is fixed. A different choice of $\lambda_0 \in \rho(A)$ leads to an equivalent norm. The completion X_{-1} of X_0 with respect to $\|\cdot\|_{-1}$ is called the extrapolation space of X_0 with respect to A. The extrapolated semigroup $(T_{-1}(t))_{t\geq 0}$ consists of the unique continuous extensions $T_{-1}(t)$ of the operators $T_0(t)$, $t \geq 0$, to X_{-1}. The semigroup $(T_{-1}(t))_{t\geq 0}$ is strongly continuous and its generator A_{-1} is the unique continuous extension of A_0 to $\mathcal{L}(X_0, X_{-1})$. Moreover, X is continuously embedded in X_{-1} and $R(\lambda, A_{-1})$ is the unique continuous extension of $R(\lambda, A)$ to X_{-1} for $\lambda \in \rho(A)$. Finally, A_0 and A are the parts of A_{-1} in X_0 and X, respectively.

We now give the definition of a mild solution of (1) as follows.

Definition. Let $x_0 \in X_0$. A function $x \in C(\mathbb{R}^+, X_0)$ is called a mild solution to (1) if it satisfies the integral equation

$$x(t) = T_0(t-s)x(s) + \int_s^t T_{-1}(t-h)(B(h)x(h) + f(h))dh$$

for all $t \geq s \geq 0$.

We consider the following homogeneous linear equation

$$\text{(2)} \qquad \begin{cases} \dfrac{dx}{dt} = (A + B(t))x(t), \ t \geq 0, \\ x(0) = x_0 \in X_0 \end{cases}$$

and assume that

($\mathbf{M}_2$): $t \mapsto B(t)x$ is strongly measurable for every $x \in X_0$,

($\mathbf{M}_3$): The operator $B(\cdot)$ is 1-periodic.

Proposition 2.2 ([7]). *Let ($\mathbf{M}_1$)-($\mathbf{M}_3$) be satisfied. Then, there exists a unique 1-periodic strongly continuous evolutionary process $(\mathcal{U}_B(t,s))_{t\geq s\geq 0}$ that satisfies*

(i) $\mathcal{U}_B(t,s) \in \mathcal{L}(X_0)$ *for all* $t \geq s \geq 0$;
(ii) $\mathcal{U}_B(t,t) = I$ *for every* $t \in \mathbb{R}$;
(iii) $\mathcal{U}_B(t,s)\mathcal{U}_B(s,r) = \mathcal{U}_B(t,r)$ *for all* $t \geq s \geq r$;
(iv) $\mathcal{U}_B(t+1, s+1) = \mathcal{U}_B(t,s)$ *for all* $t \geq s \geq 0$;
(v) *The function* $(t,s,x) \mapsto \mathcal{U}_B(t,s)x$ *is continuous in* (t,s,x);

(vi) *There are positive constants K, δ such that*

$$\|\mathcal{U}_B(t,s)\| \le K e^{\delta(t-s)} \quad \text{for all} \quad t \ge s \ge 0.$$

(vii) *Furthermore,*

$$\mathcal{U}_B(t,s)x = T_0(t-s)x + \int_s^t T_{-1}(t-h)B(h)\mathcal{U}_B(h,s)xdh, \ t \ge s \ge 0, \ x \in X_0,$$

i.e., $t \mapsto \mathcal{U}_B(t,0)x_0$ *is the unique solution of* (2).

Theorem 2.3 ([7]). *Let* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+, X)$ *and* $x_0 \in X_0$. *Then there is a unique mild solution* $x(\cdot) \in C(\mathbb{R}^+, X_0)$ *of Eq.* (1) *which satisfies the integral equation*

$$x(t) = \mathcal{U}_B(t,s)x(s) + \lim_{\xi\to\infty} \int_s^t \mathcal{U}_B(t,h)\xi R(\xi,A)f(h)dh \quad \text{for } t \ge s \ge 0.$$

Moreover, $\lim_{\xi\to\infty}\int_s^t \mathcal{U}_B(t,h)\xi R(\xi,A)f(h)dh \in X_0$ *exists uniformly for* $t \ge s$ *in compact sets in* $\mathbb{R}$.

3. Main results

3.1. Asymptotic periodic functions and their spectral characterization

We begin this subsection by recalling the concept of asymptotic periodic functions on the half line. It is noted that our definition of asymptotic periodicity is slightly different from the concept used in many previous works, and the period 1 is not a restriction, but just for the reader's convenience. All results can be easily stated for the general case of period.

Definition ([12]). A function $f \in BC(\mathbb{R}^+, X)$ is said to be asymptotic 1-periodic if

$$\lim_{t\to\infty} (f(t+1) - f(t)) = 0. \tag{3}$$

Remark 3.1. It is worth emphasizing that if f can be written in the form

$$f(t) = p(t) + q(t),$$

where p, q are continuous functions such that p is 1-periodic and $\lim_{t\to\infty} q(t) = 0$, then it satisfies (3) but the inversion is not true (see [12, Example 3.3]).

Proposition 3.2 ([12]). *The following assertions are valid*:

i) *Let* $x \in BC(\mathbb{R}^+, X_0)$. *Then,* $\sigma(x) = \emptyset$ *if and only if* $x \in C_0(\mathbb{R}^+, X_0)$;

ii) *Let* $p \in \mathbb{R}$ *and* $x \in BC(\mathbb{R}^+, X_0)$. *Then,* $\sigma(x) \subset \{e^{ip}\}$ *if and only if*

$$\lim_{t\to\infty} \left(x(t+1) - e^{ip}x(t)\right) = 0.$$

Lemma 3.3 ([12]). *Assume that* $Q(t)$, $t \in \mathbb{R}^+$ *is a family of bounded linear operators in* X_0 *that satisfies*

(i) *The function* $\mathbb{R}^+ \times X_0 \ni (t,x) \mapsto Q(t)x \in X_0$ *is continuous,*

(ii) $Q(t+1) = Q(t)$ *for all* $t \in \mathbb{R}^+$,

(iii) $\sup_{0\le t\le 1}\|Q(t)\|<\infty$.

Then, for each $x(\cdot)\in BC(\mathbb{R}^+,X_0)$ *we have*

$$\sigma(\mathcal{Q}x(\cdot))\subset\sigma(x(\cdot)),$$

where $\mathcal{Q}$ *denotes the operator in* $BC(\mathbb{R}^+,X_0)$ *defined as*

$$[\mathcal{Q}x(\cdot)](t):=Q(t)x(t),\ t\in\mathbb{R}^+.$$

3.2. Asymptotic periodic solution

Definition. A function $x(\cdot)\in BC(\mathbb{R}^+,X_0)$ is said to be an *asymptotic mild solution* of (1) if there exists a function $\epsilon(\cdot)\in C_0(\mathbb{R}^+,X)$ such that

$$x(t)=\mathcal{U}_B(t,s)x(s)+\lim_{\xi\to\infty}\int_s^t\mathcal{U}_B(t,h)\xi R(\xi,A)[f(h)+\epsilon(h)]dh$$

for all $t\ge s\ge 0$.

Now, for T is an operator in a Banach space X_0, we denote $\sigma_\Gamma(T):=\sigma(T)\cap\Gamma$. We also recall the following well known result on the spectrum of the "monodromy" operators

$$P(t):=\mathcal{U}_B(t+1,t)$$

for each $t\ge 0$. When $t=1$ we denote $P:=P(1)$. In particular, $P=\mathcal{U}_B(1,0)$ if $(\mathcal{U}_B(t,s))_{t\ge s\ge 0}$ is a 1-periodic process. Let us denote by $\mathcal{P}$ the operator of multiplication $u\mapsto\mathcal{P}u$ defined as

$$\mathcal{P}u(t)=P(t)u(t).$$

Lemma 3.4. *Let* $(\mathcal{U}_B(t,s))_{t\ge s\ge 0}$ *be a* 1*-periodic process in* X_0*. Then, for each* $t\ge 0$

$$\sigma(P(t))\backslash\{0\}=\sigma(P)\backslash\{0\}.$$

Proof. See [9, Lemma 7.2.2, p. 197]. □

The unique existence of an asymptotic mild solution of (1) is implied from Theorem 2.3, by the fact that $f\in BC(\mathbb{R}^+,X)\subset L^1_{loc}(\mathbb{R}^+,X)$. Now we prove the relation between the spectral of asymptotic mild solution x with spectral of P and f.

Lemma 3.5. *Let* $x(\cdot)\in BC(\mathbb{R}^+,X_0)$ *be an asymptotic mild solution of* (1) *and* $f\in BC(\mathbb{R}^+,X)$*. Then*

$$\sigma(x)\subset\sigma_\Gamma(P)\cup\sigma(f). \tag{4}$$

Proof. By the definition of asymptotic mild solutions there is a function $\epsilon(\cdot)\in C_0(\mathbb{R}^+,X)$ such that, for each $t\in\mathbb{R}^+$

$$x(t+1)=\mathcal{U}_B(t+1,t)x(t)+\lim_{\xi\to\infty}\int_t^{t+1}\mathcal{U}_B(t+1,h)\xi R(\xi,A)(f(h)+\epsilon(h))dh. \tag{5}$$

For $\xi>\omega$ we set $f_\xi=\xi R(\xi,A)f$. Note that $\sigma(f_\xi)\subset\sigma(f)$ and $f_\xi\in BC(\mathbb{R}^+,X_0)$.

Let us denote

$$F_\xi(t) := \int_t^{t+1} \mathcal{U}_B(t+1,h) f_\xi(h)dh.$$

Observe that the operator taking f_ξ to F_ξ commutes with S, and it is a bounded linear operator from $BC(\mathbb{R}^+, X_0)$ into itself, so by Lemma 3.3,

$$\sigma(F_\xi) \subset \sigma(f_\xi).$$

Moreover, $F_\xi \in BC(\mathbb{R}^+, X_0)$ and

$$F_\xi(t) \to F(t) := \lim_{\xi\to\infty} \int_t^{t+1} \mathcal{U}_B(t+1,h) f_\xi(h)dh \in X_0,$$

which shows that

$$\sigma(F) \subset \sigma(F_\xi) \subset \sigma(f_\xi) \subset \sigma(f).$$

Also, if we denote

$$\varepsilon(t) = \lim_{\xi\to\infty} \int_t^{t+1} \mathcal{U}_B(t+1,h)\xi R(\xi,A)\epsilon(h)dh,$$

then $\varepsilon(\cdot) \in C_0(\mathbb{R}^+, X_0)$. Hence, for the function

$$w(t) := \lim_{\xi\to\infty} \int_t^{t+1} \mathcal{U}_B(t+1,h)\xi R(\xi,A)(f(h)+\epsilon(h))dh = F(t) + \varepsilon(t)$$

we have

$$\sigma(w) = \sigma(F) \subset \sigma(f).$$

The periodicity of the evolution process $(\mathcal{U}_B(t,s))_{t\geq s}$ yields that $P(t)$ is 1-periodic, so it commutes with the translation S. Therefore, (5) gives

$$\bar{S}\bar{x} = \overline{\mathcal{P}}\bar{x} + \bar{F}.$$

Let $0 \neq \lambda_0 \notin \sigma_\Gamma(P) \cup \sigma(f)$ and V be a fixed small open neighborhood of λ_0 such that

$$V \cap (\sigma_\Gamma(P) \cup \sigma(f)) = \emptyset.$$

Using the identity

$$R(\lambda,\bar{S})\bar{S}\bar{x} = \lambda R(\lambda,\bar{S})\bar{x} - \bar{x} \quad \text{for } \lambda \in V, \quad |\lambda| \neq 1$$

we have

$$R(\lambda,\bar{S})(\overline{\mathcal{P}}\bar{x} + \bar{F}) = R(\lambda,\bar{S})\bar{S}\bar{x} = \lambda R(\lambda,\bar{S})\bar{x} - \bar{x}.$$

Together with the fact that $R(\lambda,\bar{S})\overline{\mathcal{P}}\bar{x} = \overline{\mathcal{P}}R(\lambda,\bar{S})\bar{x}$ we obtain

$$\begin{aligned}\bar{x} + R(\lambda,\bar{S})\bar{F} &= \lambda R(\lambda,\bar{S})\bar{x} - \overline{\mathcal{P}}R(\lambda,\bar{S})\bar{x} \\ &= (\lambda - \overline{\mathcal{P}})R(\lambda,\bar{S})\bar{x}.\end{aligned}$$

Since $\lambda \in V$, the operator $\lambda - \overline{\mathcal{P}}$ is invertible and its inverse is determined by $R(\lambda,\overline{\mathcal{P}})$. Therefore, for all $\lambda \in V$ such that $|\lambda| \neq 1$ we have

$$R(\lambda,\bar{S})\bar{x} = R(\lambda,\overline{\mathcal{P}})(\bar{x} + R(\lambda,\bar{S})\bar{F}).$$

Since $R(\lambda, \overline{\mathcal{P}})\bar{x}$ is analytic in V and $R(\lambda, \bar{S})\bar{F}$ is analytically extendable in a neighborhood of λ_0, the complex function $R(\lambda, \bar{S})\bar{x}$ is analytically extendable to a neighborhood of λ_0. That is $\lambda_0 \notin \sigma(x)$. This proves (4), completing the proof of the lemma. □

Theorem 3.6. *Let* $(\mathbf{M}_1)$-$(\mathbf{M}_3)$ *be satisfied. Let* $\sigma_\Gamma(P) \subset \{1\}$ *and* $x \in BC(\mathbb{R}^+, X_0)$ *be an asymptotic mild solution of* (1). *Furthermore, let* $f \in BC(\mathbb{R}^+, X)$ *in* (1) *be asymptotic* 1*-periodic. Then,* $x(\cdot)$ *is asymptotic* 1*-periodic, i.e.,*

$$\lim_{t\to\infty} (x(t+1) - x(t)) = 0.$$

Proof. Since f is asymptotic 1-periodic,

$$\sigma(f) \subset \{1\}.$$

By Lemma 3.5,

$$\sigma(x) \subset \sigma_\Gamma(P) \cup \sigma(f) \subset \{1\}.$$

Then, by Proposition 3.2 we conclude that $x(\cdot)$ is asymptotic 1-periodic. □

3.3. Example

To illustrate our results, we consider the following nondensely defined nonautonomous partial differential equation

$$(6) \quad \begin{cases} \dfrac{\partial x}{\partial t}(t,\zeta) = \dfrac{\partial^2}{\partial \zeta^2} x(t,u) - b(t)x(t,\zeta) + g(\zeta)\cdot \sin\sqrt{t} & \text{for } t \in \mathbb{R}_+,\ \zeta \in [0,\pi], \\ x(t,0) = x(t,\pi) = 0 \quad \text{for } t \in \mathbb{R}_+, \end{cases}$$

where $b(\cdot)$ is a 1-periodic function which satisfies $0 < \bar{b} < b(\cdot)$ and g is L^2-integrable on $[0,\pi]$.

We set $X := C([0,\pi],\mathbb{R})$, the Banach space of continuous functions on $[0,\pi]$, equipped with the uniform norm topology, and we define $A : \mathcal{D}(A) \subset X \to X$ by

$$\begin{cases} \mathcal{D}(A) = \left\{ z \in C^2([0,\pi],\mathbb{R}) : z(0) = z(\pi) = 0 \right\}, \\ Az = z''. \end{cases}$$

We have $(0,\infty) \subset \rho(A)$,

$$\|R(\lambda, A)\| \le \frac{1}{\lambda}, \quad \forall\, \lambda > 0,$$

and

$$X_0 := \overline{\mathcal{D}(A)} = \{ y \in C([0,\pi],\mathbb{R}) : y(0) = y(\pi) = 0\} \ne X.$$

Hence, $(\mathbf{M}_1)$ is satisfied. We will use the fact that A generates a strongly continuous exponentially semigroup $(T_0(t))_{t\ge 0}$ on X_0 with

$$\|T_0(t)\| \le e^{-t}, \quad \forall\, t \ge 0.$$

Moreover, as in [19, p. 414] the eigenvalues of A on $i\mathbb{R}$ are determined from the set of solutions of the equations

$$\lambda - 1 = -n^2,\ n = 1, 2, \ldots .$$

Obviously, there is only one root $\lambda = 0$ that lies on $i\mathbb{R}$, so $\sigma(A) \cap i\mathbb{R} = \{0\}$. Since this semigroup is compact, the spectral mapping theorem yields that $\sigma(T_0(1)) = e^{\sigma(A)} = \{1\}$.

We now consider the family $(B(t))_{t\geq 0}$ defined on X_0 by $B(t) = -b(t)I$ for every $t \geq 0$. Since $b(\cdot) \in L^1_{loc}(\mathbb{R}_+)$, $t \mapsto B(t)x$ is strongly measurable. Hence, $(\mathbf{M}_2)$ is satisfied. Clearly that $B(\cdot)$ is 1-periodic so $(\mathbf{M}_3)$ is fulfilled. We find that $A + B(t)$ generates a unique 1-periodic strongly continuous evolutionary process $(\mathcal{U}_B(t,s))_{t\geq s\geq 0}$ on X_0 defined by

$$\mathcal{U}_B(t,s) = \exp\left(-\int_s^t b(\tau)d\tau\right) T_0(t-s).$$

For the monodromy operator $P = \mathcal{U}_B(1,0) = \exp\left(-\int_0^1 b(\tau)d\tau\right) T_0(1)$ we have

$$\sigma_\Gamma(P) = \{1\}.$$

Furthermore, if we assume that $g \in X$, then the function $f(t) := \sin\sqrt{t} \cdot g(\cdot)$ is an asymptotic 1-periodic function taking values in X .

Therefore, by applying Theorem 3.6 we conclude that every asymptotic solution to (6) is asymptotic 1-periodic.

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References

[1] T. A. Burton, *Stability and periodic solutions of ordinary and functional-differential equations*, Mathematics in Science and Engineering, 178, Academic Press, Inc., Orlando, FL, 1985.

[2] C. Cuevas and M. Pinto, *Existence and uniqueness of pseudo almost periodic solutions of semilinear Cauchy problems with non dense domain*, Nonlinear Anal. **45** (2001), no. 1, Ser. A: Theory Methods, 73–83. https://doi.org/10.1016/S0362-546X(99)00330-2

[3] G. Da Prato and E. Sinestrari, *Differential operators with nondense domain*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), no. 2, 285–344 (1988).

[4] B. de Andrade and C. Cuevas, *S-asymptotically ω-periodic and asymptotically ω-periodic solutions to semi-linear Cauchy problems with non-dense domain*, Nonlinear Anal. **72** (2010), no. 6, 3190–3208. https://doi.org/10.1016/j.na.2009.12.016

[5] K. Ezzinbi and M. Jazar, *New criteria for the existence of periodic and almost periodic solutions for some evolution equations in Banach spaces*, Electron. J. Qual. Theory Differ. Equ. **2004** (2004) , No. 6, 12 pp. https://doi.org/10.14232/ejqtde.2004.1.6

[6] K. Ezzinbi and J. H. Liu, *Periodic solutions of non-densely defined delay evolution equations*, J. Appl. Math. Stoch. Anal. **15** (2002), no. 2, 113–123. https://doi.org/10.1155/S1048953302000114

[7] G. Gühring and F. Räbiger, *Asymptotic properties of mild solutions of nonautonomous evolution equations with applications to retarded differential equations*, Abstr. Appl. Anal. **4** (1999), no. 3, 169–194. https://doi.org/10.1155/S1085337599000214

[8] H. Henríquez, M. Pierri, and P. Táboas, *On S-asymptotically ω-periodic functions on Banach spaces and applications*, J. Math. Anal. Appl. **343** (2008), no. 2, 1119–1130. https://doi.org/10.1016/j.jmaa.2008.02.023

[9] D. B. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, 840, Springer, Berlin, 1981.

[10] N. T. Huy and Ngo Quy Dang, *Existence, uniqueness and conditional stability of periodic solutions to evolution equations*, J. Math. Anal. Appl. **433** (2016), no. 2, 1190–1203. https://doi.org/10.1016/j.jmaa.2015.07.059

[11] J. H. Liu, G. M. N'Guérékata, and N. V. Minh, *Topics on stability and periodicity in abstract differential equations*, Series on Concrete and Applicable Mathematics, 6, World Sci. Publ., Hackensack, NJ, 2008. https://doi.org/10.1142/9789812818249

[12] V. T. Luong, D. V. Loi, N. V. Minh, and H. Matsunaga, *A Massera theorem for asymptotic periodic solutions of periodic evolution equations*, J. Differential Equations **329** (2022), 371–394. https://doi.org/10.1016/j.jde.2022.05.010

[13] V. T. Luong, N. H. Tri, and N. V. Minh, *Asymptotic behavior of solutions of periodic linear partial functional differential equations on the half line*, J. Differential Equations **296** (2021), 1–14. https://doi.org/10.1016/j.jde.2021.05.053

[14] N. V. Minh, H. Matsunaga, N. D. Huy, and V. T. Luong, *A Katznelson-Tzafriri type theorem for difference equations and applications*, Proc. Amer. Math. Soc. **150** (2022), no. 3, 1105–1114. https://doi.org/10.1090/proc/15722

[15] N. V. Minh, G. M. N'Guérékata, and S. Siegmund, *Circular spectrum and bounded solutions of periodic evolution equations*, J. Differential Equations **246** (2009), no. 8, 3089–3108. https://doi.org/10.1016/j.jde.2009.02.014

[16] J. W. Prüss, *Periodic solutions of the thermostat problem*, in Differential equations in Banach spaces (Bologna, 1985), 216–226, Lecture Notes in Math., 1223, Springer, Berlin, 1986. https://doi.org/10.1007/BFb0099195

[17] W. M. Ruess and V. Q. Phóng, *Asymptotically almost periodic solutions of evolution equations in Banach spaces*, J. Differential Equations **122** (1995), no. 2, 282–301. https://doi.org/10.1006/jdeq.1995.1149

[18] J. B. Serrin Jr., *A note on the existence of periodic solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **3** (1959), 120–122. https://doi.org/10.1007/BF00284169

[19] C. C. Travis and G. F. Webb, *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc. **200** (1974), 395–418. https://doi.org/10.2307/1997265

[20] W. Zhang, D. Zhu, and P. Bi, *Existence of periodic solutions of a scalar functional differential equation via a fixed point theorem*, Math. Comput. Modelling **46** (2007), no. 5-6, 718–729. https://doi.org/10.1016/j.mcm.2006.12.026

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