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EQUALITY IN DEGREES OF COMPACTNESS: SCHAUDER'S THEOREM AND s-NUMBERS

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Abstract. We investigate an extension of Schauder's theorem by studying the relationship between various s-numbers of an operator T and its adjoint T^*. We have three main results. First, we present a new proof that the approximation number of T and T^* are equal for compact operators. Second, for non-compact, bounded linear operators from X to Y, we obtain a relationship between certain s-numbers of T and T^* under natural conditions on X and Y. Lastly, for non-compact operators that are compact with respect to certain approximation schemes, we prove results for comparing the degree of compactness of T with that of its adjoint T^*.

1. Introduction

In the following, we give a brief review of the background, notation, and terminology that will be relevant to this paper. Let $\mathcal{L}(X,Y)$ denote the normed vector space of all continuous operators from X to Y, X^* be the dual space of X, and $\mathcal{K}(X,Y)$ denote the collection of all compact operators from X to Y. Denote by $T^* \in \mathcal{L}(Y^*, X^*)$ the adjoint operator of $T \in \mathcal{L}(X,Y)$. The well known theorem of Schauder states that $T \in \mathcal{K}(X,Y)$ if and only if $T^* \in \mathcal{K}(Y^*, X^*)$. The proof of Schauder's theorem that uses Arzelà-Ascoli theorem is presented in most textbooks on functional analysis (see, e.g., [19]). A new and simple proof that does not depend on Arzelà-Ascoli can be found in [20]. Recalling the fact that a class of operators $\mathcal{A}(X,Y) \subset \mathcal{L}(X,Y)$ is called *symmetric* if $T \in \mathcal{A}(X,Y)$ implies $T^* \in \mathcal{A}(Y^*, X^*)$, we note that Schauder's theorem assures that the class $\mathcal{K}(X,Y)$ of compact operators between arbitrary Banach spaces X and Y is a symmetric operator ideal in $\mathcal{L}(X,Y)$.

In [18] F. Riesz proved compact operators have at most countable set of eigenvalues $\lambda_n(T)$, which arranged in a sequence, tend to zero. This result raises the question of what are the conditions on $T \in \mathcal{L}(X,Y)$ such that $(\lambda_n(T)) \in \ell_q$? Specifically, what is the rate of convergence to zero of the sequence $(\lambda_n(T))$? To answer these questions, in [15] and [17], A. Pietsch developed s-numbers

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$s_n(T)$ (closely related to singular values), which characterize the degree of compactness of T. The concept of *s-numbers* $s_n(T)$ is introduced axiomatically in [15], and their relationships to eigenvalues are given in detail in [17].

Definition. A map that assigns to every operator T a scalar sequence is said to be an s-function if the following conditions are satisfied:

(1) $||T|| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for $T \in \mathcal{L}(X,Y)$.
(2) $s_{m+n-1}(S+T) \leq s_m(T) + s_n(T)$ for $S, T \in \mathcal{L}(X,Y)$.
(3) $s_n(RTK) \leq ||R|| s_n(T) ||K||$ for $K \in \mathcal{L}(X_0, X)$, $T \in \mathcal{L}(X,Y)$, $R \in \mathcal{L}(Y, Y_o)$.
(4) If rank $(T) < n$, then $s_n(T) = 0$.
(5) $s_n(I_n) = 1$, where I_n is the identity map of ℓ_2^n.

We call $s_n(T)$ the n-th *s-number* of the operator T. Observe that $s_n(T)$ depends on T continuously since

$$|s_n(S) - s_n(T)| \leq ||S - T||.$$

In [15] it is shown that there is only one s-function on the class of all operators between Hilbert spaces. For example, if we let T be a diagonal operator acting on ℓ_2 such that

$$T(x_n) = (\lambda_n x_n), \quad \text{where } \lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \text{ then } s_n(T) = \lambda_n$$

for every s-function.

However, for Banach spaces, there are several possibilities of assigning to every operator $T : X \to Y$ a certain sequence of numbers $\{s_n(T)\}$ which characterizes the degree of approximability or compactness of T. The main examples of s-numbers to be used in this paper are approximation numbers, Kolmogorov numbers, Gelfand numbers and symmetrized approximation numbers which are all defined below.

First, for two arbitrary normed spaces X and Y, we define the collection of the finite-rank operators as follows:

$$\mathcal{F}_n(X,Y) = \{A \in \mathcal{L}(X,Y) : \text{rank}(A) \leq n\} \quad \text{and} \quad \mathcal{F}(X,Y) = \bigcup_{n=0}^{\infty} \mathcal{F}_n(X,Y)$$

which forms the smallest ideal of operators that exists.

Definition. In the following we define the s-numbers we will use.

(1) The *nth approximation number*

$$a_n(T) = \inf\{||T - A|| : A \in \mathcal{F}_n(X,Y)\}, \quad n = 0, 1, \ldots.$$

Note that $a_n(T)$ provides a measure of how well T can be approximated by finite mappings whose range is at most n-dimensional. It is clear that the sequence $\{a_n(T)\}$ is monotone decreasing and $\lim_{n\to\infty} a_n(T) = 0$ if and only if T is the limit of finite rank operators. It is known that the largest s-number is the approximation number. This is so because $a : S \to (a_n(S))$ is an s-function and if we consider $S \in \mathcal{L}(X, Y)$ and if $L \in \mathcal{F}(X, Y)$ with $\mathrm{rank}(L) < n$, then

$$s_n(S) \le s_n(L) + ||S - L|| = ||S - L||.$$

Therefore $s_n(S) \le a_n(S)$. See [7] or [15] for more details.

(2) The *nth Kolmogorov diameter* of $T \in \mathcal{L}(X)$ is defined by

$$\delta_n(T) = \inf\{||Q_G T|| : \dim G \le n\},$$

where the infimum is over all subspaces $G \subset X$ such that $\dim G \le n$ and Q_G denotes the canonical quotient map $Q_G : X \to X/G$.

(3) The *nth Gelfand number of* T, $c_n(T)$ is defined as:

$$c_n(T) = \inf\{\epsilon > 0 : ||Tx|| \le \sup_{1 \le i \le k} |\langle x, a_i \rangle| + \epsilon ||x||\},$$

where $a_i \in X^*$, $1 \le i \le k$ with $k < n$. It follows that an operator T is compact if and only if $c_n(T) \to 0$ as $n \to \infty$.

(4) The *nth symmetrized approximation number* $\tau_n(T)$ for any operator T defined between arbitrary Banach spaces X and Y is defined as follows:

$$\tau_n(T) = \delta_n(J_Y T), \quad \text{where} \quad J_Y : Y \to \ell_\infty(B_{Y^*})$$

is an embedding map. Note that above definition is equivalent to

$$\tau_n(T) = a_n(J_Y T Q_X)$$

as well as to

$$\tau_n(T) = c_n(T Q_X),$$

where $Q_X : \ell_1(B_X) \to X$ is a metric surjection onto X given by $Q_X(\xi_x) = \sum_{B_X} \xi_x x$ for $(\xi_x) \in \ell_1(B_X)$.

It is possible to compare various s-numbers such as $a_n(T)$, $\delta_n(T)$, $c_n(T)$ if one imposes some mild restrictions on X and Y. With this purpose in mind we define well known concepts of lifting and extension properties.

Definition. In the following we introduce two well-known important properties of Banach spaces. See [7] for details.

(1) We say that a Banach space X has *the lifting property* if for every $T \in \mathcal{L}(X, Y/F)$ and every $\epsilon > 0$ there exists an operator $S \in \mathcal{L}(X, Y)$ such that

$$||S|| \le (1 + \epsilon)||T|| \quad \text{and } T = Q_F S,$$

where F is a closed subspace of the Banach space Y and $Q_F : Y \to Y/F$ denotes the canonical projection.

Example 1.1. The Banach space $\ell_1(\Gamma)$ of *summable number families* $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ over an arbitrary index set Γ, whose elements $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ are characterized by $\sum_{\gamma \in \Gamma} |\lambda_\gamma| < \infty$, has the metric lifting property.

(2) A Banach space Y is said to have *the extension property* if for each $T \in \mathcal{L}(M, Y)$ there exists an operator $S \in \mathcal{L}(X, Y)$ such that $T = SJ_M$ and $||T|| = ||S||$, where M is a closed subspace of an arbitrary Banach space X and $J_M : M \to Y$ is the canonical injection.

Example 1.2. The Banach space $\ell_\infty(\Gamma)$ of *bounded number families* $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ over an arbitrary index set Γ has the metric extension property.

We mention a couple of facts to illustrate the importance of lifting and extensions properties with respect to s-numbers. If T is any map from a Banach space with metric lifting property to an arbitrary Banach space, then $a_n(T) = \delta_n(T)$ ([7], Prop. 2.2.3). It is also known that every Banach space X appears as a quotient space of an appropriate space $\ell_1(\Gamma)$ (see [7], p. 52). Furthermore, if T is any map from an arbitrary Banach space into a Banach space with metric extension property, then $a_n(T) = c_n(T)$ ([7], Prop. 2.3.3). Additionally, every Banach space Y can be regarded as a subspace of an appropriate space $\ell_\infty(\Gamma)$ (see [7], p. 60).

For non-compact operator $T \in \mathcal{L}(X, Y)$, we do not have too much information about the relationship between $s_n(T)$ with $s_n(T^*)$. In this paper, by imposing certain natural conditions on X and Y we are able to obtain a relationship between $s_n(T)$ with $s_n(T^*)$ for certain s-numbers. Moreover, using a new characterization of compactness due to Runde [20] together with the Principle of Local Reflexivity, we give a different, simpler proof of Hutton's theorem [10] establishing that for any compact map T,

$$a_n(T) = a_n(T^*) \quad \text{for all } n.$$

Next we consider operators which are not compact but compact with respect to certain approximation schemes Q. We call such operators as Q-compact and prove that for any Q-compact operator T, one has $\tau_n(T) = \tau_n(T^*)$. This result answers the question of comparing the degree of compactness for T and its adjoint T^* for non-compact operators T.

2. Comparing $s_n(T)$ and $s_n(T^*)$

Hutton in [10] used the Principle of Local Reflexivity (PLR) to prove that for $T \in \mathcal{K}(X, Y)$ we have

$$a_n(T) = a_n(T^*) \quad \text{for all } n.$$

This result fails for non-compact operators. For example, if $T = I : \ell_1 \to c_0$ is the canonical injection and $T^* : \ell_1 \to \ell_\infty$ is the natural injection, then one can show

$$1 = a_n(T) \neq a_n(T^*) = \frac{1}{2}.$$

On the other hand by considering the ball measure of non-compactness, namely,

$$\gamma(T) := \inf\{r > 0 : T(B_X) \subset \bigcup_{k=1}^{n} A_k,\ \max_{1\le k\le n} \text{diam } (A_k) < r,\ n \in \mathbb{N}\}.$$

Astala in [4] proved that if $T \in \mathcal{L}(X,Y)$, where X and Y are arbitrary Banach spaces with metric lifting and extension property, respectively, then

$$\gamma(T) = \gamma(T^*).$$

Our first result is a different, simpler proof of Hutton's theorem. We use only the characterization of compactness by Runde [20], together with the Principle of Local Reflexivity. Lindenstrass and Rosenthal [12] discovered a principle that shows that all Banach spaces are "locally reflexive" or said in another way, every bidual X^{**} is finitely representable in the original space X. The following is a stronger version of this property called *Principle of Local Reflexivity* (PLR) due to Johnson, Rosenthal and Zippin [11]:

Definition. Let X be a Banach space regarded as a subspace of X^{**}, let E and F be finite dimensional subspaces of X^{**} and X^*, respectively, and let $\epsilon > 0$. Then there exists a one-to-one operator $T : E \to X$ such that

(1) $T(x) = x$ for all $x \in X \cap E$,
(2) $f(Te) = e(f)$ for all $e \in E$ and $f \in F$,
(3) $||T||||T^{-1}|| < 1 + \epsilon$.

PLR is an effective tool in Banach space theory. For example Oja and Silja in [14] investigated versions of the principle of local reflexivity for nets of subspaces of a Banach space and gave some applications to duality and lifting theorems.

Lemma 2.1 (Lemma 1 in [20])**.** *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then $T \in \mathcal{K}(X)$ if and only if, for each $\epsilon > 0$, there is a finite-dimensional subspace F_ϵ of X such that $||Q_{F_\epsilon} T|| < \epsilon$, where $Q_{F_\epsilon} : X \to X/F_\epsilon$ is the canonical projection.*

Theorem 2.2. *Let $T \in \mathcal{K}(X)$. Then $a_n(T) = a_n(T^*)$ for all n.*

Proof. Since one always has $a_n(T^*) \le a_n(T)$, if we have $a_n(T) \le a_n(T^{**})$, then $a_n(T^{**}) \le a_n(T^*)$ would imply $a_n(T) \le a_n(T^*)$. Thus we must verify $a_n(T) \le a_n(T^{**})$. To this end, suppose $T \in \mathcal{K}(X)$, by Schauder's theorem, T^* and T^{**} are compact. Let $\epsilon > 0$, then by definition, there exists $A \in \mathcal{F}_n(X^{**})$ such that $||T^{**} - A|| < a_n(T^{**}) + \epsilon$. By Lemma 2.1, there are finite-dimensional subspaces E_ϵ of X^{**} and F_ϵ of X^* such that $||Q_{E_\epsilon} T^{**}|| < \epsilon$, where $Q_{E_\epsilon} : X^{**} \to X^{**}/E_\epsilon$ and $||Q_{F_\epsilon} T^*|| < \epsilon$, where $Q_{F_\epsilon} : X^* \to X^*/F_\epsilon$. By the Principle of Local Reflexivity (PLR), there exists a one-to-one linear operator $S : E_\epsilon \to X$ such that $||S||||S^{-1}|| < 1 + \epsilon$, $y^*(Sx^{**}) = x^{**}(y^*)$ for all $x^{**} \in E_\epsilon$ and all $y^* \in F_\epsilon$, and $S_{|E_\epsilon \cap X} = I$.

Let $J : X \to X^{**}$ be the canonical map. By the Hahn-Banach theorem, since E_ϵ is a subspace of X^{**}, $S : E_\epsilon \to X$ can be extended to a linear operator $\overline{S} : X^{**} \to X$. We now have $T \in \mathcal{L}(X)$ and $\overline{S}AJ \in \mathcal{L}(X)$ and rank $(\overline{S}AJ) = \text{rank}(A) < n$, and therefore

$$a_n(T) \leq ||T - \overline{S}AJ||.$$

To get an upper bound for $||T - \overline{S}AJ||$ we estimate $||Tx - \overline{S}AJ(x)||$ for $x \in B_X$ using an appropriate element z_j of the covering of the set $T(B_X)$. Indeed, the compactness of T implies that $T(B_X)$ is relatively compact so that one can extract a finite-dimensional subset $Y_\epsilon \subset T(B_X) \subset X$ and let $z_j = Tx_j$ be the n elements forming a basis. Let $x \in B_X$. Then we have

$$\begin{aligned} a_n(T) &\leq |Tx - \overline{S}AJ(x)|| \\ &\leq ||Tx - z_j|| + ||z_j - \overline{S}AJ(x)|| \\ &\leq \epsilon + ||z_j - \overline{S}AJ(x)|| = \epsilon + ||\overline{S}z_j - \overline{S}AJ(x)|| \\ &\leq \epsilon + (1+\epsilon)||z_j - AJ(x)|| \\ &< \epsilon + (1+\epsilon)(a_n(T^*) + \epsilon) \end{aligned}$$

since

$$\begin{aligned} ||z_j - AJ(x)|| &= ||Jz_j - AJ(x)|| \\ &\leq ||Jz_j - JTx|| + ||JTx - AJ(x)|| \\ &\leq \epsilon + ||JTx - AJx|| = \epsilon + ||T^{**}Jx - AJx|| \\ &\leq ||T^{**} - A|| \\ &< a_n(T^*) + \epsilon. \end{aligned}$$

It follows that $a_n(T) \leq a_n(T^{**})$, as promised. □

Theorem 2.3. *If $T \in \mathcal{L}(X, Y)$, where X and Y are arbitrary Banach spaces with metric lifting and extension property, respectively, then $\delta_n(T^*) = \delta_n(T)$ for all n.*

Proof. It is known that if $T \in \mathcal{L}(X, Y)$, where X and Y are arbitrary Banach spaces, then $\delta_n(T^*) = c_n(T)$ ([7], Prop. 2.5.5). We also know that if $T \in \mathcal{L}(X, Y)$, where X and Y are arbitrary Banach spaces with metric lifting and extension property, respectively, then $\delta_n(T) = a_n(T) = c_n(T)$. Hence,

$$\delta_n(T^*) = c_n(T) = a_n(T) = \delta_n(T).$$ □

Remark 2.4. As stated before, Astala in [4] proved that if $T \in \mathcal{L}(X, Y)$, where X and Y are arbitrary Banach spaces with metric lifting and extension property, respectively, then $\gamma(T) = \gamma(T^*)$, where $\gamma(T)$ denotes the measure of non-compactness of T. In [1], it is shown that $\lim_{n\to\infty} \delta_n(T) = \gamma(T)$. This relationship between Kolmogorov diameters and the measure of non-compactness together with Theorem 2.3 provide an alternative proof for the result of Astala.

Theorem 2.5. *If $T \in \mathcal{K}(X,Y)$, where X and Y are arbitrary Banach spaces with metric lifting and extension property, respectively, then $c_n(T^*) = c_n(T)$ for all n.*

Proof. If $T \in \mathcal{K}(X,Y)$, then it is known that $\delta_n(T) = c_n(T^*)$ ([7], Prop. 2.5.6). If X and Y are Banach spaces with metric lifting and extension property, respectively, then we also have $\delta_n(T) = a_n(T) = c_n(T)$. Thus, $c_n(T^*) = c_n(T)$ for all n. □

Remark 2.6. In [9] it is shown that if X has the lifting property, then X^* has the extension property. However, if Y has the extension property, then Y^* has the lifting property if and only if Y is finite-dimensional. Therefore one can observe that if X has the lifting property and Y is finite-dimensional with the extension property, then Y^* has the lifting property and X^* has the extension property, so that we have

$$\delta_n(T^*) = a_n(T^*) = c_n(T^*).$$

3. Compactness with approximation schemes

Approximation schemes were introduced in Banach space theory by Butzer and Scherer in 1968 [6] and independently by Y. Brudnyi and N. Kruglyak under the name of "approximation families" in [5]. They were popularized by Pietsch in his 1981 paper [16], for later developments we refer the reader to [1–3]. The following definition is due to Aksoy and generalizes the classical concept of approximation scheme in a way that allows using families of subsets of X instead of elements of X, which is useful when we deal with n-widths.

Definition (Generalized Approximation Scheme)**.** Let X be a Banach space. For each $n \in \mathbb{N}$, let $Q_n = Q_n(X)$ be a family of subsets of X satisfying the following conditions:

$(GA1)$ $\{0\} = Q_0 \subset Q_1 \subset \cdots \subset Q_n \subset \cdots$.
$(GA2)$ $\lambda Q_n \subset Q_n$ for all $n \in \mathbb{N}$ and all scalars λ.
$(GA3)$ $Q_n + Q_m \subseteq Q_{n+m}$ for every $n, m \in \mathbb{N}$.

Then $Q(X) = (Q_n(X))_{n\in\mathbb{N}}$ is called a *generalized approximation scheme* on X. We shall simply use Q_n to denote $Q_n(X)$ if the context is clear.

We use here the term "generalized" because the elements of Q_n may be subsets of X. Let us now give a few important examples of generalized approximation schemes.

Example 3.1.

(1) Q_n is the set of all at-most-n-dimensional subspaces of any given Banach space X.
(2) Let E be a Banach space and $X = L(E)$; let $Q_n = N_n(E)$, where $N_n(E)$ is the set of all n-nuclear maps on E [15].

(3) Let $a^k = (a_n)^{1+\frac{1}{k}}$, where (a_n) is a nuclear exponent sequence. Then Q_n on $X = L(E)$ can be defined as the set of all $\Lambda_\infty(a^k)$-nuclear maps on E [8].

Definition (Generalized Kolmogorov number)**.** Let B_X be the closed unit ball of X, $Q = Q(X) = (Q_n(X))_{n\in\mathbb{N}}$ be a *generalized approximation scheme* on X, and D be a bounded subset of X. Then the n^{th} *generalized Kolmogorov number* $\delta_n(D;Q)$ of D with respect to Q is defined by

$$\delta_n(D;Q) = \inf\{r > 0 : D \subset rB_X + A \text{ for some } A \in Q_n(X)\}. \tag{1}$$

Assume that Y is a Banach space and $T \in \mathcal{L}(Y,X)$. The n^{th} Kolmogorov number $\delta_n(T;Q)$ of T is defined as $\delta_n(T(B_Y);Q)$.

It follows that $\delta_n(T;Q)$ forms a non-increasing sequence of non-negative numbers:

$$\|T\| = \delta_0(T;Q) \geq \delta_1(T;Q) \geq \cdots \geq \delta_n(T;Q) \geq 0. \tag{2}$$

We are now able to introduce Q-compact sets and operators:

Definition (Q-compact set)**.** Let D be a bounded subset of X. We say that D is Q-*compact* if $\lim_n \delta_n(D;Q) = 0$.

Definition (Q-compact map)**.** We say that $T \in \mathcal{L}(X,Y)$ is a Q-*compact map* if $T(B_Y)$ is a Q-compact set,

$$\lim_n \delta_n(T;Q) = 0.$$

Q-compact maps are a genuine generalization of compact maps since there are examples of Q-compact maps that are not compact in the usual sense. In the following, we present two examples of Q-compact maps that are not compact. The first of these examples is known (see [1]) and it involves a projection $P : L_p[0,1] \to R_p$, where R_p denotes the closure of the span of the space of Rademacher functions. The second example is new and illustrates the fact that if B_w is a weighted backward shift on $c_0(\mathbb{N})$ with $w = (w_n)_n$ a bounded sequence not converging to 0, then B_w is a Q-compact operator which is not compact.

Example 3.2. Let $\{r_n(t)\}$ be the space spanned by the Rademacher functions. It can be seen from the Khinchin inequality [13] that

$$\ell_2 \approx \{r_n(t)\} \subset L_p[0,1] \quad \text{for all } 1 \leq p \leq \infty. \tag{3}$$

We define an approximation scheme A_n on $L_p[0,1]$ as follows:

$$A_n = L_{p+\frac{1}{n}}. \tag{4}$$

$L_{p+\frac{1}{n}} \subset L_{p+\frac{1}{n+1}}$ gives us $A_n \subset A_{n+1}$ for $n = 1, 2, \ldots$, and it is easily seen that $A_n + A_m \subset A_{n+m}$ for $n, m = 1, 2, \ldots$, and that $\lambda A_n \subset A_n$ for all λ. Thus $\{A_n\}$ is an approximation scheme. It can be shown that for $p \geq 2$ the projection $P : L_p[0,1] \to R_p$ is a non-compact Q-compact map, where R_p denotes the closure of the span of $\{r_n(t)\}$ in $L_p[0,1]$ (see [1] for details).

Next, we give another example is a Q-operator which is not compact.

Example 3.3. Consider the weighted backward shift

$$B(x_1, x_2, x_3, \dots) = (w_2x_2, w_3x_3, w_4x_4, \dots),$$

where $w = (w_n)_n$ is a sequence of non-zero scalars called a *weight sequence.* Any weighted shift is a linear operator and is bounded if and only if w is a bounded sequence.

Let $w = (w_n)_n$ be a bounded sequence of positive real numbers. The unilateral weighted shift on $c_0(\mathbb{N})$ is defined by

$$B_w(e_1) = 0 \quad \text{and} \quad B_w(e_n) = w_n e_{n-1} \quad \text{for all } \ n \geq 2.$$

Proposition 3.4. *Suppose the approximation scheme $Q = (A_n)_{n=1}^{\infty}$ of $c_0(\mathbb{N})$ is defined as $A_n = \ell_n(\mathbb{N})$ for all n. Then any bounded weighted shift on c_0 is Q-compact.*

Proof. Let B_w be any bounded and linear weighted shift on c_0. Then $w = (w_n)_n$ is a bounded weight. Let $m \geq 1$. Consider,

$$\begin{aligned}
&\delta_m(B_w(U_{c_0}), (A_n)_n) \\
&= \inf\{r > 0 : B_w(U_{c_0}) \subseteq rU_{c_0} + \ell_m\} \\
&= \inf\{r > 0 : \forall x \in U_{c_0}, \exists y \in U_{c_0}, \exists z \in \ell_m \text{ with } B_w(x) = ry + z\}.
\end{aligned}$$

Let $x = (x_n)_{n\geq 1} \in U_{c_0}$. Let us define $y = (y_n)_{n\geq 1} \in U_{c_0}$ and $z = (z_n)_{n\geq 1} \in \ell_1 \subseteq \ell_m$ such that $B_w(x) = \frac{1}{2^m}y + z$. Let $A := \{n \geq 1 : 2^m|x_nw_n| > 1\}$. The set A is finite, otherwise $(w_n)_n$ is unbounded. Set

$$\begin{cases} x_nw_n = z_{n-1}, & \\ y_{n-1} = 0, & \forall n \in A. \end{cases}$$

Observe that $(w_nx_n)_{n\in\mathbb{N}\setminus A} \in c_0$, hence there exists a subsequence $(n_k)_k$ such that $\sum_{k=1}^{\infty} |w_{n_k}x_{n_k}| < \infty$. Set

$$\begin{cases} x_{n_k}w_{n_k} = z_{n_k-1}, & \\ y_{n_k-1} = 0, & \forall k \geq 1. \end{cases}$$

Finally, set

$$\begin{cases} 2^m x_nw_n = y_{n-1}, & \\ z_{n-1} = 0, & \forall n \in \mathbb{N} \setminus \{(n_k)_k \cup A\}. \end{cases}$$

Hence, $x_nw_n = \frac{1}{2^m}y_{n-1} + z_{n-1}$ for all $n \geq 2$. In other words, $B_w(x) = \frac{1}{2^m}y + z$. Note that $y \in U_{c_0}$ and $z \in \ell_1 \subset \ell_m$. In conclusion, $\delta_m(B_w(U_{c_0}), (A_n)_n) \leq \frac{1}{2^m}$. As m goes to ∞, we obtain that $\delta_m(B_w(U_{c_0}), (A_n)_n)$ goes to 0 and B_w is Q-compact. □

It is well-known that B_w is compact if and only if $w = (w_n)_n$ is a null sequence.

Corollary 3.5. *Let B_w be a weighted backward shift on $c_0(\mathbb{N})$ with $w = (w_n)_n$ a bounded sequence not converging to 0. Consider the approximation schemes on $c_0(\mathbb{N})$ as $Q = (A_n)_{n=1}^{\infty}$ with $A_n = \ell_n(\mathbb{N})$ for all n. Then, B_w is a non-compact Q-compact operator.*

Our next objective here is to ascertain whether or not Schauder's type of theorem is true for Q-compact maps. For this purpose we use symmetrized approximation numbers of T. For our needs, we choose the closed unit ball B_Z of the Banach space Z as an index set Γ. Our proof of the Schauder's theorem for Q-compact operators will depend on the fact that $\ell_1(B_Z)$ has the lifting property and $\ell_\infty(B_Z)$ has the extension property. First we recall the following proposition.

Proposition 3.6 (Refined version of Schauder's theorem [7], p. 84)**.** *An operator T between arbitrary Banach spaces X and Y is compact if and only if*

$$\lim_{n\to\infty} \tau_n(T) = 0$$

and moreover,

$$\tau_n(T) = \tau_n(T^*).$$

Motivated by this, we give the definition of Q-compact operators using the symmetrized approximation numbers.

Definition. We say T is Q-symmetric compact if and only if

$$\lim_{n\to\infty} \tau_n(T, Q) = 0.$$

Remark 3.7. We need the following simple facts for our proof, for details we refer the reader to [7, Propositions 2.5.4-2.5.6].

(a) Recall that $\tau_n(T, Q) = c_n(TQ_X, Q)$, where $Q_X : \ell_1(B_X) \to X$.

(b) We will also abbreviate the canonical embedding

$$K_{\ell_1(B_{Y^*})} : \ell_1(B_{Y^*}) \to \ell_\infty(B_{Y^*})^*$$

by K so that $Q_{Y^*} = J_Y^* K$.

(c) Denote by $P_0 : \ell_\infty(B_{X^{**}}) \to \ell_\infty(B_X)$ the operator which restricts any bounded function on $B_{X^{**}}$ to the subset $K_X(B_X) \subset B_{X^{**}}$ so that $Q_X^* = P_0 J_{X^*}$.

(d) The relations (b) and (c) are crucial facts for the estimates of $\delta_n(T^*, Q^*)$ and $c_n(T^*, Q^*)$. In particular, we have $c_n(T^*, Q^*) \leq \delta_n(T, Q)$.

We now state and prove the following theorem which states that the degree of Q-compactness of T and T^* is the same in so far as it is measured by the symmetrized approximation numbers τ_n.

Theorem 3.8 (Schauder's theorem for Q-compact operators)**.** *Let $T \in \mathcal{L}(X, Y)$ with X, Y be arbitrary Banach spaces, and let $Q = (Q_n(X))$ be a generalized approximation scheme on X. Then*

$$\tau_n(T^*, Q^*) = \tau_n(T, Q)$$

for all n.

Proof. Let us show that $\tau_n(T^*, Q^*) = \tau_n(T, Q)$. By Remark 3.7 parts (a) and (b) we have the following estimates:

$$\begin{aligned} \tau_n(T^*, Q^*) &= c_n(T^* Q_{Y^*}, Q^*) \\ &= c_n(T^* J_Y^* K, Q^*) \\ &\leq c_n((J_Y T)^*, Q^*) \\ &\leq \delta_n(J_Y T, Q) \\ &= t_n(T, Q). \end{aligned}$$

Conversely, we have by using Remark 3.7 parts (c) and (d):

$$\begin{aligned} t_n(T, Q) &= c_n(T Q_X, Q) \\ &= \delta_n(T Q_X)^*, Q^*) \\ &= \delta_n(Q_X^* T^*, Q^*) \\ &= \delta_n(P_0 J_{X^*} T^*, Q^*) \\ &\leq \delta_n(J_{X^*} T^*, Q^*) \\ &= t_n(T^*, Q^*). \end{aligned}$$ □

Next we define approximation numbers with respect to a given scheme as follows:

Definition. Given an approximation scheme $\{Q_n\}$ on X and $T \in \mathcal{L}(X)$, the n-th approximation number $a_n(T, Q)$ with respect to this approximation scheme is defined as:

$$a_n(T, Q) = \inf\{||T - B|| : B \in \mathcal{L}(X),\ B(X) \subseteq Q_n\}.$$

Let X^* and X^{**} be the dual and second dual of X. Note that if we let $J : X \to X^{**}$ be the canonical injection and let (X, Q_n) be an approximation scheme, then $(X^{**}, J(Q_n))$ is an approximation scheme. Let $\{Q_n\}$ and $\{Q_n^{**}\} := \{J(Q_n)\}$ denote the subsets of X and X^{**}, respectively.

Definition. We say (X, Q_n) has the *Extended Local Reflexivity Property* (ELRP) if for each countable subset C of X^{**}, for each $F \in Q_n^{**}$, for some n and each $\epsilon > 0$, there exists a continuous linear map

$$P : \text{span}(F \cup C) \to X \quad \text{such that}$$

(1) $||P|| \leq 1 + \epsilon$,
(2) $P \restriction_{C \cap X} = I$ (Identity).

Note that ELRP is an analogue of local reflexivity principle which is possessed by all Banach spaces.

Theorem 3.9. *Suppose (X, Q_n) has ELRP and $T \in \mathcal{L}(X)$ has separable range. Then for each n we have $a_n(T, Q) = a_n(T^*, Q^*)$.*

Proof. Since one always have $a_n(T^*, Q^*) \leq a_n(T, Q)$ we only need to verify $a_n(T, Q) \leq a_n(T^{**}, Q^{**})$. Let $J : X \to X^{**}$ be the canonical map and U_X be the unit ball of X. Given $\epsilon > 0$, choose $B \in \mathcal{L}(X^{**})$ such that $B(X^{**}) \in Q_n^{**}$ and

$$||B - T^{**}|| < \epsilon + a_n(T^{**}, Q_n^{**}).$$

Let $\{z_j\}$ be a countable dense set in $T(X)$, thus $Tx_j = z_j$, where $x_j \in X$. Consider the set

$$K = \text{span}\{(JTx_j)_1^\infty \cup B(X^{**})\}$$

applying ELRP of X we obtain a map

$$P : K \to X \text{ such that } ||P|| \leq 1 + \epsilon \text{ and } P \restriction_{(JTx_j)_1^\infty \cap X} = I.$$

For $x \in U_X$, consider

$$\begin{aligned} ||Tx - PBJx|| &\leq ||Tx - z_j|| + ||z_j - PBJx|| \\ &\leq \epsilon + ||PJTx_j - PBJx|| \\ &\leq \epsilon + (1 + \epsilon)||JTx_j - BJx|| \\ &\leq \epsilon + (1 + \epsilon)[||JTx_j - JTx|| + ||JTx - BJx||] \\ &\leq \epsilon + (1 + \epsilon)[a_n(T^{**}, Q_n^{**}) + 2\epsilon] \end{aligned}$$

and thus

$$a_n(T, Q) \leq a_n(T^{**}, Q_n^{**}).$$

□

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