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ANALYTIC FUNCTIONS WITH CONIC DOMAINS ASSOCIATED WITH CERTAIN GENERALIZED q-INTEGRAL OPERATOR

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Abstract. In this paper, we define a new subclass of k-uniformly starlike functions of order γ $(0 \leq \gamma < 1)$ by using certain generalized q-integral operator. We explore geometric interpretation of the functions in this class by connecting it with conic domains. We also investigate q-sufficient coefficient condition, q-Fekete-Szegö inequalities, q-Bieberbach-De Branges type coefficient estimates and radius problem for functions in this class. We conclude this paper by introducing an analogous subclass of k-uniformly convex functions of order γ by using the generalized q-integral operator. We omit the results for this new class because they can be directly translated from the corresponding results of our main class.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

that are analytic in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$. Denote by $\mathcal{P}$ the class of functions p which are analytic and have positive real part in $\mathbb{D}$ with $p(0) = 1$. Let Ω be the family of Schwarz functions w which are analytic in $\mathbb{D}$ satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{D}$. If f and g are analytic functions in $\mathbb{D}$, then we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function $w \in \Omega$ such that $f(z) = g(w(z))$. We also note that if g is univalent in $\mathbb{D}$, then by subordination rules we get $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

We denote the class $\mathcal{S}$ of all functions in $\mathcal{A}$ that are univalent in $\mathbb{D}$. Also, let $\mathcal{ST}$ and $\mathcal{CV}$ denote the subclasses of $\mathcal{S}$ that are starlike and convex, respectively.

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For definitions and properties of these classes, one may refer to the survey article by first author [1].

In 1991, Goodman [13] introduced the concept of uniform convexity and uniform starlike functions in $\mathcal{S}$. In fact, he defined such uniform classes, denoted by $\mathcal{UCV}$ and $\mathcal{UST}$, by their geometrical properties. A function f in $\mathcal{A}$ is said to be uniformly convex (uniformly starlike) in $\mathbb{D}$ if f is in $\mathcal{CV}$ ($\mathcal{ST}$) and has the property that for every circular arc γ contained in $\mathbb{D}$ with center ξ also in $\mathbb{D}$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$.

In 1993, Ronning [29] proved the basis for further investigation of the classes $\mathcal{UCV}$ and $\mathcal{UST}$.

Theorem A ([29])**.** *If $f \in \mathcal{A}$, then $f \in \mathcal{UCV}$ if and only if*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{D}).$$

Applying the classic Alexander Theorem found by Alexander [5] in 1915, Ronning [29] obtained a characterization of the class $\mathcal{UST}$.

Theorem B ([29])**.** *If a function f belongs to $\mathcal{A}$, then f belongs to $\mathcal{UST}$ if and only if*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)}-1\right| \quad (z \in \mathbb{D}).$$

For $k \geq 0$, Kanas *et al.* [21] introduced the class of k-uniformly starlike functions denoted by k-$\mathcal{UST}$. Such a class consists of functions $f \in \mathcal{A}$ that satisfy the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)}-1\right| \quad (z \in \mathbb{D}).$$

We note that 1-$\mathcal{UST} \equiv \mathcal{UST}$. In 1997, Bharati *et al.* [11] introduced the class of k-uniformly starlike functions of order γ $(0 \leq \gamma < 1)$ for functions in the class k-$\mathcal{UST}(\gamma)$.

Definition 1.1 ([11])**.** Let $0 \leq \gamma < 1$ and $k \geq 0$. A function $f \in \mathcal{A}$ is said to be in k-$\mathcal{UST}(\gamma)$, called k-uniformly starlike functions of order γ, if f satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)}-1\right| + \gamma \quad (z \in \mathbb{D}).$$

We note that k-$\mathcal{UST}(0) \equiv k$-$\mathcal{UST}$. The class 1-$\mathcal{UST}(\gamma)$ was investigated in [6] and [28].

We next recall some concepts and notations of q-calculus that we need to define a new class which connects k-$\mathcal{UST}(\gamma)$ and a generalized integral operator defined by q-calculus.

Quantum calculus (or q-calculus) is a theory of calculus where smoothness is not required. A systematic study of q-differentiation and q-integration was initiated by Jackson [15, 16]. The q-derivative (or q-difference) operator of a function f, defined on a subset of $\mathbb{C}$, is defined by

$$(D_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & \text{if } z \neq 0, \\ f'(0), & \text{if } z = 0, \end{cases}$$

where $q \in (0,1)$. Note that $\lim_{q\to 1^-}(D_q f)(z) = f'(z)$ if f is differentiable at z. Under the hypothesis of the definition of q-derivative, we have the following rules:

$$\begin{aligned} D_q(af(z) \pm bg(z)) &= aD_q f(z) \pm bD_q g(z) \ (a, b \in \mathbb{C}), \\ D_q(f(z) \cdot g(z)) &= f(qz)D_q g(z) + g(z)D_q f(z). \end{aligned} \tag{1.2}$$

It easily follows that if a function f is given by (1.1), then

$$(D_q f)(z) = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}, \quad D_q(zD_q f(z)) = \sum_{n=1}^{\infty} [n]_q^2 a_n z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q}$$

is called q-number or q-bracket of n. Clearly, $\lim_{q\to 1^-}[n]_q = n$. In [16], Jackson defined q-integral of a function f as follows:

$$\int_0^x f(t)d_q t = x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n),$$

provided the series converges. It is known that q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(x+1) = [x]_q!, \tag{1.3}$$

where q-factorial $[x]_q!$ is given by

$$[x]_q! = \begin{cases} [x]_q[x-1]_q \cdots [2]_q[1]_q, & \text{if } x \geq 1, \\ 1, & \text{if } x = 0. \end{cases}$$

Note that in the limiting case when $q \to 1^-$, $\Gamma_q(x) \to \Gamma(x)$; see [14].

The q-beta function has the q-integral representation, which is a q-analogue of Euler's formula (see [16]):

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x, \quad (0 < q < 1; t, s > 0). \tag{1.4}$$

Jackson [14] also showed that the q-beta function defined by the formula

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}, \tag{1.5}$$

tends to $B(t,s)$ as $q \to 1^-$. Also, the Gauss q-binomial coefficients are given by

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{1.6}$$

For more details, one may refer to [12, 18].

In recent years, quantum calculus approach has led to a great development in geometric function theory. Ahuja and Çetinkaya [3] recently wrote a survey on the use of quantum calculus approach in mathematical sciences and its role in geometric function theory. One may also refer to the recent paper by Ahuja *et al.* [2].

Motivated by Jung *et al.* [17], Mahmood *et al.* [24] introduced the generalized q-integral operator $\chi^{\alpha}_{\beta,q}f : \mathcal{A} \to \mathcal{A}$ defined by

$$\chi^{\alpha}_{\beta,q}f(z) = \binom{\alpha+\beta}{\beta}_q \frac{[\alpha]_q}{z^\beta} \int_0^z \left(1 - \frac{qt}{z}\right)_q^{\alpha-1} t^{\beta-1} f(t) d_q t, \tag{1.7}$$

where $\alpha > 0$, $\beta > -1$, $q \in (0,1)$. Using (1.3), (1.4), (1.5), and (1.6), they observed that

$$\chi^{\alpha}_{\beta,q}f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)} \frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)} a_n z^n. \tag{1.8}$$

For special values of the parameters, the generalized integral operator (1.8) gives the following known integral operators as special cases:

(i) For $q \to 1^-$, the operator $\chi^{\alpha}_{\beta,q}f$ reduces to the integral operator $\chi^{\alpha}_{\beta}f$ defined in [17] by

$$\begin{aligned}\chi^{\alpha}_{\beta}f(z) &= \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} a_n z^n.\end{aligned}$$

(ii) For $\alpha = 1$, the operator $\chi^{\alpha}_{\beta,q}f$ yields the q-Bernardi integral operator $J_{\beta,q}f$ defined in [25] by

$$J_{\beta,q}f = \frac{[1+\beta]_q}{z^\beta} \int_0^z t^{\beta-1} f(t) d_q t = \sum_{n=1}^{\infty} \frac{[1+\beta]_q}{[n+\beta]_q} a_n z^n.$$

(iii) For $\alpha = 1$, $q \to 1^-$, the operator $\chi^{\alpha}_{\beta,q}f$ gives the Bernardi integral operator $J_\beta f$ defined in [10] by

$$J_\beta f(z) = \frac{1+\beta}{z^\beta} \int_0^z t^{\beta-1} f(t) dt = \sum_{n=1}^{\infty} \frac{1+\beta}{n+\beta} a_n z^n. \tag{1.9}$$

(iv) For $\alpha = 1$, $\beta = 0$, $q \to 1^-$, the operator $\chi^{\alpha}_{\beta,q}f$ reduces to the Alexander integral operator $J_0 f$ given in [30] by

$$J_0 f(z) = \int_0^z \frac{f(t)}{t} dt = z + \sum_{n=2}^{\infty} \frac{1}{n} a_n z^n. \tag{1.10}$$

Making use of q-integral operator, $\chi^{\alpha}_{\beta,q}f$ and k-uniformly starlike function of order γ given in Definition 1.1, we define the new class k-$\mathcal{JUST}(q; \alpha, \beta, \gamma)$.

Definition 1.2. Let $0 \leq \gamma < 1$, $q \in (0,1)$, $k \geq 0$, $\alpha > 0$, $\beta > -1$. A function $f \in \mathcal{A}$ is in the class $k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma)$ if and only if

$$\operatorname{Re}\left(\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}\right) > k\left|\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)} - 1\right| + \gamma \quad (z \in \mathbb{D}), \tag{1.11}$$

where $\chi^{\alpha}_{\beta,q}f(z)$ is given by (1.8).

In what follows, we shall first look at the geometric interpretation of (1.11). Note that

$$\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)} = 1 + ([2]_q - 1)\psi_2 a_2 z + \left[([3]_q - 1)\psi_3 a_3 - ([2]_q - 1)\psi_2^2 a_2^2\right] z^2 + \cdots, \tag{1.12}$$

where

$$\psi_n = \frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)} \quad (n \geq 2).$$

It follows that for $z = 0$, we have

$$\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)} = 1.$$

Thus $p(z) = zD_q(\chi^{\alpha}_{\beta,q}f(z))/\chi^{\alpha}_{\beta,q}f(z)$, where p belongs to the class $\mathcal{P}$. Therefore (1.11) is equivalent to

$$\operatorname{Re}\ p(z) > k|p(z) - 1| + \gamma \quad (z \in \mathbb{D}).$$

Thus, $p(z)$ takes the values in the conic domain $\Omega_{k,\gamma}$ defined by

$$\Omega_{k,\gamma} = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2} + \gamma,\ 0 \leq \gamma < 1,\ k \geq 0\right\}.$$

Note that $1 \in \Omega_{k,\gamma}$ and $\partial\Omega_{k,\gamma}$ is a curve defined by

$$\partial\Omega_{k,\gamma} = \left\{u + iv : (u-\gamma)^2 = k^2(u-1)^2 + k^2v^2,\ 0 \leq \gamma < 1,\ k \geq 0\right\}.$$

Elementary computations show that $\partial\Omega_{k,\gamma}$ represents a conic section symmetric about the real axis. Hence $\Omega_{k,\gamma}$ is an elliptic domain for $k > 1$, parabolic domain for $k = 1$, hyperbolic domain for $0 < k < 1$ and a right half plane $u > \gamma$ for $k = 0$.

Denote by $\mathcal{P}(p_{k,\gamma})$ the family of functions p such that $p \in \mathcal{P}$ and $p \prec p_{k,\gamma}$ in $\mathbb{D}$, where $p_{k,\gamma}$ maps $\mathbb{D}$ conformally onto the domain $\Omega_{k,\gamma}$.

Kanas *et al.* [20, 21] found that the function $p_{k,\gamma}(z)$ plays a role of extremal function of the class $\mathcal{P}(p_{k,\gamma})$ and is given by

$$p_{k,\gamma}(z) = \begin{cases} \frac{1+(1-2\gamma)z}{1-z}, & (k=0) \\ \frac{1-\gamma}{1-k^2}\cos\left\{\frac{2}{\pi}(\cos^{-1}k)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2-\gamma}{1-k^2}, & (0<k<1) \\ 1+\frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & (k=1) \\ \frac{1-\gamma}{k^2-1}\sin\left\{\frac{\pi}{2K(t)}\int_0^{u(z)/\sqrt{t}}\frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right\}+\frac{k^2-\gamma}{k^2-1}, & (k>1), \end{cases} \tag{1.13}$$

where $u(z) = (z-\sqrt{t})/(1-\sqrt{t}z)$, $t \in (0,1)$ and t is chosen such that $k = \cosh\frac{\pi K'(t)}{4K(t)}$ and $K(t)$ is Legendre's complete elliptic integral of the first kind and $K'(t)$ is complementary integral of $K(t)$. Furthermore, since $p_{k,\gamma}(\mathbb{D}) = \Omega_{k,\gamma}$ and $p_{k,\gamma}(\mathbb{D})$ is convex univalent in $\mathbb{D}$ (see [20]), it follows that (1.11) is equivalent to

$$\frac{zD_q(\chi^\alpha_{\beta,q}f(z))}{\chi^\alpha_{\beta,q}f(z)} \prec p_{k,\gamma}(z).$$

Remark 1.3. For special values of parameters q, α, β, γ and k, we get the following new classes as special cases of Definition 1.2; for example:

1. If $q \to 1^-$ and $\alpha = 1$, we get $k\text{-}\mathcal{JUST}(\beta,\gamma) := \lim(k\text{-}\mathcal{JUST}(q;1,\beta,\gamma))$ with Bernardi operator (1.9).
2. If $q \to 1^-$, $\alpha = 1$, $\beta = 0$ and $k = 0$, we get $\mathcal{JUST}(\gamma) := \lim(0\text{-}\mathcal{JUST}(q;1,0,\gamma))$ with Alexander operator (1.10).

In this paper, we shall investigate the class $k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma)$. In particular, we obtain q-sufficient coefficient condition, q-Fekete-Szegö inequalities, q-Bieberbach-De Branges type coefficient estimates and solve radius problem for the functions in this class. In the concluding section, we introduce another new class $k\text{-}\mathcal{JUCV}(q;\alpha,\beta,\gamma)$ and omit the results for this class, because analogous results can be directly translated from the corresponding results found in Section 2 for the class $k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma)$.

Unless otherwise stated, we assume in the reminder of the article that $0 \le \gamma < 1$, $q \in (0,1)$, $k \ge 0$, $\alpha > 0$, $\beta > -1$ and $z \in \mathbb{D}$.

2. Main results

We first obtain q-sufficient coefficient condition for the functions belonging to the class $k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma)$.

Theorem 2.1. *If a function f defined by* (1.1) *satisfies the inequality*

$$\sum_{n=2}^{\infty}\big([n]_q(k+1)-(k+\gamma)\big)\frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}|a_n| \le 1-\gamma, \tag{2.1}$$

then f belongs to $k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma)$. The result is sharp.

Proof. To show that $f \in k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma)$, it suffices to prove that

$$k\left|\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}-1\right|-\mathrm{Re}\left(\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}-1\right)\leq 1-\gamma.$$

We note that

$$\left|\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}-1\right|=\left|\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))-\chi^{\alpha}_{\beta,q}f(z)}{\chi^{\alpha}_{\beta,q}f(z)}\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}([n]_q-1)\frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}a_nz^n}{z+\sum_{n=2}^{\infty}\frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}a_nz^n}\right|$$

$$\leq\frac{\sum_{n=1}^{\infty}\left([n]_q-1\right)\frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}|a_n|}{1-\sum_{n=1}^{\infty}\frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}|a_n|}. \tag{2.2}$$

In view of (2.1), it follows that

$$1-\sum_{n=1}^{\infty}\frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}|a_n|>0.$$

Using (2.2), we have

$$\begin{aligned}
&k\left|\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}-1\right|-\mathrm{Re}\left(\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}-1\right)\\
&\leq k\left|\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}-1\right|+\left|\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}-1\right|\\
&\leq (k+1)\left|\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))-\chi^{\alpha}_{\beta,q}f(z)}{\chi^{\alpha}_{\beta,q}f(z)}\right|\\
&\leq (k+1)\left\{\frac{\sum_{n=2}^{\infty}\left([n]_q-1\right)\frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}|a_n|}{1-\sum_{n=1}^{\infty}\frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}|a_n|}\right\}\\
&\leq 1-\gamma
\end{aligned}$$

which proves (2.1).

For sharpness, consider the function $f_n:\mathbb{D}\to\mathbb{C}$ defined by

$$f_n(z)=z-\frac{(1-\gamma)\Gamma_q(\alpha+\beta+n)}{\big([n]_q(k+1)-(k+\gamma)\big)\Gamma_q(\beta+n)}\frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}z^n.$$

Since

$$\mathrm{Re}\left(\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}\right)=\mathrm{Re}\left(\frac{[n]_q(k+1)-(k+\gamma)-(1-\gamma)[n]_qz^{n-1}}{[n]_q(k+1)-(k+\gamma)-(1-\gamma)z^{n-1}}\right)$$

$$>\frac{k+\gamma}{k+1}$$

and

$$k\left|\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}-1\right| = k\left|\frac{(1-\gamma)(1-[n]_q)z^{n-1}}{[n]_q(k+1)-(k+\gamma)-(1-\gamma)z^{n-1}}\right| < \frac{k(1-\gamma)}{k+1},$$

it follows that $f_n \in k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma)$. Also, it is easy to show that the equality holds in (2.1) for the function f_n. Thus the result is sharp. □

Corollary 2.2. *If* $f(z) = z + a_n z^n$ *and*

$$|a_n| \le \frac{(1-\gamma)\Gamma_q(\alpha+\beta+n)}{\big([n]_q(k+1)-(k+\gamma)\big)\Gamma_q(\beta+n)}\frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}, \quad (n \ge 2)$$

then $f \in k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma)$.

Using Remark 1.3.1 and Remark 1.3.2, Theorem 2.1 gives the following new results.

Corollary 2.3. *If a function* f *defined by* (1.1) *is in the class* $k\text{-}\mathcal{JUST}(\beta,\gamma)$, *then*

$$\sum_{n=2}^{\infty}\big(n(k+1)-(k+\gamma)\big)\frac{1+\beta}{n+\beta}|a_n| \le 1-\gamma.$$

Corollary 2.4. *If a function* f *defined by* (1.1) *is in the class* $\mathcal{JUST}(\gamma)$, *then*

$$\sum_{n=2}^{\infty}\big(n-\gamma\big)\frac{1}{n}|a_n| \le 1-\gamma.$$

In order to determine q-Fekete-Szegö inequalities for the functions in the class $k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma)$, we need next three lemmas.

Lemma 2.5 ([7,20]). *Let* $k \ge 0$ *be fixed and* $p_{k,\gamma}$ *defined by* (1.13). *If*

$$p_{k,\gamma}(z) = 1 + P_1 z + P_2 z^2 + \cdots,$$

then

$$P_1(z) = \begin{cases} \frac{8(1-\gamma)(\arccos k)^2}{\pi^2(1-k^2)}, & 0 \le k < 1, \\ \frac{8(1-\gamma)}{\pi^2}, & k = 1, \\ \frac{\pi^2(1-\gamma)}{4\sqrt{t}(1+t)K^2(t)(k^2-1)}, & k > 1, \end{cases} \tag{2.3}$$

and

$$P_2(z) = \begin{cases} \frac{(A^2+2)}{3}P_1, & 0 \le k < 1, \\ \frac{2}{3}P_1, & k = 1, \\ \frac{4K^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}K^2(t)(1+t)}P_1, & k > 1, \end{cases}$$

where $A = \frac{2}{\pi}(\cos^{-1} k)$ *and* $t \in (0,1)$ *are chosen such that* $k = \cosh(\pi K'(t)/4K(t))$ *and* $K(t)$ *is Legendre's complete elliptic integral of the first kind and* $K'(t)$ *is complementary integral of* $K(t)$.

Lemma 2.6 ([22, Lemma 3, p. 254]). *If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ is in class $\mathcal{P}$ and η is a complex number, then*

$$|c_2 - \eta c_1^2| \leq 2 \max\{1, |2\eta - 1|\}.$$

The result is sharp for the functions $p(z) = (1 + z^2)/(1 - z^2)$ and $p(z) = (1 + z)/(1 - z)$.

Lemma 2.7 ([23, Lemma 1, p. 162]). *If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ is in class $\mathcal{P}$ and η is a real number, then*

$$|c_2 - \eta c_1^2| \leq \begin{cases} -4\eta + 2 & \text{if} \quad \eta \leq 0, \\ 2 & \text{if} \quad 0 \leq \eta \leq 1, \\ 4\eta - 2 & \text{if} \quad \eta \geq 1. \end{cases}$$

When $\eta < 0$ and $\eta > 1$, equality holds if and only if $p(z) = (1 + z)/(1 - z)$ or one of its rotations. If $0 < \eta < 1$, then equality holds if and only if $p(z) = (1 + z^2)/(1 - z^2)$ or one of its rotations. If $\eta = 0$, equality holds if and only if

$$p(z) = \frac{1 + \lambda}{2} \left(\frac{1 + z}{1 - z} \right) + \frac{1 - \lambda}{2} \left(\frac{1 - z}{1 + z} \right), \quad 0 \leq \lambda \leq 1$$

or one of its rotations. If $\eta = 1$, equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds in the case $\eta = 0$.

Theorem 2.8. *Let $k \geq 0$ and $f \in k\text{-}\mathcal{JUST}(q; \alpha, \beta, \gamma)$, where f is of the form (1.1). Then, for a complex number η, q-Fekete-Szegö inequality is given by*

$$|a_3 - \eta a_2^2| \leq \frac{P_1}{2([3]_q - 1)\psi_3} \max\{1, |2\nu - 1|\},$$

where

$$\nu = \frac{1}{2} - \frac{P_2}{2P_1} - \frac{P_1}{2([2]_q - 1)} + \eta \frac{P_1([3]_q - 1)\psi_3}{2([2]_q - 1)^2 \psi_2^2}, \tag{2.4}$$

$$\psi_n = \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)}, \quad (n = 2, 3),$$

and where P_1 and P_2 are given by Lemma 2.5.

Proof. If $f \in k\text{-}\mathcal{JUST}(q; \alpha, \beta, \gamma)$, then there is a Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{z D_q(\chi_{\beta,q}^{\alpha} f(z))}{\chi_{\beta,q}^{\alpha} f(z)} = p_{k,\gamma}(w(z)).$$

Define the function p by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{D}).$$

Since $p \in \mathcal{P}$ is a function with $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$, we get

$$p_{k,\gamma}(w(z)) = p_{k,\gamma}\left(\frac{p(z) - 1}{p(z) + 1} \right)$$

$$= p_{k,\gamma}\left(\frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right)$$

$$= 1 + \frac{P_1c_1}{2}z + \left(\frac{P_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{P_2c_1^2}{4}\right)z^2 + \cdots . \tag{2.5}$$

Comparing coefficients of (2.5) and (1.12), we get

$$a_2 = \frac{P_1c_1}{2([2]_q - 1)\psi_2}$$

and

$$a_3 = \frac{P_1}{2([3]_q - 1)\psi_3}\left\{c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(\frac{P_2}{P_1} + \frac{P_1}{[2]_q - 1}\right)\right\}.$$

For any complex number η, we have

$$a_3 - \eta a_2^2 = \frac{P_1}{2([3]_q - 1)\psi_3}\left\{c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(\frac{P_2}{P_1} + \frac{P_1}{[2]_q - 1}\right)\right\} - \eta\frac{P_1^2c_1^2}{4([2]_q - 1)^2\psi_2^2}. \tag{2.6}$$

Equation (2.6) can be written as:

$$a_3 - \eta a_2^2 = \frac{P_1}{2([3]_q - 1)\psi_3}\{c_2 - \nu c_1^2\}, \tag{2.7}$$

where ν is defined by (2.4). Applying Lemma 2.6, the proof is completed. The result is sharp for a function f given by

$$\frac{zD_q(\chi^\alpha_{\beta,q}f(z))}{\chi^\alpha_{\beta,q}f(z)} = p_{k,\gamma}(z) \quad \text{or} \quad \frac{zD_q(\chi^\alpha_{\beta,q}f(z))}{\chi^\alpha_{\beta,q}f(z)} = p_{k,\gamma}(z^2).$$

□

In view of Remark 1.3.1 and Remark 1.3.2, we get the following new results as special cases of Theorem 2.8.

Corollary 2.9. *If a function f defined by* (1.1) *is in the class k-$\mathcal{JUST}(\beta,\gamma)$, then*

$$|a_3 - \eta a_2^2| \le \frac{P_1(3+\beta)}{4(1+\beta)}\max\left\{1, \left| -\frac{P_2}{P_1} - P_1 + \eta\frac{2P_1(2+\beta)^2}{(3+\beta)(1+\beta)}\right|\right\}.$$

Corollary 2.10. *If a function f defined by* (1.1) *is in the class $\mathcal{JUST}(\gamma)$, then*

$$|a_3 - \eta a_2^2| \le \frac{3P_1}{4}\max\left\{1, \left| -\frac{P_2}{P_1} - P_1 + \eta\frac{8P_1}{3}\right|\right\}.$$

Theorem 2.11. *Let $k \ge 0$ and $f \in k$-$\mathcal{JUST}(q;\alpha,\beta,\gamma)$, where f is of the form* (1.1). *Then, for a real number η, we have*

$$|a_3 - \eta a_2^2| \le \frac{1}{([3]_q - 1)\psi_3} \times \begin{cases} P_2 + \frac{P_1^2}{[2]_q-1} - \eta\frac{P_1^2([3]_q-1)\psi_3}{([2]_q-1)^2\psi_2^2}, & \text{if } \eta \le \sigma_1, \\ P_1, & \text{if } \sigma_1 \le \eta \le \sigma_2, \\ -P_2 - \frac{P_1^2}{[2]_q-1} + \eta\frac{P_1^2([3]_q-1)\psi_3}{([2]_q-1)^2\psi_2^2}, & \text{if } \eta \ge \sigma_2, \end{cases}$$

where

$$\psi_n = \frac{\Gamma_q(\beta+n)}{\Gamma_q(\alpha+\beta+n)}\frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}, \quad (n=2,3),$$

$$\sigma_1 = \frac{([2]_q-1)^2\psi_2^2}{([3]_q-1)\psi_3}\left(\frac{P_2}{P_1^2}+\frac{1}{[2]_q-1}-\frac{1}{P_1}\right),$$

$$\sigma_2 = \frac{([2]_q-1)^2\psi_2^2}{([3]_q-1)\psi_3}\left(\frac{P_2}{P_1^2}+\frac{1}{[2]_q-1}+\frac{1}{P_1}\right),$$

and P_1 and P_2 are given by Lemma 2.5.

Proof. Using (2.4), (2.7) and Lemma 2.7, we get the proof. The bounds are sharp as can be seen by defining the following functions for $n \geq 2$ and $0 \leq \lambda \leq 1$.

$$\frac{zD_q(\chi^{\alpha}_{\beta,q}F_n(z))}{\chi^{\alpha}_{\beta,q}F_n(z)} = p_{k,\gamma}(z^{n-1}), \quad \mathcal{F}_n(0) = \mathcal{F}'_n(0)-1=0,$$

$$\frac{zD_q(\chi^{\alpha}_{\beta,q}G_\lambda(z))}{\chi^{\alpha}_{\beta,q}G_\lambda(z)} = p_{k,\gamma}\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \quad \mathcal{G}_\lambda(0) = \mathcal{G}'_\lambda(0)-1=0,$$

$$\frac{zD_q(\chi^{\alpha}_{\beta,q}H_\lambda(z))}{\chi^{\alpha}_{\beta,q}H_\lambda(z)} = p_{k,\gamma}\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \quad \mathcal{H}_\lambda(0) = \mathcal{H}'_\lambda(0)-1=0.$$

When $\eta < \psi_1$ or $\eta > \psi_2$, equality holds if and only if f is $\mathcal{F}_2$ or one of its rotations. When $\psi_1 < \eta < \psi_2$, equality holds if and only if f is $\mathcal{F}_3$ or one of its rotations. If $\eta = \psi_1$, equality holds if and only if f is $\mathcal{G}_\lambda$ or one of its rotations and if $\eta = \psi_2$, equality holds if and only if f is $\mathcal{H}_\lambda$ or one of its rotations. □

For investigating q-Bieberbach-De Branges inequalities, we need the following result called Rogogonki's Theorem.

Lemma 2.12 ([27, Theorem 2.3, p. 70])**.** *Let $p(z) = 1+\sum_{n=1}^{\infty} c_n z^n$ be subordinate to $p_{k,\gamma}(z) = 1+\sum_{n=1}^{\infty} P_n z^n$ in $\mathbb{D}$. If $p_{k,\gamma}$ is univalent in $\mathbb{D}$ and $p_{k,\gamma}(\mathbb{D})$ is convex, then*

$$|c_n| \leq P_1, \quad (n \geq 1).$$

Theorem 2.13. *If a function f of the form* (1.1) *belongs to the class k-$\mathcal{JUST}(q;\alpha,\beta,\gamma)$, then*

$$|a_2| \leq \frac{P_1}{q\psi_2} \quad \text{and} \quad |a_3| \leq \frac{qP_2+P_1^2}{q^2(1+q)\psi_3}.$$

These results are sharp for the function given by (2.12)*.*

Proof. Let $p(z) = \frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)}$. Using the relation (1.8) for $p(z) = 1+c_1z+c_2z^2+\cdots$, we have

$$([n]_q-1)\psi_n a_n = \sum_{k=1}^{n-1}\psi_k a_k c_{n-k}, \quad a_1 = 1.$$

Comparing the coefficients for $n = 2$ and $n = 3$, we get

$$a_2 = \frac{c_1}{([2]_q - 1)\psi_2} \text{ and } a_3 = \frac{c_2 + c_1\psi_2 a_2}{([3]_q - 1)\psi_3}. \tag{2.8}$$

It is obvious that

$$|a_2| = \frac{|c_1|}{([2]_q - 1)\psi_2} \le \frac{P_1}{q\psi_2}, \tag{2.9}$$

where $|c_1| \le P_1$.

Now, Lemma 2.12, (2.8) together with inequality $|c_1^2| + |c_2| \le |P_1|^2 + |P_2|$ (see [19]) yield

$$\begin{aligned} |a_3| = \left| \frac{qc_2 + c_1^2}{q^2(1+q)\psi_3} \right| &\le \frac{q(|c_2| + |c_1|^2) + (1-q)|c_1^2|}{q^2(1+q)\psi_3} \\ &\le \frac{q(|P_2| + |P_1|^2) + (1-q)|P_1^2|}{q^2(1+q)\psi_3} \le \frac{qP_2 + P_1^2}{q^2(1+q)\psi_3}. \end{aligned}$$

□

In our next result, we state and prove a q-Bieberbach-De Branges inequality.

Theorem 2.14. *If $f \in k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma)$, then*

$$|a_n| \le \frac{P_1}{([n]_q - 1)\psi_n} \prod_{k=1}^{n-2} \left(1 + \frac{P_1}{([k+1]_q - 1)}\right), \quad (n \ge 3) \tag{2.10}$$

where P_1 is given by (2.3) *and*

$$\psi_n = \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha+\beta+n)} \frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}, \quad (n \ge 3). \tag{2.11}$$

Proof. In view of Definition 1.2, we can write

$$\frac{zD_q(\chi^\alpha_{\beta,q} f(z))}{\chi^\alpha_{\beta,q} f(z)} = p(z) \prec p_{k,\gamma},$$

where $p \in \mathcal{P}$ is analytic in $\mathbb{D}$. Since $p(z) = 1 + \sum_{n=1}^\infty c_n z^n$ and $\chi^\alpha_{\beta,q} f$ given by (1.8), we have

$$z + \sum_{n=2}^\infty [n]_q \psi_n a_n z^n = \left(z + \sum_{n=2}^\infty \psi_n a_n z^n\right)\left(1 + \sum_{n=1}^\infty c_n z^n\right),$$

where ψ_n is given by (2.11).

Comparing the coefficients of z^n on both sides, we observe

$$[n]_q \psi_n a_n = a_n + c_1\psi_{n-1}a_{n-1} + c_2\psi_{n-2}a_{n-2} + \cdots + c_{n-2}\psi_2 a_2 + c_{n-1}$$

for all integer $n \ge 3$. Taking absolute value on both sides and applying Lemma 2.12, we have

$$|a_n| \le \frac{P_1}{([n]_q - 1)\psi_n}\{1 + \psi_2|a_2| + \cdots + \psi_{n-2}|a_{n-2}| + \psi_{n-1}|a_{n-1}|\}.$$

We will prove (2.10) by using mathematical induction. For $n = 2$, the result follows by (2.9). Let us assume that (2.10) is true for $n \leq m$, that is

$$\begin{aligned}|a_m| &\leq \frac{P_1}{([m]_q - 1)\psi_m}\{1 + \psi_2|a_2| + \cdots + \psi_{m-1}|a_{m-1}|\} \\ &\leq \frac{P_1}{([m]_q - 1)\psi_m}\prod_{k=1}^{m-2}\left(1 + \frac{P_1}{([k+1]_q - 1)}\right).\end{aligned}$$

Consider

$$\begin{aligned}|a_{m+1}| &\leq \frac{P_1}{([m+1]_q - 1)\psi_{m+1}}\{1 + \psi_2|a_2| + \cdots + \psi_m|a_m|\} \\ &\leq \frac{P_1}{([m+1]_q - 1)\psi_{m+1}}\left\{1 + \frac{P_1}{([2]_q - 1)} + \frac{P_1}{([3]_q - 1)}\left(1 + \frac{P_1}{([2]_q - 1)}\right)\right. \\ &\quad \left. + \cdots + \frac{P_1}{([m]_q - 1)}\prod_{k=1}^{m-2}\left(1 + \frac{P_1}{([k+1]_q - 1)}\right)\right\} \\ &= \frac{P_1}{([m+1]_q - 1)\psi_{m+1}}\prod_{k=1}^{m-1}\left(1 + \frac{P_1}{([k+1]_q - 1)}\right).\end{aligned}$$

Thus (2.10) is true for $n = m+1$. Consequently, mathematical induction shows that (2.10) holds for n, $n \geq 2$. This completes the proof. The result is sharp for a function f given by

$$\frac{zD_q(\chi^{\alpha}_{\beta,q}f(z))}{\chi^{\alpha}_{\beta,q}f(z)} = p_{k,\gamma}(z). \tag{2.12}$$

□

For different values of the parameters, Theorem 2.14 gives several new results. In particular, in view of Remark 1.3.1 and Remark 1.3.2, Theorem 2.14 gives the following results.

Corollary 2.15. *If a function f defined by* (1.1) *is in the class k-$\mathcal{JUST}(\beta,\gamma)$, then*

$$|a_n| \leq \frac{P_1(\beta + n)}{(n-1)(\beta+1)}\prod_{k=1}^{n-2}\left(1 + \frac{P_1}{k}\right), \quad (n \geq 3).$$

Corollary 2.16. *If a function f defined by* (1.1) *is in the class $\mathcal{JUST}(\gamma)$, then*

$$|a_n| \leq \frac{nP_1}{(n-1)}\prod_{k=1}^{n-2}\left(1 + \frac{P_1}{k}\right), \quad (n \geq 3).$$

We now conclude this section by exploring q-radius for the functions in the class k-$\mathcal{JUST}(q;\alpha,\beta,\gamma)$. For recent radius problems, see [4, 8, 9, 26].

Theorem 2.17. *If $f \in k$-$\mathcal{JUST}(q;\alpha,\beta,\gamma)$, then $f(\mathbb{D})$ contains an open disc of radius*

$$r = \frac{q\psi_2}{2q\psi_2 + P_1},$$

where P_1 is given by (2.3).

Proof. Let $w_0 \neq 0$ be a complex number such that $f(z) \neq w_0$ for $z \in \mathbb{D}$. Then

$$f_1(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + \cdots .$$

Since f_1 is univalent in $\mathbb{D}$, it follows that

$$\left| a_2 + \frac{1}{w_0} \right| \leq 2.$$

By using (2.9), we get

$$\left| \frac{1}{w_0} \right| \leq 2 + \frac{P_1}{q\psi_2} = \frac{2q\psi_2 + P_1}{q\psi_2}.$$

Consequently, we obtain

$$|w_0| \geq \frac{q\psi_2}{2q\psi_2 + P_1}.$$ □

3. Concluding remarks

Using the well-known formula (1.2) and replacing $\chi^{\alpha}_{\beta,q} f(z)$ in (1.11) by $zD_q(\chi^{\alpha}_{\beta,q} f(z))$, we obtain a new subclass k-$\mathcal{JUCV}(q;\alpha,\beta,\gamma)$ of k-uniformly convex functions of order γ associated with the generalized q-integral operator given by (1.7).

Definition 3.1. Let $0 \leq \gamma < 1$, $q \in (0,1)$, $k \geq 0$, $\alpha > 0$, $\beta > -1$. A function $f \in \mathcal{A}$ is in the class k-$\mathcal{JUCV}(q;\alpha,\beta,\gamma)$ if and only if

$$\mathrm{Re}\left(1 + q\frac{zD_q^2(\chi^{\alpha}_{\beta,q} f(z))}{D_q(\chi^{\alpha}_{\beta,q} f(z))}\right) > k\left| q\frac{zD_q^2(\chi^{\alpha}_{\beta,q} f(z))}{D_q(\chi^{\alpha}_{\beta,q} f(z))} \right| + \gamma \quad (z \in \mathbb{D}),$$

where $\chi^{\alpha}_{\beta,q} f(z)$ is given by (1.7) and (1.8).

Alexander-type relationship between functions of these classes is

$$\chi^{\alpha}_{\beta,q} f(z) \in k\text{-}\mathcal{JUCV}(q;\alpha,\beta,\gamma) \Leftrightarrow zD_q(\chi^{\alpha}_{\beta,q} f(z)) \in k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma);$$

that is,

$$\chi^{\alpha}_{\beta,q} f(z) \in k\text{-}\mathcal{JUST}(q;\alpha,\beta,\gamma) \Leftrightarrow \int_0^z \frac{\chi^{\alpha}_{\beta,q} f(z)}{t} dt \in k\text{-}\mathcal{JUCV}(q;\alpha,\beta,\gamma).$$

In view of the classical Alexander Theorem and the results for the class k-$\mathcal{JUST}(q;\alpha,\beta,\gamma)$, it is easy to obtain the corresponding properties for the class k-$\mathcal{JUCV}(q;\alpha,\beta,\gamma)$. Therefore, we omit the statements and proofs of the corresponding results of the class k-$\mathcal{JUCV}(q;\alpha,\beta,\gamma)$.

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