Commun. Korean Math. Soc. **38** (2023), No. 4, pp. 1075–1090
https://doi.org/10.4134/CKMS.c220355
pISSN: 1225-1763 / eISSN: 2234-3024

SOME CLASSES OF OPERATORS RELATED TO (m,n)-PARANORMAL AND $(m,n)^*$-PARANORMAL OPERATORS

Shine Lal Enose, Ramya Perumal, and Prasad Thankarajan

Abstract. In this paper, we study new classes of operators k-quasi (m,n)-paranormal operator, k-quasi $(m,n)^*$-paranormal operator, k-quasi (m,n)-class $\mathcal{Q}$ operator and k-quasi (m,n)-class $\mathcal{Q}^*$ operator which are the generalization of (m,n)-paranormal and $(m,n)^*$-paranormal operators. We give matrix characterizations for k-quasi (m,n)-paranormal and k-quasi $(m,n)^*$-paranormal operators. Also we study some properties of k-quasi (m,n)-class $\mathcal{Q}$ operator and k-quasi (m,n)-class $\mathcal{Q}^*$ operators. Moreover, these classes of composition operators on L^2 spaces are characterized.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators defined on an infinite dimensional complex separable Hilbert space $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the null space and range of T, respectively. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if $TT^* \leq T^*T$, *paranomal* if $\|Tx\|^2 \leq \|T^2x\|\|x\|$, $*$-*paranomal* if $\|T^*x\|^2 \leq \|T^2x\|\|x\|$, n^*-*paranomal* if $\|T^*x\|^n \leq \|T^nx\|\|x\|^{n-1}$ for all $x \in \mathcal{H}$ and *class* $\mathcal{Q}$ if $T^{*2}T^2 - 2T^*T + I \geq 0$ ([3, 4, 14]). An operator $T \in \mathcal{B}(\mathcal{H})$ is called a *class* $\mathcal{Q}^*$ operator if $T^{*2}T^2 - 2TT^* + I \geq 0$ [16]. It is well known that all paranormal operators are of class $\mathcal{Q}$ and all $*$-paranormal operators are in class $\mathcal{Q}^*$. For $m \in \mathbb{R}^+$, $n \geq 1$, $T \in \mathcal{B}(\mathcal{H})$ is called (m,n)-*paranomal* if $\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$ and $(m,n)^*$-*paranomal* if $\|T^*x\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$ ([2]). As an extension of (m,n)-paranomal and $(m,n)^*$-paranomal the authors studied (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ operators ([12]). An operator T is called (m,n)-*class* $\mathcal{Q}$ if $\|Tx\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right)$ for every $x \in \mathcal{H}$, and (m,n)-*class* $\mathcal{Q}^*$

Received December 8, 2022; Revised May 22, 2023; Accepted August 2, 2023.
2010 *Mathematics Subject Classification.* 47B20, 47B38.
Key words and phrases. k-quasi (m,n)-paranormal operator, k-quasi (m,n)-class $\mathcal{Q}$ operator, k-quasi (m,n)-class $\mathcal{Q}^*$ operator, composition operators.
The third author is supported by seed money project grant UO.No. 11874/2021/Admn, University of Calicut.

if $\|T^*x\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right)$ for every $x \in \mathcal{H}$ ([12]). The following inclusion holds:

$$\text{hyponormal} \subseteq \text{paranormal} \subseteq (m,n)\text{-paranormal} \subseteq (m,n)\text{-class } Q,$$

$$(m,n)^*\text{-paranormal} \subseteq (m,n)\text{-class } Q^*$$

see ([2, 4, 12]).

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. A transformation T is said to be measurable if $T^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{A}$. If T is a nonsingular measurable transformation on $(X, \mathcal{A}, \mu)$ and the Randon-Nikodym derivative $\frac{d\mu T^{-1}}{d\mu}$ denoted by h, is essentially bounded, then the composition operator C_T on $L^2(\mu)$ is defined by $C_T f = f \circ T$, $f \in L^2(\mu)$ [15]. Let $L^\infty(\mu)$ denote the space of all essentially bounded complex valued measurable functions on X. For $\pi \in L^\infty(\mu)$, the multiplication operator M_π on $L^2(\mu)$ is given by $M_\pi f = \pi f$, $f \in L^2(\mu)$. The weighted composition operator W on $L^2(X, \mathcal{A}, \mu)$ induced by T and a complex valued measurable function π is given by

$$W = \pi(f \circ T)$$

for $f \in L^2(\mu)$. Let π_k denote $\pi(\pi \circ T)(\pi \circ T^2) \cdots (\pi \circ T^{k-1})$. Then, $W^k(f) = \pi_k (f \circ T)^k$ [11]. More details on general properties of (measure based) composition operators can be found in [10, 15]. The conditional expectation operator $E(\cdot|_{T^{-1}(\mathcal{A})}) = E(f)$ is defined for each non-negative function $f \in L^p(\mu)$, $1 \leq p < \infty$ and is uniquely determined by the conditions

(i) $E(f)$ is $T^{-1}(\mathcal{A})$ measurable.
(ii) If B is any $T^{-1}(\mathcal{A})$ measurable set for which $\int_B f d\mu$ converges, then $\int_B f d\mu = \int_B E(f) d\mu$.

The conditional expectation operator E satisfies the following:

For $f, g \in L^2(\mu)$,

(i) $E(g) = g$ if and only if g is $T^{-1}(\mathcal{A})$ measurable.
(ii) If g is $T^{-1}(\mathcal{A})$ measurable, then $E(fg) = E(f)g$.
(iii) $E(fg \circ T) = (E(f))(g \circ T)$ and $E(E(f)g) = E(f)E(g)$.
(iv) $E(1) = 1$, E is the identity operator in $L^2(\mu)$ if and only if $T^{-1}(\mathcal{A}) = \mathcal{A}$, and E is the projection operator from $L^2(\mu)$ onto $\overline{C(L^2(\mu))}$.

We refer the reader to [1, 8, 9, 13] for more details on the properties of conditional expectation.

In this paper, we introduce new classes of operators, k-quasi (m,n)-class $\mathcal{Q}$ and k-quasi (m,n)-class $\mathcal{Q}^*$. In ([2]) P. Dharmarha and S. Ram introduced (m,n)-*paranomal* and $(m,n)^*$-*paranomal* operators and studied some of its properties. Here we consider *k-quasi* $(m,n)^*$-*paranomal* operators which include $(m,n)^*$-*paranomal* operators. Note that *k-quasi* $(m,n)^*$-*paranomal* operators are k-quasi (m,n)-class $\mathcal{Q}^*$ operators.

2. k-quasi $(m,n)^*$-paranormal operators and k-quasi (m,n)-paranormal operators

In this section we study some properties of k-quasi $(m,n)^*$-paranormal operators and k-quasi (m,n)-paranormal operators. Also we give matrix characterizations of these classes of operators.

Definition 2.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be k-quasi (m,n)-paranormal if

$$\|TT^kx\|^{n+1} \leq m\|T^{n+1}T^kx\|\|T^kx\|^n \text{ for all } x \in \mathcal{H}.$$

From the definition, it is clear that every (m,n)-paranormal operators are k-quasi (m,n)-paranormal operators. The reverse inclusion need not be true in general. For example, if $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then T is a k-quasi (m,n)-paranormal operator for $k \geq 2$. But T is not $(25,3)$-paranormal operator.

Definition 2.2. Let $m \in \mathbb{R}^+$ and $n,k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be k-quasi $(m,n)^*$-paranormal if

$$\|T^*T^kx\|^{n+1} \leq m\|T^{n+1}T^kx\|\|T^kx\|^n \text{ for all } x \in \mathcal{H}.$$

In particular if $k = 0$ and $m = 1$, then this class of operators coincides with the class of n^*-paranormal operators [14]. If $k = 0$ and $m = n = 1$, then k-quasi $(m,n)^*$-paranormal operators coincide with $*$-paranormal operators. The following example shows that there is an operator which is k-quasi $(m,n)^*$-paranormal but not $(m,n)^*$-paranormal, That is, the class of k-quasi $(m,n)^*$-paranormal operators is larger than the class of $(m,n)^*$-paranormal operators.

Example 2.3. Let

$$T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If $k \geq 2$, then T is a k-quasi $(m,n)^*$-paranormal operator. But T is not $(25,3)^*$-paranormal.

Now, we give some properties of k-quasi $(m,n)^*$-paranormal operators.

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is k-quasi $(m,n)^*$-paranormal if and only if*

$$(2.1)\quad m^{\frac{2}{n+1}}T^{*k}T^{*n+1}T^{n+1}T^k-(n+1)\ a^nT^{*k}TT^*T^k+m^{\frac{2}{n+1}}\ n\ a^{n+1}T^{*k}T^k \geq 0$$

for all $a \geq 0$.

Proof. Suppose that T is a k-quasi $(m,n)^*$-paranormal operator. Then by the definition,

$$m^{\frac{2}{n+1}}\langle |T^{n+1}|^2T^kx, T^kx\rangle^{\frac{1}{n+1}}\ \langle T^kx, T^kx\rangle^{\frac{n}{n+1}} \geq \langle |T^*|^2T^kx, T^kx\rangle, \forall x \in \mathcal{H}.$$

By the generalized arithmetic-geometric mean inequality, it follows that

$$\frac{1}{n+1}\langle a^{-n}m^{\frac{2}{n+1}}|T^{n+1}|^2T^kx, T^kx\rangle + \frac{n}{n+1}\langle a\ m^{\frac{2}{n+1}}T^kx, T^kx\rangle$$

$$\geq \langle a^{-n} m^{\frac{2}{n+1}} |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}} \ \langle a\ m^{\frac{2}{n+1}} T^k x, T^k x\rangle^{\frac{n}{n+1}}$$
$$= m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}} \ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}}$$
$$\geq \langle |T^*|^2 T^k x, T^k x\rangle.$$

Thus,

$$\frac{a^{-n}}{n+1}\ m^{\frac{2}{n+1}} \langle T^{*k} T^{*n+1} T^{n+1} T^k x, x\rangle + \frac{na}{n+1}\ m^{\frac{2}{n+1}} \langle T^{*k} T^k x, x\rangle$$
$$- \langle T^{*k} T T^* T^k x, x\rangle \geq 0.$$

Hence,

$$m^{\frac{2}{n+1}} T^{*k} T^{*n+1} T^{n+1} T^k - (n+1)\ a^n T^{*k} T T^* T^k + m^{\frac{2}{n+1}}\ n\ a^{n+1} T^{*k} T^k \geq 0$$

for all $a \geq 0$. Conversely, suppose that (2.1) holds. Let $x \in \mathcal{H}$ with $\langle T^{*k} T^{*n+1} T^{n+1} T^k x, x\rangle = 0$. From (2.1),

$$m^{\frac{2}{n+1}}\ n\ a\langle T^{*k} T^k x, x\rangle - (n+1)\langle T^{*k} T T^* T^k x, x\rangle \geq 0.$$

Letting $a \to 0$, we get $\langle T^{*k} T T^* T^k x, x\rangle = 0$. Hence

$$m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}} \ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \geq \langle |T^*|^2 T^k x, T^k x\rangle.$$

For $x \in \mathcal{H}$ with $\langle T^{*k} T^{*n+1} T^{n+1} T^k x, x\rangle > 0$, by taking

$$a = \left(\frac{\langle T^{*k} T^{*n+1} T^{n+1} T^k x, x\rangle}{\langle T^{*k} T^k x, x\rangle} \right)^{\frac{1}{n+1}}$$

in (2.1), we get

$$m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}} \ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \geq \langle |T^*|^2 T^k x, T^k x\rangle.$$

Hence, T is k-quasi $(m,n)^*$-paranormal. □

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi $(m,n)^*$-paranormal operator and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under T. Then $T|_{\mathcal{M}}$ is a k-quasi $(m,n)^*$-paranormal operator.*

Proof. Let $B = T\ |_{\mathcal{M}}$ and P be the orthogonal projection on to $\mathcal{M}$. Then $TP = PTP$. Hence, $B^{*j} B^j = PT^{*j} T^j P$ for all $j \in \mathbb{N}$. Since T is a k-quasi $(m,n)^*$-paranormal operator, we have

$$m^{\frac{2}{n+1}} B^{*k} B^{*n+1} B^{n+1} B^k - (n+1)\ a^n B^{*k} B B^* B^k + m^{\frac{2}{n+1}}\ n\ a^{n+1} B^{*k} B^k$$
$$= PT^{*k}(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) a^n TPT^* + m^{\frac{2}{n+1}} n a^{n+1} I) T^k P$$
$$\geq PT^{*k}(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) a^n TT^* + m^{\frac{2}{n+1}} n a^{n+1} I) T^k P \geq 0.$$

Hence, $T|_{\mathcal{M}}$ is a k-quasi $(m,n)^*$-paranormal operator. □

Theorem 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$.*

(i) *If T is a $(k+1)$-quasi $(m,n)^*$-paranormal operator, then T is a k-quasi $(m,n+1)$-paranormal operator.*

(ii) *If T is a k-quasi $(m, n)^*$-paranormal operator, then T is a $(k+1)$-quasi $(m, n)^*$-paranormal operator.*

Proof. (i) Suppose that T is a $(k+1)$-quasi $(m, n)^*$-paranormal operator. Then

$$\|T^*T^{k+1}x\|^{n+1} \leq m\|T^{n+1}T^{k+1}x\|\|T^{k+1}x\|^n \quad \text{for all } x \in \mathcal{H}.$$

Now,

$$\begin{aligned}\|T^{k+1}x\|^{2n+2} &= \langle T^*T^{k+1}x, T^kx\rangle^{n+1} \\ &\leq \|T^*T^{k+1}x\|^{n+1}\|T^kx\|^{n+1} \\ &\leq m\|T^{n+1}T^{k+1}x\|\|T^{k+1}x\|^n\|T^kx\|^{n+1}.\end{aligned}$$

Thus, $\|TT^kx\|^{n+2} \leq m\|T^{n+2}T^kx\|\|T^kx\|^{n+1}$ for all $x \in \mathcal{H}$. Hence, T is a k-quasi $(m, n+1)$-paranormal operator.

(ii) Assume that T is a k-quasi $(m, n)^*$-paranormal operator. Then

$$\|T^*T^kx\|^{n+1} \leq m\|T^{n+1}T^kx\|\|T^kx\|^n \quad \text{for all } x \in \mathcal{H}.$$

Then for $x = Tu$, we get

$$\|T^*T^{k+1}u\|^{n+1} \leq m\|T^{n+1}T^{k+1}u\|\|T^{k+1}u\|^n \text{ for all } u \in \mathcal{H}.$$

Hence, T is a $(k+1)$-quasi $(m, n)^*$-paranormal operator. □

The following theorem gives matrix representation for k-quasi $(m, n)^*$-paranormal operators in terms of $(m, n)^*$-paranormal operators.

Theorem 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\overline{\mathcal{R}(T^k)} \neq \mathcal{H}$. If T is a k-quasi $(m, n)^*$-paranormal operator, then*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k}),$$

where A is an $(m, n)^*$-paranormal operator on $\overline{\mathcal{R}(T^k)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Assume that T is a k-quasi $(m, n)^*$-paranormal operator. Then

$$\|T^*T^kx\|^{n+1} \leq m\|T^{n+1}T^kx\|\|T^kx\|^n \quad \text{for all } x \in \mathcal{H}.$$

Put $T^kx = z$ in the above equation we get

$$\|T^*z\|^{n+1} \leq m\|T^{n+1}z\|\|z\|^n.$$

Since $\mathcal{R}(T^k)$ is not dense in $\mathcal{H}$, $T = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ on $\overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$, where $A = T|_{\overline{\mathcal{R}(T^k)}}$. Therefore, $\|A^*z\|^{n+1} \leq m\|A^{n+1}z\|\|z\|^n$ for all $z \in \overline{\mathcal{R}(T^k)}$. Hence, A is an $(m, n)^*$-paranormal operator on $\overline{\mathcal{R}(T^k)}$. Let $x \in \mathcal{N}(T^{*k})$. Then

$$T^k(x) = \begin{pmatrix} A^k & \sum_{i=0}^{k-1} A^iBC^{k-1-i} \\ 0 & C^k \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix} \in \overline{\mathcal{R}(T^k)}.$$

By ([6, Corollary 7]), $C^k = 0$. Also $\sigma(T) = \sigma(A) \cup \{0\}$. □

Now we give some characterizations of k-quasi (m, n)-paranormal operators.

Theorem 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is k-quasi (m, n)-paranormal if and only if*

$$m^{\frac{2}{n+1}} T^{*k} T^{*n+1} T^{n+1} T^k - (n+1)\ a^n T^{*k} T^* T T^k + m^{\frac{2}{n+1}}\ n\ a^{n+1} T^{*k} T^k \geq 0 \tag{2.2}$$

for all $a \geq 0$.

Proof. Suppose that T is a k-quasi (m, n)-paranormal operator. Then by the definition,

$$m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}}\ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \geq \langle |T|^2 T^k x, T^k x\rangle, \forall x \in \mathcal{H}.$$

By the generalized arithmetic-geometric mean inequality, it follows that

$$\frac{1}{n+1}\langle a^{-n} m^{\frac{2}{n+1}} |T^{n+1}|^2 T^k x, T^k x\rangle + \frac{n}{n+1}\langle a\ m^{\frac{2}{n+1}} T^k x, T^k x\rangle \geq \langle |T|^2 T^k x, T^k x\rangle.$$

Hence,

$$m^{\frac{2}{n+1}} T^{*k} T^{*n+1} T^{n+1} T^k - (n+1)\ a^n T^{*k} T^* T T^k + m^{\frac{2}{n+1}}\ n\ a^{n+1} T^{*k} T^k \geq 0$$

for all $a \geq 0$. Assume that (2.2) holds. Let $x \in \mathcal{H}$ be such that

$$\langle T^{*k} T^{*n+1} T^{n+1} T^k x, x\rangle = 0.$$

Then from (2.2),

$$m^{\frac{2}{n+1}}\ n\ a\langle T^{*k} T^k x, x\rangle - (n+1)\langle T^{*k} T^* T T^k x, x\rangle \geq 0$$

for every $a \geq 0$. Letting $a \to 0$, we get $\langle T^{*k} T^* T T^k x, x\rangle = 0$. Hence

$$m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}}\ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \geq \langle |T|^2 T^k x, T^k x\rangle.$$

Now, let $x \in \mathcal{H}$ be such that $\langle T^{*k} T^{*n+1} T^{n+1} T^k x, x\rangle > 0$. Put

$$a = \left(\frac{\langle T^{*k} T^{*n+1} T^{n+1} T^k x, x\rangle}{\langle T^{*k} T^k x, x\rangle}\right)^{\frac{1}{n+1}}$$

in (2.1), we get

$$m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}}\ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \geq \langle |T|^2 T^k x, T^k x\rangle.$$

Hence, T is k-quasi (m, n)-paranormal. ☐

The following theorem gives matrix characterizations for k-quasi (m, n)-paranormal operators in terms of (m, n)-paranormal operators.

Theorem 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{R}(T^k)$ is not dense in $\mathcal{H}$. The following are equivalent:*

1. *T is a k-quasi (m, n)-paranormal operator.*
2. *$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$, where A is an (m, n) paranormal operator on $\overline{\mathcal{R}(T^k)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.*

Proof. Assume that T is a k-quasi (m,n)-paranormal operator. Since $\mathcal{R}(T^k)$ is not dense in $\mathcal{H}$, $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$. Since T is a k-quasi (m,n)-paranormal operator, we have

$$\begin{aligned} &\langle (m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)\ a^n T^*T + m^{\frac{2}{n+1}}\ n\ a^{n+1} I)x, x\rangle \\ &= \langle (m^{\frac{2}{n+1}} A^{*n+1}A^{n+1} - (n+1)\ a^n A^*A + m^{\frac{2}{n+1}}\ n\ a^{n+1} I)x, x\rangle \geq 0 \end{aligned}$$

for all $x \in \overline{\mathcal{R}(T^k)}$. Hence, A is an (m,n)-paranormal operator on $\overline{\mathcal{R}(T^k)}$. Let $x \in \mathcal{N}(T^{*k})$. Then

$$T^k(x) = \begin{pmatrix} A^k & \sum_{i=0}^{k-1} A^i B C^{k-1-i} \\ 0 & C^k \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} \in \overline{\mathcal{R}(T^k)}.$$

Hence, $C^k = 0$. By ([6, Corollary 7]), $\sigma(T) = \sigma(A) \cup \{0\}$. Conversely, let $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$, where A is an (m,n)-paranormal operator on $\overline{\mathcal{R}(T^k)}$ and $C^k = 0$. Thus

$$T^k = \begin{pmatrix} A^k & \sum_{i=0}^{k-1} A^i B C^{k-1-i} \\ 0 & 0 \end{pmatrix}$$

and $T^kT^{*k} = \begin{pmatrix} A^kA^{*k} + \sum_{i=0}^{k-1} A^iBC^{k-1-i}(\sum_{i=0}^{k-1} A^iBC^{k-1-i})^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$, where $S = A^kA^{*k} + \sum_{i=0}^{k-1} A^iBC^{k-1-i}(\sum_{i=0}^{k-1} A^iBC^{k-1-i})^*$. Since A is an (m,n) paranormal operator, we have

$$\begin{aligned} &T^kT^{*k}(m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)a^n T^*T + m^{\frac{2}{n+1}} n a^{n+1} I)T^kT^{*k} \\ &= \begin{pmatrix} S(m^{\frac{2}{n+1}} A^{*n+1}A^{n+1} - (n+1)a^n A^*A + m^{\frac{2}{n+1}} n a^{n+1} I)S & 0 \\ 0 & 0 \end{pmatrix} \geq 0. \end{aligned}$$

Let $D = T^{*k}(m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)a^n T^*T + m^{\frac{2}{n+1}} n a^{n+1} I)T^k$. Then $T^kDT^{*k} \geq 0$. Let $x \in \mathcal{H}$. Then $x = y + z$, where $y \in \overline{\mathcal{R}(T^{*k})}$, $z \in \mathcal{N}(T^k)$. Since $y \in \overline{\mathcal{R}(T^{*k})}$, there exists a sequence (x_n) in $\mathcal{H}$ such that $T^{*k}(x_n) \to y$. Since $z \in \mathcal{N}(T^k)$, $Dz = 0$ and $\langle Dx, x\rangle = \langle Dy, y\rangle \geq 0$. Hence, T is a k-quasi (m,n)-paranormal operator. □

3. k-quasi (m,n)-class $\mathcal{Q}$ and k-quasi (m,n)-class $\mathcal{Q}^*$ operators

In this section, we study some extensions of k-quasi (m,n)-paranormal and k-quasi $(m,n)^*$-paranormal operators namely k-quasi (m,n)-class $\mathcal{Q}$ and k-quasi (m,n)-class $\mathcal{Q}^*$. In ([12]), the authors studied (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ operators. It is evident that these classes are independent.

Definition 3.1. Let $m \in \mathbb{R}^+$ and $n, k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a k-quasi (m,n)-class $\mathcal{Q}$ operator if

$$T^{*k}\left(m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)T^k \geq 0.$$

In particular if $k = 1$, then T is said to be a quasi (m, n)-class $\mathcal{Q}$ operator. If $m = n = 1$, then this class of operators coincides with k-quasi class $\mathcal{Q}$ operators [5].

Definition 3.2. Let $m \in \mathbb{R}^+$ and $n, k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a k-quasi (m, n)-class $\mathcal{Q}^*$ operator if

$$T^{*k}\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)TT^* + m^{\frac{2}{n+1}}\ n\ I\right)T^k \geq 0.$$

In particular if $k = 1$, then T is said to be a *quasi* (m, n)-class $\mathcal{Q}^*$ operator.

Now we give some characterizations of k-quasi (m, n)-class $\mathcal{Q}$ operators.

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is a k-quasi (m, n)-class $\mathcal{Q}$ operator if and only if $\frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{k+n+1}x\|^2 + n\|T^kx\|^2\right) \geq \|T^{k+1}x\|^2$ for all $x \in \mathcal{H}$.*

Proof. Let T be a k-quasi (m, n)-class $\mathcal{Q}$ operator. By definition, we have

$$\langle T^{*k}\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)T^kx, x\rangle \geq 0\ \forall x \in \mathcal{H}.$$

Therefore,

$$\begin{aligned}&m^{\frac{2}{n+1}}\ \langle T^{*k+n+1}T^{k+n+1}x, x\rangle - (n+1)\ \langle T^{*k+1}T^{k+1}x, x\rangle\\ &+ m^{\frac{2}{n+1}}\ n\ \langle T^{*k}T^kx, x\rangle\ \geq 0\end{aligned}$$

if and only if $\frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{k+n+1}x\|^2 + n\|T^kx\|^2\right) \geq \|T^{k+1}x\|^2$ for all $x \in \mathcal{H}$. □

Theorem 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\lambda^{\frac{-m}{n+1}}T$ is a k-quasi (m, n)-class $\mathcal{Q}$ operator, for all $\lambda > 0$ if and only if T is k-quasi (m, n)-paranormal.*

Proof. Let $\lambda^{\frac{-m}{n+1}}T$ be a k-quasi (m, n)-class $\mathcal{Q}$ operator, for all $\lambda > 0$. Then, by definition, we have

$$\begin{aligned}&(\lambda^{\frac{-m}{n+1}}T)^{*k}\left[m^{\frac{2}{n+1}}(\lambda^{\frac{-m}{n+1}}T)^{*n+1}(\lambda^{\frac{-m}{n+1}}T)^{n+1} - (n+1)\lambda^{\frac{-2m}{n+1}}T^*T + m^{\frac{2}{n+1}}n\ I\right]\\ &(\lambda^{\frac{-m}{n+1}}T)^k \geq 0,\ \lambda > 0.\end{aligned}$$

Then

$$\begin{aligned}&(\lambda^{\frac{-2mk}{n+1}})T^{*k}\left[m^{\frac{2}{n+1}}\lambda^{-2m}T^{*n+1}T^{n+1} - (n+1)\lambda^{\frac{-2m}{n+1}}T^*T + m^{\frac{2}{n+1}}nI\right]\\ &T^k \geq 0,\ \lambda > 0\\ \Leftrightarrow\ &(\lambda^{\frac{-2mk}{n+1}})\lambda^{-2m}T^{*k}\left[m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\lambda^{\frac{2mn}{n+1}}T^*T + m^{\frac{2}{n+1}}n\lambda^{2m}I\right]\\ &T^k \geq 0,\ \lambda > 0\\ \Leftrightarrow\ &T^{*k}\left[m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)(\lambda^{\frac{2m}{n+1}})^nT^*T + m^{\frac{2}{n+1}}n(\lambda^{\frac{2m}{n+1}})^{n+1}I\right]\\ &T^k \geq 0,\ \lambda > 0\\ \Leftrightarrow\ &T^{*k}\left[m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)a^nT^*T + m^{\frac{2}{n+1}}na^{n+1}I\right]T^k \geq 0,\ a > 0\end{aligned}$$

if and only if T is a k-quasi (m,n)-paranormal operator. □

Theorem 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi (m,n)-class $\mathcal{Q}$ operator and $A \in \mathcal{B}(\mathcal{H})$ be an isometric operator such that $AT = TA$. Then TA is a quasi (m,n)-class $\mathcal{Q}$ operator.*

Proof. Let $S = TA$. Since $AT = TA$, $A^*A = I$ and T is a quasi (m,n)-class $\mathcal{Q}$ operator, we have

$$\begin{aligned}
&m^{\frac{2}{n+1}} S^{*n+2}S^{n+2} - (n+1)S^{*2}S^2 + m^{\frac{2}{n+1}}\ n\ S^*S \\
&= m^{\frac{2}{n+2}} (A^*T^*)^{n+2}(TA)^{n+2} - (n+1)(A^*T^*)^2(TA)^2 + m^{\frac{2}{n+1}}\ n\ A^*T^*TA \\
&= m^{\frac{2}{n+1}} T^{*n+2}T^{n+2} - (n+1)T^{*2}T^2 + m^{\frac{2}{n+1}}\ n\ T^*T \geq 0.
\end{aligned}$$

Hence $S = TA$ is a quasi (m,n)-class $\mathcal{Q}$ operator. □

Theorem 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi (m,n)-class $\mathcal{Q}$ operator and T is unitarily equivalent to an operator $B \in \mathcal{B}(\mathcal{H})$. Then B is a k-quasi (m,n)-class $\mathcal{Q}$ operator.*

Proof. Since T is unitarily equivalent to B, there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $B = U^*TU$.

Now

$$\begin{aligned}
&B^{*k}\left(m^{\frac{2}{n+1}} B^{*n+1}B^{n+1} - (n+1)B^*B + m^{\frac{2}{n+1}}\ n\ I\right)B^k \\
&= U^*T^{*k}U\left[m^{\frac{2}{n+1}} U^*T^{*n+1}T^{n+1}U - (n+1)U^*T^*TU + m^{\frac{2}{n+1}}\ nU^*U\right]U^*T^kU \\
&= U^*T^{*k}U\left[U^*\left(m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ nI\right)U\right]U^*T^kU \\
&= U^*T^{*k}\left(m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)T^kU.
\end{aligned}$$

Since T is a quasi (m,n)-class $\mathcal{Q}$ operator, we get

$$U^*T^{*k}\left(m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)T^kU \geq 0.$$

Hence, B is a k-quasi (m,n)-class $\mathcal{Q}$ operator. □

Theorem 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi (m,n)-class $\mathcal{Q}$ operator. If $\overline{\mathcal{R}(T^k)} = \mathcal{H}$, then T is an (m,n)-class $\mathcal{Q}$ operator.*

Proof. Let $y \in \mathcal{H}$. Since $\overline{\mathcal{R}(T^k)} = \mathcal{H}$, there exists a sequence (x_i) in $\mathcal{H}$ such that $T^k(x_i)$ converges to $y \in \mathcal{H}$. Since T is a k-quasi (m,n)-class $\mathcal{Q}$ operator,

$$\left\langle\left[T^{*k}\left(m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)T^k\right]x_i, x_i\right\rangle \geq 0.$$

Then, $\langle (m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I)T^kx_i, T^kx_i\rangle \geq 0$. Hence, $\langle (m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I)y, y\rangle \geq 0$. That is, T is an (m,n)-class $\mathcal{Q}$ operator. □

Theorem 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi (m, n)-class $\mathcal{Q}$ operator and $\overline{\mathcal{R}(T^k)} \neq \mathcal{H}$. If*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k}),$$

then A is an (m, n)-class $\mathcal{Q}$ operator on $\overline{\mathcal{R}(T^k)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Since T is a k-quasi (m, n)-class $\mathcal{Q}$ operator, we have

$$m^{\frac{2}{n+1}} \left(\|T^{k+n+1}y\|^2 + n\|T^k y\|^2 \right) \geq (n+1)\|T^{k+1}y\|^2.$$

Let $z = T^k y$. Then

$$m^{\frac{2}{n+1}} \left(\|T^{n+1}z\|^2 + n\|z\|^2 \right) \geq (n+1)\|Tz\|^2.$$

Since $A = T|_{\overline{\mathcal{R}(T^k)}}$, $m^{\frac{2}{n+1}} \left(\|A^{n+1}z\|^2 + n\|z\|^2 \right) \geq (n+1)\|Az\|^2$ for all $z \in \overline{\mathcal{R}(T^k)}$. Hence, A is an (m, n)-class $\mathcal{Q}$ operator on $\overline{\mathcal{R}(T^k)}$. Let $x \in \mathcal{N}(T^{*k})$. Then

$$T^k(x) = \begin{pmatrix} A^k & \sum_{i=0}^{k} A^i B C^{k-1-i} \\ 0 & C^k \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} \in \overline{\mathcal{R}(T^k)}.$$

Hence, $C^k = 0$. By ([6, Corollary 7]), we get $\sigma(T) = \sigma(A) \cup \{0\}$. □

Now we give some characterizations of k-quasi (m, n)-class $\mathcal{Q}^*$ operators.

Theorem 3.9. *Let $T \in \mathcal{B}(\mathcal{H})$. T is a k-quasi (m, n)-class $\mathcal{Q}^*$ operator if and only if $\frac{m^{\frac{2}{n+1}}}{n+1} \left(\|T^{k+n+1}x\|^2 + n\|T^k x\|^2 \right) \geq \|T^* T^k x\|^2$ for all $x \in \mathcal{H}$.*

Proof. The result follows by a similar argument as in Theorem 3.3. □

Theorem 3.10. *Let $T \in \mathcal{B}(\mathcal{H})$. $\lambda^{\frac{-m}{n+1}} T$ is a k-quasi (m, n)-class $\mathcal{Q}^*$ operator, for all $\lambda > 0$ if and only if T is a k-quasi $(m, n)^*$ paranormal operator.*

Proof. The result follows by a similar argument as in Theorem 3.4. □

It is clear that the following results hold for k-quasi (m, n)-class $\mathcal{Q}^*$ operators.

(i) If $T \in \mathcal{B}(\mathcal{H})$ is a quasi (m, n)-class $\mathcal{Q}^*$ operator and $A \in \mathcal{B}(\mathcal{H})$ is an isometric operator such that $AT = TA$, then TA is a quasi (m, n)-class $\mathcal{Q}^*$ operator.

(ii) If $T \in \mathcal{B}(\mathcal{H})$ is a quasi (m, n)-class $\mathcal{Q}^*$ operator and T is unitarily equivalent to an operator $B \in \mathcal{B}(\mathcal{H})$, then B is a k-quasi (m, n)-class $\mathcal{Q}^*$ operator.

(iii) If $T \in \mathcal{B}(\mathcal{H})$ is a k-quasi (m, n)-class $\mathcal{Q}^*$ operator and $\overline{\mathcal{R}(T^k)} = \mathcal{H}$, then T is an (m, n)-class $\mathcal{Q}^*$ operator.

4. k-quasi (m,n)-class $\mathcal{Q}$ and k-quasi (m,n)-class $\mathcal{Q}^*$ composition operators

In this section, we give measure theoretical characterizations of k-quasi (m,n)-class $\mathcal{Q}$ and k-quasi (m,n)-class $\mathcal{Q}^*$ composition operators on L^2-spaces. Study of these classes of operator in the view point of composition operator helps to create more examples for the above classes of operators.

Proposition 4.1 ([1,7])**.** *Let P be the projection from $L^2(X,\mathcal{A},\mu)$ onto $\overline{\mathcal{R}(C_T)}$. Then the following results holds for every $f \in L^2(\mu)$*

(i) $C_T^* f = h \cdot E(f) \circ T^{-1}$.
(ii) $C_T^k f = f \circ T^k$, $C_T^{*k} f = h_k E(f) \circ T^{-k}$.
(iii) $C_T C_T^* f = (h \circ T) P f$, $C_T^* C_T = hf$.

Theorem 4.2. *C_T is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if*

$$m^{\frac{2}{n+1}}(h_{k+n+1} + n\ h_k) \geq (n+1) h_{k+1}.$$

Proof. By definition, C_T is a k-quasi (m,n)-class $\mathcal{Q}$ if and only if

$$\left\langle (m^{\frac{2}{n+1}} C_T^{*k+n+1} C_T^{k+n+1} - (n+1) C_T^{*k+1} C_T^{k+1} + m^{\frac{2}{n+1}}\ n\ C_T^{*k} C_T^k) f, f \right\rangle \geq 0$$

for every $f \in L^2(\mu)$. Now

$$\begin{aligned} C_T^{*k+n+1} C_T^{k+n+1} f &= C_T^{*k+n+1}(f \circ T^{k+n+1}) \\ &= h_{k+n+1} E(f \circ T^{k+n+1}) \circ T^{-(k+n+1)} \\ &= h_{k+n+1} f. \end{aligned}$$

Also, $C_T^{*k+1} C_T^{k+1} f = h_{k+1} f$ and $C_T^{*k} C_T^k f = h_k f$. Hence, C_T is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}}(h_{k+n+1} + n\ h_k) \geq (n+1) h_{k+1}.$$

□

Theorem 4.3. *C_T^* is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if*

$$m^{\frac{2}{n+1}}\left(h_{k+n+1} \circ T^{k+n+1} + n\ h_k \circ T^k\right) \geq (n+1) h_{k+1} \circ T^{k+1}.$$

Proof. By definition, C_T^* is a k-quasi (m,n)-class $\mathcal{Q}$ if and only if

$$\left\langle (m^{\frac{2}{n+1}} C_T^{k+n+1} C_T^{*k+n+1} - (n+1) C_T^{k+1} C_T^{*k+1} + m^{\frac{2}{n+1}}\ n\ C_T^k C_T^{*k}) f, f \right\rangle \geq 0$$

for every $f \in L^2(\mu)$. We have

$$\begin{aligned} C_T^{k+n+1} C_T^{*k+n+1} f &= C_T^{k+n+1}\left(h_{k+n+1} E(f) \circ T^{-(k+n+1)}\right) \\ &= \left(h_{k+n+1}\ E(f) \circ T^{-(k+n+1)}\right) \circ T^{k+n+1} \\ &= h_{k+n+1} \circ T^{k+n+1} E(f) \\ &= h_{k+n+1} \circ T^{k+n+1} f. \end{aligned}$$

Similarly, we get $C_T^{k+1} C_T^{*k+1} f = h_{k+1} \circ T^{k+1} f$ and $C_T^k C_T^{*k} f = h_k \circ T^k f$.

Hence, C_T^* is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}}\left(h_{k+n+1}\circ T^{k+n+1}+n\ h_k\circ T^k\right)\geq (n+1)h_{k+1}\circ T^{k+1}.\qquad\square$$

Example 4.4. Let $X=\mathbb{N}\cup\{0\}$, $\mathcal{A}=P(X)$ and μ be the measure defined by

$$\mu(A)=\sum_{k\in A} m_k,$$

where

$$m_k=\begin{cases}1 & \text{if } k=0,\\ \frac{1}{4^{k-1}} & \text{if } k\geq 1.\end{cases}$$

Let $T:\,X\to X$ defined by

$$T(k)=\begin{cases}0 & k=0,1,\\ k-1 & k\geq 2.\end{cases}$$

Then for $q>1$, we have

$$T^q(k)=\begin{cases}0 & k=0,1,2,\dots,q,\\ k-q & k\geq q+1.\end{cases}$$

Therefore, $h(k)=\frac{\mu T^{-1}(\{k\})}{\mu\{k\}}=\begin{cases}2 & k=0,\\ \frac{1}{4} & k\geq 1.\end{cases}$

Then, for $q>1$ we have

$$h_q(k)=\begin{cases}2+\frac{1}{4}+\frac{1}{4^2}+\cdots+\frac{1}{4^{q-1}} & k=0,\\ \frac{1}{4^q} & k\geq 1.\end{cases}$$

If $m\geq 2$ and $n=3$, then $m^{\frac{1}{2}}(h_6+3h_2)\geq 4h_3$ for $k=2$. Hence C_T is a 2-quasi (m,n)-class $\mathcal{Q}$ operator.

Theorem 4.5. *Let C_T be the composition operator of T on $L^2(\mu)$. Then*

(i) *C_T is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if and only if*

$$m^{\frac{2}{n+1}}(h_{k+n+1}+n\ h_k)\geq (n+1)h_{k+1}\circ T^{k+1}.$$

(ii) *C_T^* is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if and only if*

$$m^{\frac{2}{n+1}}\left(h_{k+n+1}\circ T^{k+n+1}+n\ h_k\circ T^k\right)\geq (n+1)h_{k+1}.$$

Proof. (i) C_T is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$\left\langle (m^{\frac{2}{n+1}}C_T^{*k+n+1}C_T^{k+n+1}-(n+1)C_T^{k+1}C_T^{*k+1}+m^{\frac{2}{n+1}}\ n\ C_T^{*k}C_T^{k})f,f\right\rangle\geq 0$$

for every $f\in L^2(\mu)$. Now

$$\begin{aligned}C_T^{*k+n+1}C_T^{k+n+1}f&=C_T^{*k+n+1}(f\circ T^{k+n+1})\\&=h_{k+n+1}E(f\circ T^{k+n+1})\circ T^{-(k+n+1)}\\&=h_{k+n+1}f.\end{aligned}$$

Similarly, we get $C_T^{*k}C_T^k f=h_k f$. Also,

$$C_T^{k+1}C_T^{*k+1}f=C_T^{k+1}(h_{k+1}E(f)\circ T^{k+1})$$

$$= (h_{k+1}E(f)\circ T^{k+1})\circ T^{k+1}$$
$$= h_{k+1}\circ T^{k+1}f.$$

Hence, C_T is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$m^{\frac{2}{n+1}}(h_{k+n+1}+n\ h_k)\geq (n+1)h_{k+1}\circ T^{k+1}.$$

(ii) C_T^* is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if

$$\Big\langle (m^{\frac{2}{n+1}}C_T^{k+n+1}C_T^{*k+n+1}-(n+1)C_T^{*k+1}C_T^{k+1}+m^{\frac{2}{n+1}}\ n\ C_T^kC_T^{*k})f,f\Big\rangle \geq 0$$

for every $f\in L^2(\mu)$. Now,

$$\begin{aligned} C_T^{k+n+1}C_T^{*k+n+1}f &= C_T^{k+n+1}\Big(h_{k+n+1}E(f)\circ T^{-(k+n+1)}\Big)\\ &= \Big(h_{k+n+1}\ E(f)\circ T^{-(k+n+1)}\Big)\circ T^{k+n+1}\\ &= h_{k+n+1}\circ T^{k+n+1}E(f)\\ &= h_{k+n+1}\circ T^{k+n+1}f. \end{aligned}$$

Similarly, we get $C_T^kC_T^{*k}f=h_k\circ T^kf$. Also,

$$\begin{aligned} C_T^{*k+1}C_T^{k+1}f &= C_T^{*k+1}(f\circ T^{k+1})\\ &= h_{k+1}E(f\circ T^{k+1})\circ T^{-(k+1)}\\ &= h_{k+1}f. \end{aligned}$$

Hence, C_T^* is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$m^{\frac{2}{n+1}}\left(h_{k+n+1}\circ T^{k+n+1}+n\ h_k\circ T^k\right)\geq (n+1)h_{k+1}. \qquad \square$$

Example 4.6. If we choose $m\geq 2428$ and $n=3$ in Example 4.4, we see that C_T is a 2-quasi (m,n)-class $\mathcal{Q}^*$ operator.

Now we give characterizations for k-quasi (m,n)-class $\mathcal{Q}$ weighted composition operators on $L^2(\mu)$.

Proposition 4.7 ([1])**.** *If W is a weighted composition operator induced by T and π, then the following statements hold.*

(i) $W^*W(f)=hE(|\pi|^2)\circ T^{-1}(f)$,
(ii) $WW^*(f)=\pi(h\circ T)E(\overline{\pi}f)$,
(iii) $W^{*k}W^k(f)=h_kE_k(|\pi_k|^2)\circ T^{-k}(f)$ *and*
(iv) $W^kW^{*k}f=\pi_k(h_k\circ T^k)E(\bar{\pi}_kf)$,

where $\pi_k=\pi(\pi\circ T)(\pi\circ T^2)\cdots(\pi\circ T^{k-1})$ *and* E_k *is the conditional expectation.*

For $f\in L^2(\mu)$, let $J_kf=h_kE_k(|\pi_k|^2)\circ T^{-k}(f)$ and $L_kf=\pi_k(h_k\circ T^k)E(\bar{\pi}_kf)$.

Theorem 4.8. *Let W be the weighted composition operator induced by T on $L^2(\mu)$. Then W is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if*

$$m^{\frac{2}{n+1}}(J_{k+n+1}+nJ_k)\geq (n+1)J_{k+1}.$$

Proof. By definition, W is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if

$$\left\langle (m^{\frac{2}{n+1}} W^{*k+n+1}W^{k+n+1} - (n+1)W^{*k+1}W^{k+1} + m^{\frac{2}{n+1}}\ n\ W^{*k}W^{k})f, f \right\rangle \geq 0$$

for every $f \in L^2(\mu)$. Now,

$$\begin{aligned} W^{*k+n+1}W^{k+n+1}f &= h_{k+n+1}\ E_{k+n+1}(|\pi_{k+n+1}|^2) \circ T^{-(k+n+1)}f \\ &= J_{k+n+1}f. \end{aligned}$$

Also, $W^{*k+1}W^{k+1}f = h_{k+1}E_{k+1}(|\pi_{k+1}|^2) \circ T^{-(k+1)}f = J_{k+1}f$ and $W^{*k}W^{k}f = h_kE_k(|\pi_k|^2) \circ T^{-k}f = J_kf$. Hence, W is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if $m^{\frac{2}{n+1}}(J_{k+n+1} + nJ_k) \geq (n+1)J_{k+1}$. □

Theorem 4.9. *W^* is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if*

$$m^{\frac{2}{n+1}}(L_{k+n+1} + nL_k) \geq (n+1)L_{k+1}.$$

Proof. W^* is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if

$$\left\langle (m^{\frac{2}{n+1}} W^{k+n+1}W^{*k+n+1} - (n+1)W^{k+1}W^{*k+1} + m^{\frac{2}{n+1}}\ n\ W^{k}W^{*k})f, f \right\rangle \geq 0$$

for every $f \in L^2(\mu)$.

We have

$$W^{k+n+1}W^{*k+n+1}f = \pi_{k+n+1}(h_{k+n+1} \circ T^{k+n+1})E(\overline{\pi_{k+n+1}}f) = L_{k+n+1}f.$$

Similarly,

$$W^{k+1}W^{*k+1}f = \pi_{k+1}(h_{k+1} \circ T^{k+1})E(\overline{\pi_{k+1}}f) = L_{k+1}f$$

and

$$W^kW^{*k}f = \pi_k(h_k \circ T^k)E(\bar{\pi}_kf) = L_kf.$$

Hence, W^* is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}}(L_{k+n+1} + nL_k) \geq (n+1)L_{k+1}.$$ □

Theorem 4.10. *Let W be the weighted composition operator induced by T on $L^2(\mu)$. Then*

(i) *W is k-quasi (m,n)-class $\mathcal{Q}^*$ if and only if*

$$m^{\frac{2}{n+1}}(J_{k+n+1} + nJ_k) \geq (n+1)L_{k+1}.$$

(ii) *W^* is k-quasi (m,n)-class $\mathcal{Q}^*$ if and only if*

$$m^{\frac{2}{n+1}}(L_{k+n+1} + nL_k) \geq (n+1)J_{k+1}.$$

Proof. (i) W is k-quasi (m,n)-class $\mathcal{Q}^*$ if and only if

$$\left\langle (m^{\frac{2}{n+1}} W^{*k+n+1}W^{k+n+1} - (n+1)W^{k+1}W^{*k+1} + m^{\frac{2}{n+1}}\ n\ W^{*k}W^{k})f, f \right\rangle \geq 0$$

for every $f \in L^2(\mu)$. Now,

$$\begin{aligned} W^{*k+n+1}W^{k+n+1}f &= h_{k+n+1}\ E_{k+n+1}(|\pi_{k+n+1}|^2) \circ T^{-(k+n+1)}f \\ &= J_{k+n+1}f. \end{aligned}$$

Similarly,

$$W^{*k}W^kf = h_k\ E_k(|\pi_k|^2)\circ T^{-k}f = J_kf.$$

Also,

$$W^{k+1}W^{*k+1}f = \pi_{k+1}(h_{k+1}\circ T^{k+1})E(\overline{\pi_{k+1}}f) = L_{k+1}f.$$

Hence, W is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$m^{\frac{2}{n+1}}(J_{k+n+1}+nJ_k)\geq (n+1)L_{k+1}.$$

(ii) W^* is k-quasi (m,n)-class $\mathcal{Q}^*$ if and only if

$$\left\langle (m^{\frac{2}{n+1}}W^{k+n+1}W^{*k+n+1}-(n+1)W^{*k+1}W^{k+1}+m^{\frac{2}{n+1}}\ n\ W^kW^{*k})f,f\right\rangle \geq 0$$

for every $f\in L^2(\mu)$. Also,

$$W^{k+n+1}W^{*k+n+1}f = \pi_{k+n+1}(h_{k+n+1}\circ T^{k+n+1})E(\overline{\pi_{k+n+1}}f) = L_{k+n+1}f.$$

Similarly,

$$W^kW^{*k}f = \pi_k(h_k\circ T^k)E(\bar{\pi_k}f) = L_kf.$$

Also, $W^{*k+1}W^{k+1}f = h_{k+1}\ E_{k+1}(|\pi_{k+1}|^2)\circ T^{-(k+1)}f = J_{k+1}f$. Hence, W^* is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if and only if $m^{\frac{2}{n+1}}(L_{k+n+1}+nL_k)\geq (n+1)J_{k+1}$. □

References

[1] J. T. Campbell and J. E. Jamison, *On some classes of weighted composition operators*, Glasgow Math. J. **32** (1990), no. 1, 87–94. https://doi.org/10.1017/S0017089500009095

[2] P. Dharmarha and S. Ram, *(m,n)-paranormal operators and $(m,n)^*$-paranormal operators*, Commun. Korean Math. Soc. **35** (2020), no. 1, 151–159. https://doi.org/10.4134/CKMS.c180452

[3] B. P. Duggal, C. S. Kubrusly, and N. Levan, *Contractions of a class Q and invariant subspaces*, Bull. Korean Math. Soc. **42** (2005), no. 1, 169–177. https://doi.org/10.4134/BKMS.2005.42.1.169

[4] T. Furuta, *On the class of paranormal operators*, Proc. Japan Acad. **43** (1967), 594–598. http://projecteuclid.org/euclid.pja/1195521514

[5] V. R. Hamiti, *On k-quasi class Q operators*, Bull. Math. Anal. Appl. **6** (2014), no. 3, 31–37.

[6] J. K. Han, H. Y. Lee, and W. Y. Lee, *Invertible completions of 2×2 upper triangular operator matrices*, Proc. Amer. Math. Soc. **128** (2000), no. 1, 119–123. https://doi.org/10.1090/S0002-9939-99-04965-5

[7] D. J. Harrington and R. J. Whitley, *Seminormal composition operators*, J. Operator Theory **11** (1984), no. 1, 125–135.

[8] J. D. Herron, *Weighted conditional expectation operators*, Oper. Matrices **5** (2011), no. 1, 107–118. https://doi.org/10.7153/oam-05-07

[9] A. Lambert, *Hyponormal composition operators*, Bull. London Math. Soc. **18** (1986), no. 4, 395–400. https://doi.org/10.1112/blms/18.4.395

[10] E. A. Nordgren, *Composition operators on Hilbert spaces*, in Hilbert space operators (Proc. Conf., Calif. State Univ., Long Beach, Calif., 1977), 37–63, Lecture Notes in Math., 693, Springer, Berlin, 1978.

[11] S. Panayappan, *Non-hyponormal weighted composition operators*, Indian J. Pure Appl. Math. **27** (1996), no. 10, 979–983.

[12] P. Ramya, T. Prasad, and E. Shine Lal, *On (m,n)-class Q and (m,n)-class Q^* operators*, Palesteine J. Math. **12** (2023), no. 2, 353–360.

[13] M. M. Rao, *Conditional measure and applications*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, NewYork, 1993.

[14] M. H. M. Rashid, *On n-$*$-paranormal operators*, Commun. Korean Math. Soc. **31** (2016), no. 3, 549–565. https://doi.org/10.4134/CKMS.c150195

[15] R. K. Singh and J. S. Manhas, *Composition operators on function spaces*, North-Holland Mathematics Studies, 179, North-Holland, Amsterdam, 1993.

[16] Y. Yang and C. J. Kim, *Contractions of class Q^**, Far East J. Math. Sci. (FJMS) **27** (2007), no. 3, 649–657.

Shine Lal Enose
Department of Mathematics
University College, Thiruvananthapuram
Kerala, 695034, India
Email address: shinelal.e@gmail.com

Ramya Perumal
Department of Mathematics
N.S.S College, Nemmara
Kerala, 678508, India
Email address: ramyagcc@gmail.com

Prasad Thankarajan
Department of Mathematics
University of Calicut, Malapuram
Kerala, 673635, India
Email address: prasadvalapil@gmail.com