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MULTIPLICITY RESULTS OF CRITICAL LOCAL EQUATION RELATED TO THE GENUS THEORY

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Abstract. Using variational methods, Krasnoselskii's genus theory and symmetric mountain pass theorem, we introduce the existence and multiplicity of solutions of a parameteric local equation. At first, we consider the following equation

$$\begin{cases} -div[a(x,|\nabla u|)\nabla u] = \mu(b(x)|u|^{s(x)-2} - |u|^{r(x)-2})u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, μ is a positive real parameter, p, r and s are continuous real functions on $\bar{\Omega}$ and $a(x,\xi)$ is of type $|\xi|^{p(x)-2}$. Next, we study boundedness and simplicity of eigenfunction for the case $a(x,|\nabla u|)\nabla u = g(x)|\nabla u|^{p(x)-2}\nabla u$, where $g \in L^\infty(\Omega)$ and $g(x) \geq 0$ and the case $a(x,|\nabla u|)\nabla u = (1+\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u$ such that $p(x) \equiv p$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Our main result is focused on the solutions for the following boundary value problem:

$$(1)\qquad \begin{cases} -div[a(x,|\nabla u|)\nabla u] = \mu(b(x)|u|^{s(x)-2} - |u|^{r(x)-2})u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu > 0$ and $a(x,\xi)$ is of type $|\xi|^{p(x)-2}$, where p, r and s are continuous real functions on $\bar{\Omega}$ in which

$$(2)\qquad 1 < s(x) < r(x) < p(x) < p^*(x),$$

where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p(x) < N$ for all $x \in \bar{\Omega}$, and $b : \bar{\Omega} \to [0,\infty)$ is a continuous function which satisfies in

$$(3)\qquad b(x)^{\frac{r(x)}{r(x)-s(x)}} \in L^1(\Omega).$$

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$\Delta_{p(x)}$ denote the $p(x)$-Laplacian operator defined by

$$\Delta_{p(x)} z := div(|\nabla z|^{p(x)-2} \nabla z).$$

We refer to [6, 8, 9, 11, 14] for additional results in $p(x)$-Laplacian systems.

During the last two decades, $p(x)$-Laplacian problems have deserved the attention of many researchers. Moreover, the field of partial differential problems involving variable exponent condition is of interested topics of mathematician.

Recently, the study of the variable exponent Sobolov spaces has been an interesting topic. Materials requiring this theory have been investigated experimentally since the middle of the last century when the preoccupation for electrotech logical fluids arose. In [2], the authors investigated the following Kirchhoff type problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \triangle_{p(x)} u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Based on the Krasnoselskii's genus theory, they established the existence and multiplicity of solutions.

We also refer to the work of Allali et al. [5] which extends the problem (4) to biharmonic operator under Neumann boundary condition. Under the same methods of [2] they obtained the existence and multiplicity of solutions for problem (4). By using the mountain-pass theorem and the Nehari manifold technique, the authors in [17] studied the existence of solutions for problem (1).

It is studied in [1] that nonlinear eigenvalue problems for the p-Laplacian operator subject to different kinds of boundary conditions on a bounded domain. In fact they considered Dirichlet problem, no-flux problem, Neumann problem, Robin problem and Steklov problem and investigated the regularity of the first eigenvalue.

The operator $-div[\phi(x, |\nabla u|)\nabla u]$, when $\phi(x,t) = |t|^{p(x)-2}$ was studied by I. H. Kim and Y. H. Kim in [11], and many authors after it. For more details we refer to [17] and [15].

In this paper, we find at least $\gamma(l)$ pairs of different critical points for (1). First introduce some basic preliminary results.

2. Preliminaries

First we recall some definition and the necessary functional framework for the study of the problem (1), specially the Lebesgue and Sobolev spaces with variable exponent are given.

Set

$$C_+(\Omega) := \{s \in C(\bar{\Omega}) : \min_{x \in \bar{\Omega}} s(x) > 1\}.$$

For any $s \in C(\bar{\Omega})$, set

$$s^- := \inf_{x\in\Omega} s(x) \quad \text{and} \quad s^+ := \sup_{x\in\Omega} s(x).$$

Define

$$\begin{aligned} &L_{s(x)}(\Omega) \\ &:= \{u : \Omega \to \mathbb{R} \text{ is measurable} : \int_\Omega |u|^{s(x)} dx < +\infty \text{ for } s \in C_+(\bar{\Omega})\} \end{aligned}$$

and the norm

$$|u|_{s(x)} := \inf \left\{ \alpha > 0 : \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{s(x)} dx \leq 1 \right\}.$$

$L_{s(x)}(\Omega)$ is a separable and reflexive Banach space [7]. For more details about these variable exponent Lebesgue space see into [11,15,17].

The modular of the $L_{s(x)}(\Omega)$ is defined by the mapping $\sigma_{s(x)} : L^{s(x)}(\Omega) \to \mathbb{R}$, where

$$\sigma_{s(x)}(u) := \int_\Omega |u(x)|^{s(x)} dx.$$

Proposition 2.1 ([18]). *$(L_{s(x)}(\Omega), |u|_{s(x)})$ is a separable, uniformly convex, reflexive Banach space and its conjugate space is $(L_{s'(x)}(\Omega), |u|_{s'(x)})$, where*

$$\frac{1}{s(x)} + \frac{1}{s'(x)} = 1, \ \forall x \in \Omega.$$

Moreover, for any $u \in L_{s(x)}(\Omega)$ and $h \in L_{s'(x)}(\Omega)$, we have

$$\left| \int uh dx \right| \leq \left(\frac{1}{s^-} + \frac{1}{s'^-} \right) |u|_{s(x)} |h|_{s'(x)} \leq 2|u|_{s(x)}|h|_{s'(x)}. \tag{5}$$

Proposition 2.2 ([4]). *If $u, u_n \in L_{s(x)}(\Omega)$, we have*

$$|u|_{s(x)} < 1 \Rightarrow |u|_{s(x)}^{s^+} \leq \sigma_{s(x)}(u) \leq |u|_{s(x)}^{s^-}, \tag{6}$$

$$|u|_{s(x)} > 1 \Rightarrow |u|_{s(x)}^{s^-} \leq \sigma_{s(x)}(u) \leq |u|_{s(x)}^{s^+}, \tag{7}$$

$$|u|_{s(x)} < 1 \ (\mathit{resp}, \ = 1; > 1) \Leftrightarrow \sigma_{s(x)}(u) < 1 \ (\mathit{resp}, \ = 1; > 1), \tag{8}$$

$$|u_n|_{s(x)} \to 0 \ (\mathit{resp}, \ \to +\infty) \Leftrightarrow \sigma_{s(x)}(u_n) \to 0 \ (\mathit{resp}, \ \to +\infty), \tag{9}$$

$$\lim_{n\to\infty} |u_n - u|_{s(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \sigma_{s(x)}(u_n - u) = 0. \tag{10}$$

The Sobolev space $W^{1,s(x)}(\Omega)$ is defined by

$$W^{1,s(x)}(\Omega) := \left\{ u \in L_{s(x)}(\Omega) : |\nabla u| \in L_{s(x)}(\Omega) \right\},$$

which is a separable reflexive Banach space. For more details, we refer to [6,10,12,16].

$W^{1,s(x)}(\Omega)$ is equipped with the norm $\|u\|_{1,s(x)} = \|u\|_{s(x)} + \|\nabla u\|_{s(x)}$. On $W^{1,s(x)}(\Omega)$ we can consider the equivalent norms [16]:

$$\|u\|_{s(x)} = |\nabla u|_{s(x)}.$$

$W_0^{1,s(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with the norm:

$$\|u\| = \inf\left\{\alpha > 0 : \int_\Omega \left(\left|\frac{\nabla u(x)}{\alpha}\right|^{s(x)}\right) dx \leq 1\right\}.$$

It is well known that

$$W_0^{1,s(x)}(\Omega) := \left\{u : u\Big|_{\partial\Omega} = 0,\ u \in L^{s(x)}(\Omega),\ |\nabla u| \in L^{s(x)}(\Omega)\right\}.$$

For more details, we refer to [7, 17].

Proposition 2.3 (Sobolev Embedding [14]). *If $s, s' \in C_+(\bar{\Omega})$ and $1 < s'(x) < s^*(x)$ and $x \in \bar{\Omega}$, then we have the continuous embedding*

$$W_0^{1,s(x)}(\Omega) \hookrightarrow L_{s'(x)}(\Omega)$$

which is continuous and compact. There is a constant $C_0 > 0$ such that

$$\|u_n\|_{s'(x)} \leq C_0\|u_n\|.$$

Proposition 2.4 (Poincare Inequality [6]). *There is a constant $C > 0$ such that*

$$|u|_{s(x)} \leq C\|\nabla u\|_{s(x)} \tag{11}$$

for all $u \in W_0^{1,s(x)}(\Omega)$.

3. Existence results

Now we establish the existence of infinitely many weak solution to problem (1).

Definition 3.1. Let U be a real Banach space. Set

$$R := \{B \subset U - \{0\} : B \text{ is compact and } B = -B\}.$$

Let $B \in R$ and we define the *genus* of B as follows:

$$\gamma(B) := \inf\{m \geq 1, \exists f \in C(B, \mathbb{R}^m \setminus \{0\}) : f \text{ is odd}\}$$

and $\gamma(B) = \infty$ if does not exist such a map f. $\gamma(\emptyset) = 0$ by definition. For more details, we refer to [5].

(H_1) $a : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ is a continuous function such that

$$c_1 t^{p(x)-2} \leq a(x,t) \leq c_2 t^{p(x)-2}$$

for all $(t, x) \in]0, +\infty[\times\bar{\Omega}$, $c_1, c_2 > 0$ and $p \in C(\bar{\Omega})$ in which $1 < p(x) < p^*(x) < \frac{Np(x)}{N-p(x)}$ for all $x \in \bar{\Omega}$.

(H_2) $b : \bar{\Omega} \to [0, \infty)$ is a function in $L^\infty(\bar{\Omega})$.

Definition 3.2. $u \in W_0^{1,p(x)}(\Omega)$ is called a *weak solution* of (1) if

$$\begin{aligned}&\int_\Omega a(x,|\nabla u(x)|)\nabla u(x)\nabla h(x)dx\\ &= \mu\int_\Omega [b(x)|u(x)|^{s(x)-2}u(x)h(x) - |u(x)|^{r(x)-2}u(x)h(x)]dx\end{aligned}$$

for all $h \in W_0^{1,p(x)}(\Omega)$. In what follows

$$A_0(x,z) := \int_0^z a(x,t)tdt,$$

and

$$A : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$$

by

$$A(u) := \int_\Omega A_0(x,|\nabla u(x)|)dx.$$

Lemma 3.3 ([11])**.** *If the condition (H_1) is verified, then $A \in C^1(W_0^{1,p(x)}(\Omega),\mathbb{R})$ and we have*

$$\langle A'(u),h\rangle = \int_\Omega a(x,|\nabla u(x)|)\nabla u(x)\cdot\nabla h(x)dx.$$

Definition 3.4. The mapping a' is of type (S_+) if $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ as $n \to \infty$ and $\limsup_{n\to\infty}\langle a'(u_n) - a(u), u_n - u\rangle \le 0$ implies $u_n \to u$ in $W_0^{1,s(x)}(\Omega)$ strongly as $n \to \infty$.

Lemma 3.5 ([11])**.** *Assume that (H_1) hold. We can easily see that*

$$A' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$$

is strictly monotone and of (S_+) type.

Consider energy functional (1) by

$$\tau(u) = \int_\Omega A_0(x,|\nabla u|)dx - \mu\int_\Omega \frac{b(x)}{s(x)}|u|^{s(x)}dx + \mu\int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx.$$

From [11] and Lemma 3.2 in Chapter 3 of [15], $\tau(u)$ is well defined and C^1 on $W_0^{1,p(x)}(\Omega)$. We have

$$\begin{aligned}\tau'(u)(h) = &\int_\Omega a(x,|\nabla u(x)|)\nabla u(x)\cdot\nabla h(x)dx\\ &-\mu\int_\Omega b(x)|u(x)|^{s(x)-2}u(x)\cdot h(x) + \mu\int_\Omega |u(x)|^{r(x)-2}u(x)\cdot h(x)dx\end{aligned}$$

for all $u,h \in W_0^{1,p(x)}(\Omega)$.

Let $\Omega \subset \mathbb{R}^N (N > 3)$ be a bounded domain. Let μ be a positive real parameter, p, r, s continuous functions on $\bar{\Omega}$ in which

$$\begin{aligned} 1 < s^- \leq s(x) \leq s^+ < r^- \leq r(x) \leq r^+ \\ < p^- \leq p(x) \leq p^+ < p^*(x) = \frac{Ns(x)}{N - s(x)}, \end{aligned} \tag{12}$$

where $p(x) < N$ for any $x \in \bar{\Omega}$.

Definition 3.6. The functional τ is called that satisfies in the *Palais-Smale condition* at level c, $(PS)_c$ if for every sequence $(u_n) \subset W_0^{1,p(x)}(\Omega)$ satisfying $\tau(u_n) \to c$ and $\tau'(u_n) \to 0$ as $n \to \infty$, there is a convergence subsequence of (u_n).

Theorem 3.7 ([5])**.** *Let $\tau \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and*

(a) *$\tau(u)$ satisfies the Palais-Smale condition, $\tau(u)$ is even and bounded from below;*

(b) *There exists a $T \in R$ such that $\gamma(T) = m$ and $\sup_{x\in T} \tau(x) < \tau(0)$.*

Then problem (1) *has at least m pairs of distinct critical points, and their corresponding critical values are less than $\tau(0)$.*

Theorem 3.8. *Assuming that (H_1), (H_2) and* (12) *hold. Then there are at least m pairs of different critical point for* (1).

Lemma 3.9. *Under assumptions (H_1), (H_2) and* (12)*, τ is coercive on $W_0^{1,p(x)}(\Omega)$ and bounded from below.*

Proof. Considering $\omega := s(x)$, $\nu := r(x)$, $\pi := \frac{\mu}{r(x)}$, $e := u(x)$ and $\theta = \frac{2\mu b(x)}{s(x)}$ in the following inequality [15]

$$\theta|e|^{\omega} - \pi|e|^{\nu} \leq C\theta(\frac{\theta}{\pi})^{\frac{\omega}{\nu-\omega}} \tag{13}$$

for any $e \in \mathbb{R}$, $\theta, \pi > 0$ and $0 < \omega < \nu$, where $C = C(\omega, \nu) > 0$,

$$\begin{aligned} \frac{2\mu b(x)}{s(x)}|u|^{s(x)} - \frac{\mu}{r(x)}|u|^{r(x)} &\leq C\left(\frac{2\mu b(x)}{s(x)}\right)\left(\frac{\frac{2\mu b(x)}{s(x)}}{\frac{\mu}{r(x)}}\right)^{\frac{s(x)}{r(x)-s(x)}} \\ &= C\mu\left(\frac{2b(x)}{s(x)}\right)^{\frac{r(x)}{r(x)-s(x)}} (r(x))^{\frac{s(x)}{r(x)-s(x)}} \end{aligned}$$

since $\left(\frac{2}{s(x)}\right)^{\frac{r(x)}{r(x)-s(x)}} (r(x))^{\frac{s(x)}{r(x)-s(x)}}$ is bounded, so there is $k_1 > 0$ in which

$$\frac{2\mu b(x)}{s(x)}|u|^{s(x)} - \frac{\mu}{r(x)}|u|^{r(x)} \leq k_1\mu(b(x))^{\frac{r(x)}{r(x)-s(x)}}.$$

By (13)

$$\int \left(\frac{2\mu b(x)}{s(x)}|u|^{s(x)} - \frac{\mu}{r(x)}|u|^{r(x)}\right) dx \leq k_2,$$

where k_2 is a suitable positive constant. On $W_0^{1,p(x)}(\Omega)$, we can consider the equivalent norm $|\nabla u|_{p(x)}$ replace with $\|u\|$. For any $u \in W_0^{1,p(x)}(\Omega)$ by (H_1) and (H_2),

$$\begin{aligned}
\tau(u) = & \int A_0(x, |\nabla u|)dx - \mu \int_\Omega \frac{b(x)}{s(x)}|u|^{s(x)}dx + \mu \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx \\
= & \int_\Omega \left[\int_0^{|\nabla u|} A(x,t)tdt \right] - \mu \int_\Omega \frac{b(x)}{s(x)}|u|^{s(x)}dx + \mu \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx \\
\geq & \int_\Omega \left[C \int_0^{|\nabla u|} t^{p(x)-1}dt \right] dx + \mu \int_\Omega \frac{b(x)}{s(x)}|u|^{s(x)}dx \\
& - 2\mu \int_\Omega \frac{b(x)}{s(x)}|u|^{s(x)}dx + \mu \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx \\
\geq & \frac{C}{p^+} \int_\Omega |\nabla u|^{p(x)}dx + \mu \frac{\|b\|_\infty}{s^+} \int |u|^{s(x)}dx - k_2.
\end{aligned}$$

If $\sigma_p(u) := \int_\Omega |u|^{p(x)}$, by (11) we have two cases:

i) If $\sigma_p(x) > 1$,

$$\tau(u) \geq \frac{C}{p^+}\|u\|^{p^-} + \frac{\mu\|b\|_\infty}{s^+}\|u\|^{s^-} - k_2.$$

Since (12), so τ is coercive and bounded from below.

ii) If $\sigma_p(x) < 1$,

$$\tau(u) \geq \frac{C}{p^+}\|u\|^{p^+} + \frac{\mu\|b\|_\infty}{s^+}\|u\|^{s^+} - k_2$$

since (12), so again τ is coercive and bounded from below. □

Lemma 3.10. *τ satisfies the $(PS)_C$ condition.*

Proof. Set $D(u) := \int \frac{b(x)}{s(x)}|u|^{s(x)}dx - \int \frac{1}{r(x)}|u|^{r(x)}dx$. From now on set $\tau(u) := A(u) - \mu D(u)$. By Lemma 21 from Chapter 3 in [14] and Proposition 3.10 in [17], D' satisfies condition (S_+). By using the compact Sobolev embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$, $b \in L^\infty(\Omega)$, we can see that D' is compact. So for any $(PS)_c$-sequence $(u_n)_n$ in $W_0^{1,p(x)}$; i.e.,

$$\tau(u_n) \to c \;\text{ and }\; \tau'(u_n) \to 0 \;\text{ as }\; n \to \infty,$$

from Lemma 3.5, we can see that $A' : W_0^{1,p(x)}(\Omega) \to W^{1,p(x)}(\Omega)$ satisfies condition (S_+), so τ' satisfies condition (S_+). Since τ' satisfies condition (S_+) and $W_0^{1,s(x)}(\Omega)$ is reflective, it is enough to show that $\{u_n\}$ is bounded in $W_0^{1,s(x)}(\Omega)$ [17].

But τ is coercive, so $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Therefore, τ satisfies the $(PS)_C$ condition. □

Proof of Theorem 3.8. We have $W_0^{1,p^+}(\Omega) \subset W_0^{1,p(x)}(\Omega)$. Consider a Schauder basis $(e_n)_{n=1}^{\infty}$ for $W_0^{1,p^+}(\Omega)$ [1,3,13]. Set

$$X_m := \operatorname{span}\{e_1, e_2, \dots, e_m\}.$$

Since $X_m \subset W_0^{1,p^+(x)}(\Omega) \subset L_{s(x)}(\Omega)$, it implies $X_m \subset L_{s(x)}(\Omega)$. Similarly, $X_m \subset L_{r(x)}(\Omega)$. Thus, the norms $\|u\|$, $\|u\|_{p(x)}$, $\|u\|_{s(x)}$ and $\|u\|_{r(x)}$ are equivalent on X_m since X_m is finite dimension [2,3,13]. Consequently, there are positive constants C_1, C_2 and C_3 in which

$$C_1\|u\| \le \|u\|_{s(x)} \text{ for all } u \in X_m,$$
$$C_2\|u\|_{r(x)} \le \|u\| \text{ for all } u \in X_m,$$

and

$$C_3\|u\|_{p(x)} \le \|u\| \text{ for all } u \in X_m.$$

We use r^*, s^* in which "r^* is r^- or r^+", "s^* is s^- or s^+" is for using which depend on $|u|_r$ and $|u|_s$. Let $u \in X_m$. Then by (H_1) and (H_2), we obtain

$$\begin{aligned}
\tau(u) &= \int_\Omega \int_0^{|\nabla u|} a(x,t)tdt + \mu \int_\Omega \frac{|u|^{r(x)}}{r(x)}dx - \mu \int_\Omega \frac{b(x)}{s(x)}|u|^{s(x)}dx \\
&\le \int_\Omega \left[\alpha_1 \int_0^{|\nabla u|} t^{p(x)-1}dt\right] dx + \mu \int_\Omega \frac{|u|^{r(x)}}{r(x)}dx - \mu \int_\Omega \frac{b(x)}{s(x)}|u|^{s(x)}dx \\
&\le \frac{\alpha_1}{p^-}\int_\Omega |\nabla u|^{p(x)} + \frac{\mu}{r^-}\int_\Omega |u|^{r(x)} - \frac{\mu\|b\|_\infty}{s^+}\int_\Omega |u|^{s(x)}dx \\
&\le \alpha_2\left(\|u\|^{p^-} + \|u\|^{r^*}\right) - C\|u\|^{s^*} \\
&\le \|u\|^{s^*}\left(\alpha_2\left(\|u\|^{p^- - s^*} + \|u\|^{r^* - s^*}\right) - C\right),
\end{aligned}$$

such that $\alpha_1 > 0$ and $\alpha_2 = \max\{\frac{\alpha_1}{p^-}, \frac{\mu}{r^-}\}$. There exists $r_1 \in (0,1)$ small enough in which $r^{s^*} < 1$ and

$$\alpha_2 r_1^{p^- - s^*} + \alpha_2 r_1^{r^* - s^*} \le \frac{C}{2}.$$

Consider $T := S_r^m = \{u \in X_m \mid \|u\| = r_1\}$. Then

$$\tau(u) \le r_1^{s^*}\left(\alpha_2 r_1^{p^- - s^*} + \alpha_2 r_1^{r^* - s^*} - C\right) \ \forall u \in T.$$

Hence,

$$\sup_T \tau(u) \le 1\left(\frac{C}{2} - C\right) = -\frac{C}{2} < 0 = \tau(0).$$

Since S_r^m and S^{m-1} are homomorphic so $\gamma(S_r^m) = m$.

τ is even, so by Theorem 3.7, τ has least m pairs of different critical points. □

Corollary 3.11. *If* (12) *holds, then there are infinitely many solutions for* (1).

Proof. Since m is arbitrary, so there are infinitely many critical points of τ. □

4. Regularity results on eigenfunctions

Lemma 4.1 ([1]).

(i) *Let $n \geq 2$. Then for any $a, b \in \mathbb{R}^N$*

$$|b|^n \geq |a|^n + n|a|^{n-2} a \cdot (b-a) + K(n)|a-b|^n. \tag{14}$$

(ii) *Let $1 < n < 2$. Then for any $a, b \in \mathbb{R}^N$,*

$$|b|^n \geq |a|^n + n|a|^{n-2} a \cdot (b-a) + K(n)\frac{|a-b|^2}{(|a|+|b|)^{2-n}}. \tag{15}$$

(iii) *For any $a \neq b$, $n > 2$,*

$$|b|^n > |a|^n + n|a|^{n-2} a \cdot (b-a).$$

In this inequality $K(n)$ is a constant depending only on n.

The following results show that nonnegative elements of $W^{1,p}(\Omega)$ can be approximated by a sequence of nonnegative functions in $C(\bar{\Omega}) \cap W^{1,p}(\Omega)$. Hence $1 < p < \infty$ is constant.

Lemma 4.2 ([1]). *For $e \in C(\mathbb{R})$ such that $e' \in L^\infty(\mathbb{R})$, we have*

(a) *$u \in W^{1,p}(\Omega)$ implies $e \circ u \in W^{1,p}(\Omega)$.*
(b) *$u \in W_0^{1,p}(\Omega)$ and $e(0) = 0$, implies $e \circ u \in W_0^{1,p}(\Omega)$.*

Now we study boundedness of eigenfunction of the problem (1) in typical conditions. We consider the case $a(x, |\nabla u|)\nabla u = g(x)|\nabla u|^{p(x)-2}\nabla u$, where $g \in L^\infty(\Omega)$ and $g(x) \geq 0$, that is

$$\begin{cases} -div(g(x)|\nabla u|^{p(x)-2}\nabla u) = \mu m(x)|u|^{p(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

The pair $(u, \mu) \in W_0^{1,p(x)} \times \mathbb{R}^+$ is a weak solution of (16) if

$$\int g(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla h dx = \mu \int m(x)|u|^{n(x)-2}u \cdot h dx, \ \forall h \in W_0^{1,p(x)} \tag{17}$$

for any continuous functions $a : \bar{\Omega} \to (0, \infty)$ and $m : \bar{\Omega} \to (0, \infty)$. Denote by

$$g^- := \inf_{x\in\Omega} g(x) \quad \text{and} \quad g^+ := \sup_{x\in\Omega} g(x),$$
$$m^- := \inf_{x\in\Omega} m(x) \quad \text{and} \quad m^+ := \sup_{x\in\Omega} m(x),$$

in which $g^- < g^+ < m^- < m^+$. Let $p(x) \equiv p$. Then the following theorem asserts that the nonnegative eigenfunction (16) is in $L^\infty(\Omega)$.

Theorem 4.3. *Let $(u, \mu) \in W_0^{1,p} \times \mathbb{R}^+$ be an eigensolution (of the weak form) for (16). Then $u \in L^\infty(\Omega)$.*

Proof. By Proposition 2.3 it suffices to consider the case $p \leq N$. Let $u \geq 0$. Set $h_{R(x)} := \min\{u(x), R\}$ such that $R > 0$. Letting $e(x) = x$ if $x \leq R$ and $e(x) = R$ if $x > R$, by using Theorem 4.2, we have $h_R \in W_0^{1,p} \cap L^\infty(\Omega)$. For $\alpha > 0$ define $Z := h_R^{\alpha p+1}$, then $\nabla Z = (\alpha p + 1) h_R^{\alpha p} \cdot \nabla h_R$ which it implies that $Z \in W_0^{1,p} \cap L^\infty(\Omega)$. Let Z be a test function in (16), then we have

$$\begin{aligned}(\alpha p+1)\int_\Omega g(x)|\nabla u|^{p-2}\nabla u\cdot\nabla h_R\cdot h_R^{\alpha p}dx &= \mu\int m(x)|u|^{p-2}u\cdot h_R^{\alpha p+1}dx\\ &\leq \mu\int m(x)|u|^{(\alpha+1)p}dx\end{aligned}$$

or

$$\frac{(\alpha p+1)}{(\alpha+1)^p}\int g(x)|\nabla h_R^{\alpha+1}|^p dx \leq \mu\int m(x)|u|^{(\alpha+1)p}dx,$$

since

$$(\alpha p+1)\int g(x)(|\nabla h_R|h_R^\alpha)^p = \frac{\alpha p+1}{(\alpha p+1)^p}\int|\nabla h_R^{\alpha+1}|^p.$$

Then

$$\begin{aligned}&\frac{(\alpha p+1)}{(\alpha+1)^p}\int\left[g(x)|\nabla h_R^{\alpha+1}|^p + m(x)|h_R^{\alpha+1}|^p\right]dx\\ &\leq \mu\int m(x)|u|^{(\alpha+1)p} + \frac{\alpha p+1}{(\alpha p+1)^p}\int m(x)|h_R^{\alpha+1}|^p dx.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{(\alpha p+1)}{(\alpha+1)^p}\left[g^-\int|\nabla h_R^{\alpha+1}|^p + g^-\int|h_R^{\alpha+1}|^p dx\right]\\ &\leq \frac{(\alpha p+1)}{(\alpha+1)^p}\left[g^-\int|\nabla h_R^{\alpha+1}|^p + m^-\int|h_R^{\alpha+1}|^p dx\right]\\ &\leq \frac{(\alpha p+1)}{(\alpha+1)^p}\left[\int g(x)|\nabla h_R^{\alpha+1}|^p dx + \int m(x)|h_R^{\alpha+1}|^p dx\right].\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{(\alpha p+1)}{(\alpha+1)^p}\left[g^-\int|\nabla h_R^{\alpha+1}|^p + g^-\int|h_R^{\alpha+1}|^p dx\right]\\ &\leq \mu m^+\int|u|^{(\alpha+1)p} + \frac{\alpha p+1}{(\alpha+1)^p}m^+\int|h_R^{\alpha+1}|^p dx\\ &\leq \left(\mu + \frac{\alpha p+1}{(\alpha+1)^p}\right)m^+\int u^{(\alpha+1)p}dx,\end{aligned}$$

so

$$\|h_R^{\alpha+1}\|^p \leq \left(\mu\frac{(\alpha+1)^p m^+}{(\alpha p+1)g^-} + \frac{m^+}{g^-}\right)\|u\|_{(\alpha+1)p}^{(\alpha+1)p}.$$

By Proposition 2.3, there exists a constant $K_1 > 0$ such that

$$\|h_R^{\alpha+1}\|_{p^*} \leq K_1\|h_R^{\alpha+1}\|.$$

Take $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = 2p$ if $p = N$. Thus,

$$\begin{aligned}\|h_R\|_{(\alpha+1)p^*} &\le \|h_R^{\alpha+1}\|_{p^*}^{\frac{1}{(\alpha+1)}}\\ &\le K_1^{\frac{1}{\alpha+1}}\left(\mu\frac{(\alpha+1)^p m^+}{(\alpha p+1)g^-}+\frac{m^+}{g^-}\right)^{\frac{1}{p(\alpha+1)}}\|u\|_{(\alpha+1)p}.\end{aligned}$$

There is a constant $K_2 > 0$ such that

$$\left(\mu\frac{(\alpha+1)^p m^+}{(\alpha p+1)g^-}+\frac{m^+}{g^-}\right)^{\frac{1}{p\sqrt{\alpha+1}}} \le K_2,\ \forall \alpha > 0.$$

Thus

$$\|h_R\|_{(\alpha+1)p^*} \le K_1^{\frac{1}{\alpha+1}} K_2^{\frac{1}{\sqrt{\alpha+1}}}\|u\|_{(\alpha+1)p}.$$

Approaching $R \to \infty$ and by Fatou's Lemma, we have

$$\|u\|_{(\alpha+1)p^*} \le K_1^{\frac{1}{\alpha+1}} K_2^{\frac{1}{\sqrt{\alpha+1}}}\|u\|_{(\alpha+1)p}. \tag{18}$$

Choose α_1 such that $(\alpha_1+1)p = p^\star$. Then from (18)

$$\|u\|_{(\alpha_1+1)p^\star} \le K_1^{\frac{1}{\alpha_1+1}} K_2^{\frac{1}{\sqrt{\alpha_1+1}}}\|u\|_{p^\star}.$$

Next we choose α_2 such that $(\alpha_2+1)p = (\alpha_1+1)p^\star$. Then taking $\alpha_2 = \alpha$ in (18),

$$\begin{aligned}\|u\|_{(\alpha_2+1)p^\star} &\le K_1^{\frac{1}{\alpha_2+1}} K_2^{\frac{1}{\sqrt{\alpha_2+1}}}\|u\|_{(\alpha_2+1)p}\\ &= K_1^{\frac{1}{\alpha_2+1}} K_2^{\frac{1}{\sqrt{\alpha_2+1}}}\|u\|_{(\alpha_1+1)p^\star}.\end{aligned}$$

Hence,

$$\|u\|_{(\alpha_n+1)p^\star} \le K_1^{\frac{1}{\alpha_2+1}} K_2^{\frac{1}{\sqrt{\alpha_n+1}}}\|u\|_{(\alpha_{n-1}+1)p^\star},$$

where the sequence $\{\alpha_n\}$ is chosen so that $(\alpha_n+1)p = (\alpha_{n-1}+1)p^\star$, $\alpha_0 = 0$. We see $\alpha_{n+1} = (\frac{p^\star}{p})^n$ and

$$\|u\|_{(\alpha_n+1)p^\star} \le K_1^{\sum_{i=1}^n \frac{1}{\alpha_i+1}} K_2^{\sum_{i=1}^n \frac{1}{\sqrt{\alpha_i+1}}}\|u\|_{p^\star}.$$

As $\frac{p}{p^\star} < 1$, there is $K > 0$ in which

$$\|u\|_{(\alpha_n+1)p^\star} \le K\|u\|_{p^\star}\ \forall n \in \mathbb{N},$$

where $(\alpha_{n+1})p^\star \to \infty$ as $n \to \infty$.

We aim to show that $u \in L^\infty(\Omega)$.

Suppose that $u \notin L^\infty(\Omega)$. Then there exist $\epsilon > 0$ and a set E of positive measure in Ω such that $|u(x)| > K\|u\|_{p^\star} + \epsilon = \beta$ for any $x \in E$. Then

$$\liminf_{n\to\infty}\|u\|_{r_n} \ge \liminf_{n\to\infty}\left(\int_E \beta^{r_n}\right)^{\frac{1}{r_n}} = \liminf_{n\to\infty}\beta|E|^{\frac{1}{r_n}} = \beta > k\|u\|_{p^\star},$$

which is a contradiction. □

We set $a(x,|\nabla u|)\nabla u = (1+|\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u$ and

$$a_0(x,|h|) = \frac{1}{p(x)}\left[(1+|h|^2)^{\frac{p(x)}{2}} - 1\right]$$

for all $h \in \mathbb{R}^n$. We study the following problem

$$\begin{cases} -div\left(1+|\nabla u|^2\right)^{\frac{p(x)-2}{2}} \nabla u = \mu m(x)|u|^{p(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

the pair $(u,\mu) \in W_0^{1,p(x)} \times \mathbb{R}^+$ is a weak solution of (19) if

$$\int \left((1+|\nabla u|^2)^{\frac{p(x)-2}{2}}\right) \nabla u \cdot \nabla h = \mu \int m(x)|u|^{p(x)-2}u \cdot h dx,\ \forall h \in W_0^{1,p(x)}. \tag{20}$$

Similar for $p(x) \equiv p$, the following theorem asserts that the nonnegative eigenfunction (19) is in $L^\infty(\Omega)$.

Theorem 4.4. *Let $(u,\mu) \in W_0^{1,p} \times \mathbb{R}^+$ be an eigensolution of the weak form (19) but when $p(x) \equiv p$. Then $u \in L^\infty(\Omega)$.*

Proof. By Proposition 2.3 it suffices to consider the case $p \le N$. Let $u \ge 0$. Set $h_{R(x)} := \min\{u(x), R\}$ such that $R > 0$. Letting $e(x) = x$ if $x \le R$ and $e(x) = R$ if $x > R$, by using Theorem 4.2 we have $h_R \in W_0^{1,p} \cap L^\infty(\Omega)$. For $\alpha > 0$ define $Z := h_R^{\alpha p+1}$, then $\nabla Z = (\alpha p+1)h_R^{\alpha p} \cdot \nabla h_R$. Which it implies that $Z \in W_0^{1,p} \cap L^\infty(\Omega)$. Let Z be a test function in (19), then we have

$$\begin{aligned}(\alpha p+1)\int \left((1+|\nabla u|^2)^{\frac{p-2}{2}}\right)\nabla u \cdot \nabla h_R \cdot h_R^{\alpha p} dx &= \mu \int m(x)|u|^{p-2}u \cdot h_R^{\alpha p+1} dx \\ &\le \mu \int m(x)|u|^{(\alpha+1)p} dx\end{aligned}$$

or

$$\begin{aligned}&(\alpha p+1)\int \left((|\nabla u|^2)^{\frac{p-2}{2}}\right)\nabla u \cdot \nabla h_R \cdot h_R^{\alpha p} dx \\ &= (\alpha p+1)\int |\nabla u|^{\frac{p-2}{2}}\nabla u \cdot \nabla h_R \cdot h_R^{\alpha p} dx \\ &\le \mu m^+ \int |u|^{(\alpha+1)p} dx,\end{aligned}$$

then

$$\begin{aligned}(\alpha p+1)\int (|\nabla h_R| h_R^{\alpha})^p &= \frac{\alpha p+1}{(\alpha p+1)^p}\int |\nabla h_R^{\alpha+1}|^p \\ &\le \mu m^+ \int |u|^{(\alpha+1)p} dx,\end{aligned}$$

and then

$$\frac{(\alpha p+1)}{(\alpha+1)^p}\int_\Omega |\nabla h_R^{\alpha+1}|^p + |h_R^{\alpha+1}|^p dx$$

$$\leq \left(\mu m^+ + \frac{\alpha p+1}{(\alpha+1)^p}\right)\int_\Omega u^{(\alpha+1)p}dx,$$

thus

$$\|h_R^{\alpha+1}\|^p \leq \left(\mu m^+\frac{(\alpha+1)^p}{(\alpha p+1)}+1\right)\|u\|_{(\alpha+1)p}^{(\alpha+1)p}.$$

By Proposition 2.3, there exists a constant $K_1 > 0$ such that

$$\|h_R^{\alpha+1}\|_{p^*} \leq K_1\|h_R^{\alpha+1}\|,$$

we take $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = 2p$ if $p = N$. Thus

$$\begin{aligned}\|h_R\|_{(\alpha+1)p^*} &\leq \|h_R^{\alpha+1}\|_{p^*}^{\frac{1}{(\alpha+1)}}\\ &\leq K_1^{\frac{1}{\alpha+1}}\left(\mu\frac{(\alpha+1)^p m^+}{(\alpha p+1)}+1\right)^{\frac{1}{p(\alpha+1)}}\|u\|_{(\alpha+1)p}.\end{aligned}$$

Then, we can find a constant $K_2 > 0$ such that

$$\left(\mu\frac{(\alpha+1)^p m^+}{(\alpha p+1)}+1\right)^{\frac{1}{p\sqrt{\alpha+1}}} \leq K_2$$

for any $\alpha > 0$. Thus

$$\|h_R\|_{(\alpha+1)p^*} \leq K_1^{\frac{1}{\alpha+1}}K_2^{\frac{1}{\sqrt{\alpha+1}}}\|u\|_{(\alpha+1)p}.$$

Approaching $R \to \infty$, Fatou's Lemma implies that

$$\|u\|_{(\alpha+1)p^*} \leq K_1^{\frac{1}{\alpha+1}}K_2^{\frac{1}{\sqrt{\alpha+1}}}\|u\|_{(\alpha+1)p}.$$

Choose k_1 such that $(\alpha_1+1)p = p^\star$. Then we have

$$\|u\|_{(\alpha_1+1)p^\star} \leq K_1^{\frac{1}{\alpha_1+1}}K_2^{\frac{1}{\sqrt{\alpha_1+1}}}\|u\|_{p^\star}.$$

Next we choose α_2 such that $(\alpha_2+1)p = (\alpha_1+1)p^\star$. Then taking $\alpha_2 = \alpha$, we have

$$\begin{aligned}\|u\|_{(\alpha_2+1)p^\star} &\leq K_1^{\frac{1}{\alpha_2+1}}K_2^{\frac{1}{\sqrt{\alpha_2+1}}}\|u\|_{(\alpha_2+1)p}\\ &= K_1^{\frac{1}{\alpha_2+1}}K_2^{\frac{1}{\sqrt{\alpha_2+1}}}\|u\|_{(\alpha_1+1)p^\star}.\end{aligned}$$

Then

$$\|u\|_{(\alpha_n+1)p^\star} \leq K_1^{\frac{1}{\alpha_2+1}}K_2^{\frac{1}{\sqrt{\alpha_2+1}}}\|u\|_{(\alpha_{n-1}+1)p^\star},$$

where the sequence $\{\alpha_n\}$ is chosen such that $(\alpha_n+1)p = (\alpha_{n-1}+1)p^\star$, $\alpha_0 = 0$. And we see that $\alpha_{n+1} = (\frac{p^\star}{p})^n$. Hence,

$$\|u\|_{(\alpha_n+1)p^\star} \leq K_1^{\sum_{i=1}^n \frac{1}{\alpha_i+1}}K_2^{\sum_{i=1}^n \frac{1}{\sqrt{\alpha_i+1}}}\|u\|_{p^\star}.$$

As $\frac{p}{p^\star} < 1$, there is $K > 0$ such that for any $n = 1, 2, \ldots$

$$\|u\|_{(\alpha_n+1)p^\star} \leq K\|u\|_{p^\star},$$

with $(\alpha_{n+1})p^\star \to \infty$ as $n \to \infty$. Then $\|u\|_\infty \le K\|u\|_{p^\star}$. □

5. Linear relationship between u and h

Let us define the following quantities

$$\mu^* := \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int A_0(x,|\nabla u|)dx}{\int \frac{b(x)}{s(x)}|u|^{s(x)}dx + \int \frac{1}{r(x)}|u|^{r(x)}dx},$$

where $A_0(x,z) := \int_0^z a(x,t)tdt$, and

$$\mu_1 = \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int A(x,|\nabla u|).|\nabla u|^2 dx}{\int b(x)|u|^{s(x)}dx + \int |u|^{r(x)}dx}.$$

In [11] it is shown that μ_1 is smallest eigenvalue.

In this section we will show for problem (16) and (19) that the first eigenvalue μ_1 is simple and only eigen functions associated whit μ_1 denote change sign. Let us denote the quantity of the problem (16).

$$\mu^* = \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int \frac{g(x)}{p(x)}\left[|\nabla u|^{p(x)} - 1\right]dx}{\int \frac{m(x)}{p(x)}|u|^{p(x)}dx}. \tag{21}$$

The smallest eigenvalue of problem (16) is

$$\mu_1 = \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int g(x)|\nabla u|^{p(x)}dx}{\int m(x)|u|^{p(x)}dx}. \tag{22}$$

We denote the quantity of the problem (19),

$$\mu^* = \inf_{u\in X\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}\left[(1+|\nabla u|^2)^{\frac{p(x)}{2}} - 1\right]dx}{\int_\Omega \frac{m(x)}{p(x)}|u|^{p(x)}dx}. \tag{23}$$

The smallest eigenvalue of problem (19) is

$$\mu_1 = \inf_{u\in X\setminus\{0\}} \frac{\int_\Omega (1+|\nabla u|^2)^{\frac{p(x)}{2}}\cdot|\nabla u|^2 dx}{\int_\Omega m(x)|u|^{p(x)}dx}. \tag{24}$$

For special case $p(x)\equiv p$, we consider the following theorems.

Lemma 5.1 ([1]). *Let u be an eigenfunction associated with μ_1. Then either $u>0$ or $u<0$ in Ω.*

Theorem 5.2. *The principal eigenvalue μ_1 of problem (16) is simple, i.e., if u and h are two eigenfunctions associated with μ_1, then $u = Kh$, where K is a constant.*

Proof. By Lemma 5.1, we can assume that u and h are positive in Ω. Let $\eta = \frac{(u+t)^p-(h+t)^p}{(u+t)^{p-1}}$ and $\theta = \frac{(h+t)^p-(u+t)^p}{(h+t)^{p-1}}$, where t is a positive parameter. Then

$$\nabla\eta = \left\{1 + (p-1)(\frac{h+t}{u+t})^p\right\}\nabla u - p(\frac{h+t}{u+t})^{p-1}\nabla h,$$

since u and h are bounded. From Theorem 4.3, $\nabla\eta \in L^p(\Omega)$ and thus $\eta \in W^{1,p}(\Omega)$. By symmetry, θ and $\nabla\theta$ have a similar expression when u and h interchanged.

Set $u_t = u + t$ and $h_t = h + t$. We have

$$\begin{aligned}
&\mu_1 \int_\Omega m(x) u^{p-1} \frac{u_t^p - h_t^p}{u_t^{p-1}} - \mu_1 \int_\Omega m(x) h^{p-1} \frac{h_t^p - u_t^p}{h_t^{p-1}} \\
&= \mu_1 \int_\Omega m(x) \left[\frac{u^{p-1}}{u_t^{p-1}} - \frac{h^{p-1}}{h_t^{p-1}} \right] (u_t^p - h_t^p) dx \\
&= \int_\Omega g(x) \Big[\{1 + (p-1)(\frac{h_t}{u_t})^p\} |\nabla u_t|^p + \{1 + (p-1)(\frac{u_t}{v_t})^p\} |\nabla h_t|^p \Big] dx \\
&\quad - \int_\Omega g(x) \left[p(\frac{h_t}{u_t})^{p-1} |\nabla u_t|^{p-2} \nabla u_t \cdot \nabla h_t + p(\frac{u_t}{h_t})^{p-1} |\nabla h_t|^{p-2} \nabla h_t \cdot \nabla u_t \right] \\
&= \int_\Omega g(x) (u_t^p - h_t^p)(|\nabla \ln\ u_t|^p - |\nabla \ln\ h_t|^p) dx \\
&\quad - p \int_\Omega g(x) h_t^p |\nabla \ln\ u_t|^{p-2} \nabla \ln\ u_t (\nabla \ln\ h_t - \nabla \ln\ u_t) dx \\
&\quad - p \int_\Omega g(x) u_t^p |\nabla \ln\ u_t|^{p-2} \nabla \ln\ h_t (\nabla \ln\ u_t - \nabla \ln\ h_t) dx.
\end{aligned}$$

Set $x_1 = u_t \nabla \ln\ h_t$, $y_1 = u_t \nabla \ln\ u_t$, $x_2 = h_t \nabla \ln\ u_t$, $y_2 = h_t \nabla \ln\ h_t$ and viceversa, inequality (15) in Lemma 4.1 implies that

$$\begin{aligned}
L_t = &\int_\Omega g(x) (u_t^p - h_t^p)(|\nabla \ln\ u_t|^p - |\nabla \ln\ h_t|^p) dx \\
&- p \int_\Omega g(x) h_t^p |\nabla \ln\ u_t|^{p-2} \nabla \ln\ u_t (\nabla \ln\ h_t - \nabla \ln\ u_t) dx \\
&- p \int_\Omega g(x) u_t^p |\nabla \ln\ u_t|^{p-2} \nabla \ln\ h_t (\nabla \ln\ u_t - \nabla \ln\ h_t) dx \geq 0.
\end{aligned}$$

Dominated convergence theorem implies that

$$\lim_{t \to 0^+} \mu_1 \int_\Omega \left[\frac{u^{p-1}}{u_t^{p-1}} - \frac{h^{p-1}}{h_t^{p-1}} \right] (u_t^p - h_t^p) dx = 0. \tag{25}$$

Theorem 4.3 of [1] implies that u and h are in $C^{1,\alpha}(\bar{\Omega})$. For the case $p \geq 2$, from to inequality (14) in Lemma 4.1 we have

$$\begin{aligned}
0 &\leq K(p) \int_\Omega \left(\frac{1}{h_t^p} + \frac{1}{u_t^p} \right) |h_t \nabla u_t - u_t \nabla h_t|^p dx \\
&\leq L_t \leq \mu_1 \int_\Omega \left[\frac{u^{p-1}}{u_t^{p-1}} - \frac{h^{p-1}}{h_t^{p-1}} \right] (u_t^p - h_t^p) dx
\end{aligned}$$

for every $t > 0$. Recalling (25), for $t \to 0^+$, from Fatou's Lemma we have

$$\lim h_t \nabla u_t - u_t \nabla h_t = 0 \text{ a.e, in } \Omega$$

thus

$$h\nabla u = u\nabla h \text{ a.e, in } \Omega.$$

We obtain immediately that $\nabla(\frac{u}{h}) = 0$, which shows $u = Kh$, where K is a constant. $\square$

Conclusion

Here we obtained the existence and multiplicity of solutions for problem (1), by using the symmetric mountain pass theorem via genus theory. Moreover, we studied regularity results, the simplicity and boundedness of the eigen functions for problems (16) and (19).

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