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ON GRADED N-IRREDUCIBLE IDEALS OF COMMUTATIVE GRADED RINGS

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Abstract. Let R be a commutative graded ring with nonzero identity and n a positive integer. Our principal aim in this paper is to introduce and study the notions of graded n-irreducible and strongly graded n-irreducible ideals which are generalizations of n-irreducible and strongly n-irreducible ideals to the context of graded rings, respectively. A proper graded ideal I of R is called graded n-irreducible (respectively, strongly graded n-irreducible) if for each graded ideals $I_1, \ldots, I_{n+1}$ of R, $I = I_1 \cap \cdots \cap I_{n+1}$ (respectively, $I_1 \cap \cdots \cap I_{n+1} \subseteq I$) implies that there are n of the I_i 's whose intersection is I (respectively, whose intersection is in I). In order to give a graded study to this notions, we give the graded version of several other results, some of them are well known. Finally, as a special result, we give an example of a graded n-irreducible ideal which is not an n-irreducible ideal and an example of a graded ideal which is graded n-irreducible, but not graded $(n-1)$-irreducible.

1. Introduction

In this article, all rings under consideration are assumed to be commutative with nonzero identity and all modules are assumed to be nonzero unital. R will always represent such a ring, M will represent such an R-module. Also, G will represent a commutative additive monoid with identity element 0. By a graded ring R, we mean a ring graded by G, that is, a direct sum of subgroups R_α of R such that $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for every $\alpha, \beta \in G$. The set $h(R) = \cup_{\alpha \in G} R_\alpha$ is the set of homogeneous elements of R. A nonzero element $x \in R$ is called homogeneous if it belongs to one of the R_α, homogeneous of degree α if $x \in R_\alpha$. An ideal I of R is said to be a *graded ideal* (or sometimes called *homogeneous ideal*) if the homogeneous components of every element of I belong to I, equivalently, if I is generated by homogeneous elements. Let I be a graded ideal. I is said to be a graded prime ideal if $x \in I$ or $y \in I$ whenever $xy \in I$ for some $x, y \in h(R)$. Note that, when G is a torsionless monoid (that is, if G is cancellative and the group of fractions of G, $\langle G \rangle = \{a - b/a, b \in G\}$, is a torsionfree abelian group),

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then I is graded prime if and only if I is a prime ideal. Recently, there have been various generalizations of graded prime ideals in several papers. Among the many recent generalizations of the notion of graded prime ideals in the literature, we find the following, defined first by M. Refai and K. F. Al Zoubi [9, Definition 2.13]. A proper graded ideal I of a commutative graded ring R is graded irreducible if $I = J \cap K$ for some graded ideals I and K of R implies that either $I = J$ or $I = K$. A proper ideal I of R is said to be strongly graded irreducible if for each graded ideals J, K of R, $J \cap K \subseteq I$ implies that $J \subseteq I$ or $K \subseteq I$. A graded ideal I is said to be graded maximal if $I \neq R$ and if it is maximal among graded ideals, equivalently, if R/I is a graded field, that is, if every nonzero homogeneous element of R/I is invertible.

Our aim in this paper is to study graded n-irreducible and strongly graded n-irreducible ideals which are generalizations of n-irreducible and strongly n-irreducible ideals to the context of graded rings, respectively. Let n be a positive integer. According to [13], a proper ideal I of a ring R is said to be an n-irreducible (respectively, strongly n-irreducible) ideal if there are n of the I_i's whose intersection is I (respectively, whose intersection is contained in I) whenever $I_1 \cap \cdots \cap I_{n+1} = I$ (respectively, $I_1 \cap \cdots \cap I_{n+1} \subseteq I$) for some ideals $I_1, \ldots, I_{n+1}$ of R. The notion of graded 2-irreducible ideals has been introduced in [11, Definition 2.20]. An ideal I is called graded 2-irreducible if whenever $I = J \cap K \cap L$ for some graded ideals J, K and L of R, then $J \cap L \subset I$ or $K \cap L \subset I$ or $J \cap K \subset I$.

Our work is motivated by several definitions and concepts which we will recall some of them. In [11], the authors introduced a generalization of graded prime ideals called graded 2-absorbing ideals, and this idea is generalized also by the authors in a paper [2] to a concept called graded n-absorbing ideals. According to [2, Definition 2.1], a proper graded ideal I of R is called a graded n-absorbing ideal if there are n of the x_i's whose product is in I whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in h(R)$. Thus a graded 1-absorbing ideal is just a graded prime ideal. We refer the readers to [2] for the module version of graded n-absorbing ideals. For any graded ideal I of R, the graded radical of I is defined as follows:

$$Gr(I) = \left\{ a = \sum_{g \in G} a_g \ \in R : \forall g \in G,\ \exists\, n_g > 0 \text{ such that } a_g^{n_g} \in I \right\}.$$

Note that $Gr(I)$ is a graded ideal of R. We refer the reader to [10, Proposition 2.4] for the basic properties of the graded radical. According to [9, Definition 1.5], a proper graded ideal I is said to be graded primary if $a \in I$ or $b \in Gr(I)$ whenever $a, b \in h(R)$ with $ab \in I$. The concept of graded 2-absorbing primary ideals, as a generalization of graded primary ideals, was introduced and investigated in [11].

Let n be a positive integer. A proper graded ideal I of a graded ring R is called graded n-irreducible if there are n of the I_i's whose intersection is I

whenever $I = I_1 \cap \cdots \cap I_{n+1}$ for some graded ideals $I_1, \ldots, I_{n+1}$ of R. Obviously, any graded irreducible ideal is a graded n-irreducible ideal. A proper graded ideal I of a graded ring R is called strongly graded n-irreducible if for each graded ideals $I_1, \ldots, I_{n+1}$ of R, $I_1 \cap \cdots \cap I_{n+1} \subseteq I$ implies that there are n of the I_i's whose intersection is in I. Clearly, any strongly graded irreducible ideal is a strongly graded n-irreducible ideal.

Our paper is organized as follows. In Section 2, we give the basic properties of graded n-irreducible and strongly graded n-irreducible ideals. Among others results in this section, we will prove that a strongly graded n-irreducible ideal is a graded n-irreducible ideal, and that the intersection of n strongly graded irreducible ideals is a strongly graded n-irreducible ideal. Nevertheless, the product of $(n+1)$ graded comaximal ideals need not be a graded n-irreducible ideal. Besides the above results, the stability of strongly graded n-irreducible (and hence graded n-irreducible) ideals with respect to various graded ring-theoretic constructions such as the graded localization and factor of graded rings, we also characterize the strongly graded n-irreducible ideals in the direct product of a finite number of graded rings. In particular, as a main result of this section, the relationship between strongly graded n-irreducible ideals and graded n-absorbing ideals is considered, see Theorem 2.15, Example 2.14.

We devote Section 3 to the study of graded n-irreducible and the strongly graded n-irreducible ideals in several classes of commutative graded rings. Among several other results, we prove that a nonzero graded ideal I of a graded principal ideal domain R (gr-PID for short) is graded n-irreducible if and only if I is strongly graded n-irreducible if and only if I is graded n-absorbing primary if and only if $I = R\left(p_1^{l_1} \ldots p_m^{l_m}\right)$ for some distinct homogeneous prime elements $p_1, \ldots, p_m$ of R and some natural numbers $l_1, \ldots, l_m$ such that $m \leq n$. As a consequence, an example for which a graded ring R has a graded ideal which is graded n-irreducible, but not graded $(n-1)$-irreducible is given. Also, we prove that a proper graded ideal I of a graded von Neumann regular ring R is graded n-irreducible if and only if I is graded n-absorbing. We close this section by proving that if a proper ideal I of a graded Noetherian ring is graded n-irreducible, then either I is graded irreducible or I is the intersection of exactly n graded irreducible ideals of R.

Finally, in order to give the graded version of the above results, we give the graded version of several other results, some of which are well known, see Theorems 2.10, 2.11, 2.12, 2.13, 3.1, and 3.3.

2. Graded n-irreducible and strongly graded n-irreducible ideals

Before we start our study, we first recall some basic properties and terminology related to graded ring theory. Unless otherwise stated, G will denote a commutative additive monoid with an identity element denoted by 0. Let R be a graded ring. If I is a graded ideal of a graded ring R, then R/I is a graded ring, where $(R/I)_\alpha := (R_\alpha + I)/I$. Suppose that A, B are graded rings and

$R = A \times B$. Then R is a graded ring by $R_g = A_g \times B_g$ for all $g \in G$. Also, it can be easily seen that an ideal I of R is a graded ideal if and only if $I = J \times K$ for some graded ideals J of A and K of B. Let R be a graded ring and I, J graded ideals of R and x_g a homogeneous element of R. Then, it is well known that $I + J$, IJ and $I \cap J$ are graded ideals of R. If g is a cancellable element in G, $(I : x_g) = \{a \in R : ax_g \in I\}$ is a graded ideal of R.

Let G be a group, R be a graded ring, and S be a multiplicatively closed subset of homogeneous elements of R. Then $S^{-1}R$ is a graded ring by $\left(S^{-1}R\right)_g = \left\{\frac{a}{s} : a \in R_h,\ s \in S \cap R_{h-g}\right\}$ for all $g \in G$. If I is a graded ideal of R, then it can be easily seen that $S^{-1}I$ is a graded ideal of $S^{-1}R$.

Suppose that R is a graded ring and M is an R-module. By a graded R-module M, we mean an R-module graded by G, that is, a direct sum of subgroups M_α of M such that $R_\alpha M_\beta \subseteq M_{\alpha+\beta}$ for every $\alpha, \beta \in G$. The set $h(M) = \cup_{\alpha \in G} M_\alpha$ is the set of homogeneous elements of M. A submodule N of M is called graded if $N = \oplus_{\alpha \in G}(N \cap M_\alpha)$, equivalently, if N is generated by homogeneous elements.

Let R and R' be two graded rings, a ring homomorphism $f : R \to R'$ is called graded if $f(R_\alpha) \subseteq R'_\alpha$ for all $\alpha \in G$. A graded ring isomorphism is a bijective graded ring homomorphism. For more information and other terminology on graded rings and modules, we refer [7] and [8] to the reader.

Definition 2.1. Let n be a positive integer. A proper graded ideal I of a graded ring R is said to be graded n-irreducible (respectively, strongly graded n-irreducible) ideal if there are n of the I_i's whose intersection is I (respectively, whose intersection is contained in I) whenever $I_1 \cap \cdots \cap I_{n+1} = I$ (respectively, $I_1 \cap \cdots \cap I_{n+1} \subseteq I$) for some graded ideals $I_1, \ldots, I_{n+1}$ of R.

By definition, one can easily see that every n-irreducible graded ideal of R is also a graded n-irreducible ideal. However, the converse is not always true as shown by the following example.

Example 2.2. Consider the ideal $I = (2)$ of the Gaussian integer graded ring $\mathbb{Z}[i]$ with $G = \mathbb{Z}_2$. Then $I = (2)$ is a graded prime ideal of $\mathbb{Z}[i]$ since 2 is a homogeneous element of R and then by Proposition 2.3 given next, I is graded irreducible. But, I is not irreducible.

We next give some basic properties of graded n-irreducible ideals and strongly graded n-irreducible ideals.

Proposition 2.3. *Let I be a graded ideal of a graded ring R, and let m and n be positive integers. Then,*

(1) *If I is strongly graded n-irreducible, then I is graded n-irreducible.*
(2) *$I = \{0\}$ is graded n-irreducible if and only if I is strongly graded n-irreducible.*
(3) *If I is a graded prime ideal, then I is strongly graded n-irreducible.*
(4) *There is a minimal strongly graded n-irreducible ideal of R over I.*

(5) *If I is graded n-irreducible, then I is graded m-irreducible for all integer $m \geq n$.*
(6) *I is a graded n-irreducible ideal if and only if there are n of the I_i's whose intersection is I whenever $I = I_1 \cap \cdots \cap I_m$ for some graded ideals $I_1, \ldots, I_m$ of R with $m > n$.*
(7) *If I is strongly graded n-irreducible and J is a graded ideal of R such that $J \subseteq I$, then I/J is a strongly graded n-irreducible ideal of R/J.*

Proof. (1) Suppose that I is strongly graded n-irreducible and let $I_1, \ldots, I_{n+1}$ be $(n+1)$ graded ideals of R such that $I_1 \cap \cdots \cap I_{n+1} = I$. Then $I_1 \cap \cdots \cap I_{n+1} \subseteq I$, and therefore there are n of the I_i's whose intersection is in I. Otherwise, I is in any intersection of n graded ideals of I_i's, and it then follows that I is the intersection of n graded ideals of the I_i's.

(2) The sufficient condition follows directly from (1). Conversely, suppose that I is graded n-irreducible. Let $I_1, \ldots, I_{n+1}$ be $(n+1)$ graded ideals of R such that $I_1 \cap \cdots \cap I_{n+1} \subseteq I = \{0\}$, then $I_1 \cap \cdots \cap I_{n+1} = I$. It follows by hypothesis that there are n of the I_i's whose intersection is I, and therefore this intersection is in I.

(3) Suppose that I is graded prime and let $I_1, \ldots, I_{n+1}$ be graded ideals of R such that $I_1 \cap \cdots \cap I_{n+1} \subseteq I$. Then $(I_1 \cap \cdots \cap I_n) \cap I_{n+1} \subseteq I$. Therefore, either $(I_1 \cap \cdots \cap I_n) \subseteq I$ or $I_{n+1} \subseteq I$. If $(I_1 \cap \cdots \cap I_n) \subseteq I$, we are done. If $I_{n+1} \subseteq I$, then any intersection of I_{n+1} with $(n-1)$ graded ideals of the other I_i's is in I. This implies that I is strongly graded n-irreducible, as asserted.

(4) Let $Z = \{J \mid J$ is a strongly graded n-irreducible ideal of R containing $I\}$. Since every graded maximal ideal is strongly graded n-irreducible by (3), $Z \neq \emptyset$. Let $\{J_i\}_{i \in L}$ be a chain in Z, then since any intersection of graded ideals is a graded ideal, $J = \bigcap_{i \in L} J_i$ is a strongly graded n-irreducible ideal containing I. By Zorn's lemma Z has a minimal element.

The proofs of (5) and (6) are clear.

(7) Let $I_1, \ldots, I_{n+1}$ be graded ideals of R containing J such that $(I_1/J) \cap \cdots \cap (I_{n+1}/J) \subseteq I/J$. Hence, $I_1 \cap \cdots \cap I_{n+1} \subseteq I$. Therefore, there are n of the I_i's whose intersection is in I. Without loss of generality, assume that $I_1 \cap I_2 \cap \cdots \cap I_n \subseteq I$, then $(I_1/J) \cap \cdots \cap (I_n/J) \subseteq I/J$. Consequently, I/J is strongly graded n-irreducible. $\square$

Remark 2.4. Let I be a proper graded ideal of a graded ring R. By Lemma 2.2(5), we have that a graded n-irreducible ideal is also a graded m-irreducible ideal for all integers $m \geq n$. If I is a graded n-irreducible ideal of R for some positive integer n, then just like the ungraded case we can define gr-$\omega_R(I) = \min\{n \mid I$ is a graded n-irreducible ideal of $R\}$; else, set gr-$\omega_R(I) = \infty$. It is appropriate to define gr-$\omega_R(R) = 0$. Thus for any graded ideal I of R, gr-$\omega_R(I) \in \mathbb{N} \cup \{\infty\}$ with gr-$\omega_R(I) = 1$ if and only if I is a graded irreducible ideal of R and gr-$\omega_R(I) = 0$ if and only if $I = R$.

Proposition 2.5. *If I_j is a graded n_j-irreducible ideal of a graded ring R for each $1 \leq j \leq m$, then $I_1 \cap \cdots \cap I_m$ is a graded n-irreducible ideal of R for $n = n_1 + \cdots + n_m$. In particular, if $I_1, \ldots, I_n$ are n strongly graded irreducible ideals of a graded ring R, then $\bigcap_{i=1}^{n} I_i$ is a strongly graded n-irreducible ideal of R.*

Proof. It is obvious. □

Proposition 2.6. *Let R be a graded ring, and let $P_1, \ldots, P_{n+1}$ be pairwise comaximal graded prime ideals of R. Then $P_1 \ldots P_{n+1}$ is not a graded n-irreducible ideal of R.*

Proof. Note that, since the P_i's are pairwise comaximal, $P_1 \ldots P_{n+1} = P_1 \cap \cdots \cap P_{n+1}$. The assertion follows by way of contradiction. □

Corollary 2.7. *If R is a graded ring such that every proper graded ideal of R is graded n-irreducible, then R has at most n graded maximal ideals.*

Proof. Suppose that I and J are distinct graded maximal ideals in R. Then $I + J$ is also a graded ideal, and $I \subset I + J \subseteq R$. Since I is graded maximal, the first inclusion is strict and $I + J = R$. Therefore I and J are comaximal and it remains to use Proposition 2.6. □

In order to prove Theorem 2.15, we next give the graded version of several theorems some of which are well known. But we begin first by introducing the following definitions.

Definition 2.8. Let n be a positive integer. A proper graded ideal I of a graded ring R is said to be graded n-absorbing primary if either $x_1 x_2 \cdots x_{n+1} \in I$ or the product of x_{n+1} with $(n-1)$ of the x_i's is in $Gr(I)$ whenever $x_1 x_2 \cdots x_{n+1}$ for $x_1, x_2, \ldots, x_{n+1} \in h(R)$.

Definition 2.9. A graded ideal I of R is said to be graded maximal with respect to the exclusion of S if there is no graded ideal of R which contains I such that $I \cap S = \emptyset$.

The next result presents the graded version of [5, Theorem 1].

Theorem 2.10. *Let S be a multiplicatively closed set of homogeneous elements of R and let I be a graded ideal of R, which is graded maximal with respect to the exclusion of S. Then I is graded prime.*

Proof. Let $ab \in I$ for some two homogeneous elements $a, b \in R$. We must claim that a or b lies in I. Suppose the contrary. Then the ideal (I, a) generated by I and a is a strictly larger graded ideal than I and therefore intersects S. Since S is a multiplicative set of homogeneous elements of R, there exists a homogeneous element $s_1 \in S$ of the form $s_1 = i_1 + xa$, where i_1 and xa are homogeneous elements of the same degree. Likewise, $s_2 \in S$ is of the form $s_2 = i_2 + yb$, where i_2 and yb are homogeneous elements of the same degree. But then $s_1s_2 = (i_1 + xa)(i_2 + yb) \in I$, which contradicts the fact that I is graded maximal with respect to the exclusion of S. Hence, I is graded prime. $\square$

The next result presents the graded version of [4, Theorem 2.1].

Theorem 2.11. *Let $I \subseteq P$ be two graded ideals of a graded ring R, where P is a graded prime ideal. Then the following statements are equivalents*:

(1) *P is a minimal graded prime ideal of I.*
(2) *$h(R) \setminus P$ is a multiplicative closed set of homogeneous elements that is maximal with respect to missing I.*
(3) *For each homogeneous element $x \in P$, there is a homogeneous element $y \notin P$ and a nonnegative integer i such that $yx^i \in I$.*

Proof. (1) $\Rightarrow$ (2) Develop $h(R) \setminus P$ to a multiplicatively closed set of homogeneous elements S that is maximal with respect to missing I. If Q is a graded ideal containing I that is graded maximal with respect to the exclusion of S, then by Theorem 2.10, Q is graded prime. Note that $Q \cap (h(R) \setminus P) = \emptyset$. Now, since, by hypothesis, P is a minimal graded prime ideal of I, we have that $Q = P$. Hence, $h(R) \setminus P = S$.

(2) $\Rightarrow$ (3) Take a nonzero homogeneous element $x \in P$ and let $S = \{yx^i : y \in h(R) \setminus P, i = 0, 1, 2, \ldots\}$. So S is a multiplicatively closed set of homogeneous elements that properly contains $h(R) \setminus P$. So there is some $y \in h(R) \setminus P$ and a nonnegative i such that $yx^i \in I$.

(3) $\Rightarrow$ (1) Suppose that $I \subset Q \subseteq P$, where Q is a graded prime ideal. If there exists some $x = \sum_{g\in G} x_g \in P \setminus Q$, since P is a graded ideal, for all $g \in G$, $x_g \in P \setminus Q$. Then there is a $y \notin h(P)$ and a positive integer i such that $yx_g^i \in I \subset Q$. Since Q is graded prime, $y \in Q$ or $x_g^i \in Q$, a contradiction. Therefore $Q = P$. $\square$

The next result presents the graded version of [1, Theorem 2.5].

Theorem 2.12. *Let I be a graded n-absorbing ideal of a graded ring R. Then there are at most n graded prime ideals of R minimal over I.*

Proof. We may suppose that $n \geq 2$ since a graded 1-absorbing ideal is a graded prime ideal. Suppose that $P_1, \ldots, P_{n+1}$ are distinct graded prime ideals of R minimal over I. Thus, for each $1 \leq i \leq n$ there is an $x_i \in P_i \setminus (\cup_{k\neq i} P_k) \cup P_{n+1}$. Since all the P_i's are graded, we may suppose that the x_i's are homogeneous. By Theorem 2.11, for each $1 \leq i \leq n$, there is a homogeneous element $y_i \in R\setminus P_i$ such that $y_i x_i^{n_i} \in I$ for some integer $n_i \geq 1$. We may assume that all the y_i's have the same degree such that $y_i x_i^{n_i} \in I$ for each $1 \leq i \leq n$. Since $I \subseteq P_{n+1}$ is a graded n-absorbing ideal of R and $x_i \notin P_{n+1}$ for each $1 \leq i \leq n$, we have that $y_i x_i^{n-1} \in I$ for each $1 \leq i \leq n$, and hence $(y_1 + \cdots + y_n)\, x_1^{n-1} \cdots x_n^{n-1} \in I$. Since $x_i \in P_i \setminus (\cup_{k\neq i} P_k)$ and $y_i x_i^{n-1} \in I \subseteq P_1 \cap \cdots \cap P_n$ for each $1 \leq i \leq n$, we have $y_i \in (\cap_{k\neq i} P_k) \setminus P_i$ for each $1 \leq i \leq n$, and thus $(y_1 + \cdots + y_n) \notin P_i$ for each $1 \leq i \leq n$. Hence $(y_1 + \cdots + y_n) \prod_{k\neq i} x_k^{n-1} \notin P_i$ for each $1 \leq i \leq n$; so

$(y_1 + \cdots + y_n) \prod_{k \neq i} x_k^{n-1} \notin I$ for each $1 \leq i \leq n$, and thus $x_1^{n-1} \cdots x_n^{n-1} \in I \subseteq P_{n+1}$. Now, since I is a graded n-absorbing ideal of R, we have that $x_i \in P_{n+1}$ for some $1 \leq i \leq n$, which is a contradiction. Therefore there are at most n graded prime ideals of R minimal over I, as desired. □

The next result presents the graded version of [6, Corollary 2.7].

Theorem 2.13. *Let I be a graded n-absorbing primary ideal of R. Then $Gr(I) = P_1 \cap P_1 \cap \cdots \cap P_i$, where $1 \leq i \leq n$ and P_i's are the only distinct graded prime ideals of R that are minimal over I.*

Proof. It follows directly from Theorem 2.12, since if I is a graded n-absorbing primary ideal of R, it is clear that $Gr(I)$ is a graded n-absorbing ideal of R. □

Note that the concepts of graded n-irreducible ideals and of graded n-absorbing ideals defined first by M. Hamoda and A. Eid Ashour (see [2, Definition 2.1]) are different in general as shown by the following example.

Example 2.14. Let $R = \mathbb{Z}[i]$ be the Gaussian integers ring with $G = \mathbb{Z}_2$ and consider the ideal $I = (12)$ of $\mathbb{Z}[i]$. I is a graded ideal since 12 is a homogeneous element of R. Now, since $2.2.3 \in I$, but $2.2 \notin I$ and $2.3 \notin I$, then I is not a graded 2-absorbing ideal of R. On the other hand, Theorem 3.5 given hereinafter in Section 3 ensures that I is a graded 2-irreducible ideal of R. However, in the following theorem, we show that these concepts are comparable in some cases.

Theorem 2.15. *Let I be a graded radical ideal of a ring R, i.e., $Gr(I) = I$. The following assertions are equivalent*:

(1) *I is strongly graded n-irreducible*;
(2) *I is graded n-absorbing*;
(3) *I is graded n-absorbing primary*;
(4) *I is an intersection of exactly n graded prime ideals of R.*

Proof. (1) ⇒ (2) Assume that I is strongly graded n-irreducible. Let $I_1, \ldots,$ I_{n+1} be graded ideals of R such that $I_1 \cdots I_{n+1} \subseteq I$. Hence, $\bigcap_{i=1}^{n+1} I_i \subseteq Gr(\bigcap_{i=1}^{n+1} I_i) \subseteq Gr(I) = I$. Since I is strongly graded n-irreducible, then there are n of the I_i's whose intersection is in I. So, there are n of the I_i's whose product is in I. Consequently I is graded n-absorbing.

(2) ⇒ (3) It is trivial.

(3) ⇒ (4) It follows from Theorem 2.13.

(4) ⇒ (1) As a particular case of Proposition 2.3(3), we have that any graded prime ideal is strongly graded irreducible. Then the result follows from Proposition 2.5. □

Example 2.16. Consider the ideal $I = (p_1 p_2 \cdots p_n)$ of the Gaussian integer graded ring $\mathbb{Z}[i]$ with $G = \mathbb{Z}_2$, where the p_i's are some distinct homogeneous prime elements of $\mathbb{Z}[i]$ and n is a positive integer. Then $I = (p_1) \cap (p_2) \cap \cdots \cap (p_n)$ is exactly the intersection of n graded prime ideals of $\mathbb{Z}[i]$. Moreover, $Gr(I) = I$.

Then in light of Theorem 2.15, I is graded n-irreducible (strongly graded n-irreducible).

Recall from the start of this section that the intersection of any two graded ideals is a graded ideal. We next clarify the situation for the stability of graded n-irreducible and strongly graded n-irreducible ideals in various graded ring-theoretic constructions.

Theorem 2.17. *Let $f : R \to S$ be a surjective graded ring homomorphism, and let I be a proper graded ideal of R containing* $\ker(f)$. *Then,*

(1) *If I is a strongly graded n-irreducible ideal of R, then $f(I)$ is a strongly graded n-irreducible ideal of S.*

(2) *I is a graded n-irreducible ideal of R if and only if $f(I)$ is a graded n-irreducible ideal of S. In particular, this holds if f is a graded ring isomorphism.*

Proof. Since f is a surjective graded ring homomorphism, $f(J \cap R) = J$ for each graded ideal J of S and $f(K \cap L) = f(K) \cap f(L)$, and $f(K) \cap R = K$ for every graded ideals K, L of R which contain $\ker(f)$.

(1) Assume that I is a strongly graded n-irreducible ideal of R. If $f(I) = S$, then $I = f(I) \cap R = R$, which is a contradiction. Let $J_1, \dots, J_{n+1}$ be graded ideals of S such that $J_1 \cap \cdots \cap J_{n+1} \subseteq f(I)$. Hence, $(J_1 \cap R) \cap \cdots \cap (J_{n+1} \cap R) \subseteq f(I) \cap R = I$, and so there are n of the $(J_i \cap R)$'s whose intersection is in I. Without loss of generality, suppose that $(J_1 \cap R) \cap \cdots \cap (J_n \cap R) \subseteq I$, therefore $J_1 \cap \cdots \cap J_n \subseteq f(I)$. Then $f(I)$ is strongly graded n-irreducible.

(2) The direct implication is similar to the part (1). Conversely, assume that $f(I)$ is a graded n-irreducible ideal of S. Let $I_1, \dots, I_{n+1}$ be graded ideals of R such that $I = I_1 \cap \cdots \cap I_{n+1}$. Then $f(I) = f(I_1) \cap \cdots \cap f(I_{n+1})$. Therefore, because the image of every graded ideal is a graded ideal, there are n of the $f(I_i)$'s whose intersection is $f(I)$ since $f(I)$ is a graded n-irreducible ideal of S. Without loss of generality, assume that $f(I) = f(I_1) \cap \cdots \cap f(I_n)$. Then $I = f(I) \cap R = I_1 \cap \cdots \cap I_n$. So I is graded n-irreducible, as asserted. □

Define a *graded ring extension* to be a graded ring homomorphism $R \to S$, which makes S a graded R-module. The next result presents a direct corollary of the previous theorem.

Corollary 2.18. (1) *Let $R \subseteq S$ be a graded ring extension and J a graded n-irreducible ideal of S. Then $J \cap R$ is a graded n-irreducible ideal of R.*

(2) *Let $I \subseteq J$ be graded ideals of R. Then J is a graded n-irreducible ideal if and only if J/I is a graded n-irreducible ideal of R/I.*

Let R be a ring and let $\{x_1, x_2, \dots\}$ be (commuting) algebraically independent indeterminates over R. For $m = (m_1, \dots, m_n) \in \mathbb{N}^n$, let $x^m = x_1^{m_1} \cdots x_n^{m_n}$. Then the polynomial ring $P = R[x_1, \dots, x_n]$ is graded by $\mathbb{N}$ via $P_s = \left\{\sum_{m \in \mathbb{N}^n} r_m x^m \,|\, r_m \in R \text{ and } \sum_{i=1}^n m_i = s\right\}$. Note that respecting this graduation every ideal of R is a graded ideal of the polynomial ring P since

$P_0 = R$. The following result examines the extensions of n-irreducible ideals of R in the polynomial ring $R[X]$.

Corollary 2.19. *Let I be an ideal of a ring R. Then,*

(1) *(I, X) is a graded n-irreducible ideal of $R[X]$ if and only if I is an n-irreducible ideal of R.*
(2) *$I[X]$ is a graded n-irreducible ideal of $R[X]$ if and only if I is an n-irreducible ideal of R.*

Proof. (1) It follows directly from Corollary 2.18(2) since $(I, X)/(X) \cong I$ (graded ring isomorphism since X is a homogeneous element) in $R[X]/(X) \cong R$ (graded ring isomorphism).

(2) By Corollary 2.18(1), if $I[X]$ is a graded n-irreducible ideal of $R[X]$, then I is a graded n-irreducible ideal of R.

Conversely, suppose that I is a n-irreducible ideal of R and let $I_1, \ldots, I_{n+1}$ be graded ideals of $R[X]$ such that $I[X] = I_1 \cap \cdots \cap I_{n+1}$. For all $1 \leq i \leq n+1, (I_i \cap R)[X] \subseteq I_i$. So, by taking $I'_i = I_i \cap R$, we have that $I[X] = I'_1[X] \cap \cdots \cap I'_{n+1}[X] = \left(I'_1 \cap \cdots \cap I'_{n+1}\right)[X]$, therefore $I = I[X] \cap R = \left(I'_1 \cap \cdots \cap I'_{n+1}\right) = \left(I'_1 \cap \cdots \cap I'_{n+1}\right)[X] \cap R$. Since I is n-irreducible, then there are n of the I'_i's whose intersection is I. Without loss of generality, assume that $I = I'_1 \cap \cdots \cap I'_n$. Hence, $I[X] = I'_1[X] \cap \cdots \cap I'_n[X]$ and therefore $I[X] = I_1 \cap \cdots \cap I_n$. As a result, $I[X]$ is a graded n-irreducible ideal of $R[X]$. $\square$

Let G be a group and S be a multiplicatively closed subset of homogeneous elements of a graded ring R. In the next result, consider the natural graded ring homomorphism $f : R \to S^{-1}R$ defined by $f(x) = x/1$. For each graded ideal I of the graded ring $S^{-1}R$, we consider $I^c = \{x \in R \mid x/1 \in I\} = I \cap R$, which is a graded ideal of R and $C^{gr} = \left\{I^c \mid I \text{ is a graded ideal of } S^{-1}R\right\}$.

Theorem 2.20. *Let G be a group, R be a graded ring and S be a multiplicatively closed set of homogeneous elements of R. Then there is a one-to-one correspondence between the strongly graded n-irreducible ideals of $S^{-1}R$ and strongly graded n-irreducible ideals of R contained in C^{gr} which do not meet S.*

Proof. Let I be a strongly graded n-irreducible ideal of $S^{-1}R$. It is easy to see that $I^c \neq R$, $I^c \in C^{gr}$ and $I^c \cap S = \emptyset$. Let $I_1, \ldots, I_{n+1}$ be $(n+1)$ graded ideals of R such that $I_1 \cap \cdots \cap I_{n+1} \subseteq I^c$. Then $\left(S^{-1}I_1\right) \cap \cdots \cap \left(S^{-1}I_{n+1}\right) = S^{-1}\left(I_1 \cap \cdots \cap I_{n+1}\right) \subseteq S^{-1}\left(I^c\right) = I$. Therefore there are n of the $S^{-1}I_i$'s whose intersection is in I since I is strongly graded n-irreducible. Hence, there are n of the I_i's whose intersection is in I^c. As a result, I^c is a strongly graded n-irreducible ideal of R.

Conversely, let I be a strongly graded n-irreducible ideal of R such that $I \cap S = \emptyset$ and $I \in C^{gr}$. Since $I \cap S = \emptyset$, $S^{-1}I \neq S^{-1}R$. Let $I_1 \cap I_2 \cap \cdots \cap I_{n+1} \subseteq S^{-1}I$, where $I_1, I_2, \ldots, I_{n+1}$ are graded ideals of $S^{-1}R$. Therefore $(I_1^c) \cap (I_2^c) \cap \cdots \cap \left(I_{n+1}^c\right) = \left(I_1 \cap I_2 \cap \cdots \cap I_{n+1}\right)^c \subseteq \left(S^{-1}I\right)^c$. Now, since

$I \in C^{gr}$, $\left(S^{-1}I\right)^c = I$ and so $(I_1^c)\cap(I_2^c)\cap\cdots\cap\left(I_{n+1}^c\right) \subseteq I$. As a result, there are n of the I_i^c's whose intersection is in I. Without loss of generality, suppose that $(I_1^c)\cap(I_2^c)\cap\cdots\cap(I_n^c) \subseteq I$. Therefore, $S^{-1}(I_1^c)\cap S^{-1}(I_2^c)\cap\cdots\cap S^{-1}(I_n^c) \subseteq S^{-1}I$. Hence, $S^{-1}I$ is a strongly graded n-irreducible ideal of $S^{-1}R$. □

Corollary 2.21. *Let G be a group, R be a graded ring and S be a multiplicatively closed set of homogeneous elements of R. If I is a graded primary and strongly graded n-irreducible ideal of R which does not meet S, then $S^{-1}I$ is a strongly graded n-irreducible ideal of $S^{-1}R$.*

Proof. Since I is a graded primary ideal of R, and $I \cap S = \emptyset$, by [9, Proposition 1.15(iii)], we have that $(I : s) = I$ for each $s \in S$, so then $\left(S^{-1}I\right)^c = I$ and hence $I \in C^{gr}$. Now, it remains to use Theorem 2.20 to get that $S^{-1}I$ is a strongly graded n-irreducible ideal of $S^{-1}R$. □

We close this section by a result concerning the strongly graded n-irreducible ideals in the product of a finite number of graded rings. Recall that a graded ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ for some graded ideals I_i of R_i.

Proposition 2.22. *Let I_1 be a strongly graded n-irreducible ideal of a graded ring R_1 and I_2 be a strongly graded m-irreducible ideal of a graded ring R_2. Then $J = I_1 \times I_2$ is a strongly graded $(n+m)$-irreducible ideal of the graded ring $R = R_1 \times R_2$.*

Proof. If $J = I_1 \times R_2$ for some strongly graded n-irreducible ideal I_1 of R_1 or $R_1 \times I_2$ for some strongly graded m-irreducible ideal I_2 of R_2, it is easy to see that J is a strongly graded n-irreducible or strongly graded m-irreducible ideal of $R_1 \times R_2$. Therefore, assume that $J = I_1 \times I_2$ for some strongly graded n-irreducible ideal I_1 of R_1 and some strongly graded m-irreducible ideal I_2 of R_2. Then $I_1' = I_1 \times R_2$ is a strongly graded n-irreducible ideal of $R_1 \times R_2$ and $I_2' = R_1 \times I_2$ is a strongly graded m-irreducible ideal of $R_1 \times R_2$. Hence, $I_1' \cap I_2' = I_1 \times I_2 = J$ is a strongly graded $(n+m)$-irreducible ideal of $R_1 \times R_2$ by Proposition 2.5. □

Corollary 2.23. *Let I_k be a strongly graded n_k-irreducible ideal of a graded ring R_k for each integer $1 \leq k \leq m$. Let $R = R_1 \times \cdots \times R_m$. Then $I_1 \times \cdots \times I_m$ is a strongly graded n-irreducible ideal of R with $n = n_1 + \cdots + n_m$.*

3. Extension to specific graded rings

We devote this section to the study of the transfer of graded n-irreducible and strongly graded n-irreducible ideals in several special classes of graded commutative rings. We begin by introducing the notion of *graded arithmetical* ring. A graded ring R is said to be a *graded arithmetical* ring if for each graded ideals I, J and K of R, $(I+J)\cap K = (I\cap K)+(J\cap K)$. Just like the ungraded case, this condition is equivalent to the condition that for each graded ideals I, J and K of R, $(I\cap J)+K = (I+K)\cap(J+K)$.

Let G be a group. Recall from [12] that a graded integral domain is said to be a graded principal ideal domain (gr-PID, for short) if every graded ideal of R is principal and a graded integral domain is said to be a graded Dedekind domain if every graded ideal is projective. Note that every graded principal ideal domain (gr-PID) is a graded Dedekind domain, see [12, Corollaries 1.2(1.2.1)]. Also, every graded Dedekind domain is a graded arithmetical domain.

In order to prove the main result of this section "Theorem 3.5", we first give the following result which presents the graded version of [6, Theorem 2.8] (for $\phi = \emptyset$).

Theorem 3.1. *Let G be a group and R be a graded ring. Suppose that for every $1 \leq i \leq k, I_i$ is a graded n_i-absorbing primary ideal of R such that $Gr(I_i) = P_i$ is a graded n_i-absorbing ideal of R, respectively. Set $n := n_1 + n_2 + \cdots + n_k$. The following conditions hold*:

(1) *$I_1 \cap I_2 \cap \cdots \cap I_k$ is a graded n-absorbing primary ideal of R.*

(2) *$I_1 I_2 \cdots I_k$ is a graded n-absorbing primary ideal of R.*

Proof. (1) Let $L = I_1 \cap I_2 \cap \cdots \cap I_k$. Then using [10, Proposition 2.4(4)], we have $Gr(L) = P_1 \cap P_2 \cap \cdots \cap P_k$. Assume that $a_1 a_2 \cdots a_{n+1} \in L$ for some $a_1, a_2, \ldots, a_{n+1} \in h(R)$ and $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin Gr(L)$ for every $1 \leq i \leq n$. Since G is a group, using [2, Theorem 2.7], $Gr(L) = P_1 \cap P_2 \cap \cdots \cap P_k$ is graded n-absorbing, then $a_1 a_2 \cdots a_n \in P_1 \cap P_2 \cap \cdots \cap P_k$. We have to show that $a_1 a_2 \cdots a_n \in L$. For every $1 \leq i \leq k, P_i$ is graded n_i-absorbing and $a_1 a_2 \cdots a_n \in P_i$, then there exist elements $1 \leq \alpha_1^i, \alpha_2^i, \ldots, \alpha_{n_i}^i \leq n$ such that $a_{\alpha_1^i} a_{\alpha_2^i} \cdots a_{\alpha_{n_i}^i} \in P_i$. If $\alpha_e^c = \alpha_f^d$ for two couples (c, e) and (d, f), then

$$\begin{aligned} &a_{\alpha_1^1} a_{\alpha_2^1} \cdots a_{\alpha_{n_1}^1} \cdots a_{\alpha_1^c} a_{\alpha_2^c} \cdots a_{\alpha_e^c} \cdots a_{\alpha_{n_c}^c} \cdots \\ &a_{\alpha_1^d} a_{\alpha_2^d} \cdots \widehat{a_{\alpha_f^d}} \cdots a_{\alpha_{n_d}^d} \cdots a_{\alpha_1^k} a_{\alpha_2^k} \cdots a_{\alpha_{n_k}^k} \in Gr(L). \end{aligned}$$

Hence, $a_1 \cdots \widehat{a_{\alpha_f^d}} \cdots a_n a_{n+1} \in Gr(L)$, a contradiction. So the α_j^i's are distinct. Therefore

$$\left\{ a_{\alpha_1^1}, a_{\alpha_2^1}, \ldots, a_{\alpha_{n_1}^1}, a_{\alpha_1^2}, a_{\alpha_2^2}, \ldots, a_{\alpha_{n_2}^2}, \ldots, a_{\alpha_1^k}, a_{\alpha_2^k}, \ldots, a_{\alpha_{n_k}^k} \right\} = \{a_1, a_2, \ldots, a_n\}.$$

If $a_{\alpha_1^i} a_{\alpha_2^i} \cdots a_{\alpha_{n_i}^i} \in I_i$ for every $1 \leq i \leq k$, then

$$a_1 a_2 \cdots a_n = a_{\alpha_1^1} a_{\alpha_2^1} \cdots a_{\alpha_{n_1}^1} a_{\alpha_1^2} a_{\alpha_2^2} \cdots a_{\alpha_{n_2}^2} \cdots a_{\alpha_1^k} a_{\alpha_2^k} \cdots a_{\alpha_{n_k}^k} \in L,$$

as desired. Hence, we may assume that $a_{\alpha_1^1} a_{\alpha_2^1} \cdots a_{\alpha_{n_1}^1} \notin I_1$. Since I_1 is graded n_1-absorbing primary and

$$a_{\alpha_1^1} a_{\alpha_2^1} \cdots a_{\alpha_{n_1}^1} a_{\alpha_1^2} a_{\alpha_2^2} \cdots a_{\alpha_{n_2}^2} \cdots a_{\alpha_1^k} a_{\alpha_2^k} \cdots a_{\alpha_{n_k}^k} a_{n+1} = a_1 \cdots a_{n+1} \in I_1$$

we have $a_{\alpha_1^1} \cdots \widehat{a_{\alpha_t^1}} \cdots a_{\alpha_{n_1}^1} a_{\alpha_1^2} a_{\alpha_2^2} \cdots a_{\alpha_{n_2}^2} \cdots a_{\alpha_1^k} a_{\alpha_2^k} \cdots a_{\alpha_{n_k}^k} a_{n+1} \in P_1$ for some $1 \leq t \leq n_1$. On the other hand, since $a_{\alpha_1^i} a_{\alpha_2^i} \cdots a_{\alpha_{n_i}^i} \in I_i \subseteq Gr(I_i) = P_i$ for every $2 \leq i \leq k$,

$$a_{\alpha_1^1} \cdots \widehat{a_{\alpha_t^1}} \cdots a_{\alpha_{n_1}^1} a_{\alpha_1^2} a_{\alpha_2^2} \cdots a_{\alpha_{n_2}^2} \cdots a_{\alpha_1^k} a_{\alpha_2^k} \cdots a_{\alpha_{n_k}^k} a_{n+1} \in P_2 \cap \cdots \cap P_k.$$

As a result, $a_{\alpha_1^1} \cdots \widehat{a_{\alpha_t^1}} \cdots a_{\alpha_{n_1}^1} a_{\alpha_1^2} a_{\alpha_2^2} \cdots a_{\alpha_{n_2}^2} \cdots a_{\alpha_1^k} a_{\alpha_2^k} \cdots a_{\alpha_{n_k}^k} a_{n+1} \in Gr(L)$, which is a contradiction. Likewise, we deduce that $a_{\alpha_1^i} a_{\alpha_2^i} \cdots a_{\alpha_{n_i}^i} \in I_i$ for every $2 \leq i \leq k$. Then $a_1 a_2 \cdots a_n \in L$.

The proof of (2) is omitted since it is similar to that of the part (1). □

The following result is a direct corollary of the previous theorem. We refer the reader to [9] for more information about the notion of graded primary ideals.

Corollary 3.2. *Let G be a group and R be a graded ring with $1 \neq 0$ and let $P_1, P_2, \ldots, P_n$ be graded prime ideals of R. Suppose that for every $1 \leq i \leq n$, $P_i^{t_i}$ is a graded P_i-primary ideal of R, where t_i is a positive integer. Then $P_1^{t_1} \cap P_2^{t_2} \cap \cdots \cap P_n^{t_n}$ and $P_1^{t_1} P_2^{t_2} \cdots P_n^{t_n}$ are graded n-absorbing primary ideals of R. In particular, $P_1 \cap P_2 \cap \cdots \cap P_n$ and $P_1 P_2 \cdots P_n$ are graded n-absorbing primary ideals of R.*

Recall from [7] that a graded ring R is said to be graded Noetherian if it satisfies the ascending chain condition on graded ideals of R. The following result presents the graded version of [6, Theorem 2.15].

Theorem 3.3. *Let G be a group and R be a graded Noetherian integral domain that is not a graded field. The following conditions are equivalent:*

(1) *R is a graded Dedekind domain;*
(2) *A nonzero proper graded ideal I of R is a graded n-absorbing primary ideal of R if and only if $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded maximal ideals $M_1, M_2, \ldots, M_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$;*
(3) *If I is a nonzero graded n-absorbing primary ideal of R, then $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded maximal ideals $M_1, M_2, \ldots, M_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$;*
(4) *A nonzero proper graded ideal I of R is a graded n-absorbing primary ideal of R if and only if $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded prime ideals $P_1, P_2, \ldots, P_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$;*
(5) *If I is a nonzero graded n-absorbing primary ideal of R, then $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded prime ideals $P_1, P_2, \ldots, P_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$.*

Proof. The proof is organized as follows, the proof of (1) $\Rightarrow$ (4) is similar to that of (1) $\Rightarrow$ (2) and it is omitted. (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (5) are straightforward. It remains to claim (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Suppose that R is a graded Dedekind domain that is not a graded field. Then, by [12, Lemma 1.1(3)], every nonzero graded prime ideal of R is graded maximal. Let I be a nonzero graded n-absorbing primary ideal of R. Since R is a graded Dedekind domain, by [12, Lemma 1.1(4)], there are distinct graded maximal ideals $M_1, M_2, \ldots, M_i$ of R ($i \geq 1$) such that $I =$

$M_1^{t_1}M_2^{t_2}\cdots M_i^{t_i}$ in which t_j's are positive integers. Hence, $Gr(I) = M_1 \cap M_2 \cap \cdots \cap M_i$. Since I is graded n-absorbing primary and every graded prime ideal of R is graded maximal, by Theorem 2.13, $Gr(I)$ is the intersection of at most n graded maximal ideals of R. Therefore, $i \leq n$. For the reverse implication, assume that $I = M_1^{t_1}M_2^{t_2}\cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct maximal ideals $M_1, M_2, \ldots, M_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$. Then I is n-absorbing primary by Corollary 3.2.

(5) $\Rightarrow$ (1) Let M be an arbitrary graded maximal ideal of R and I be a graded ideal of R such that $M^2 \subset I \subset M$. Hence $Gr(I) = M$ and so I is graded M-primary. Then I is graded n-absorbing primary, and thus by assertion (5) we have that $I = P_1^{t_1}P_2^{t_2}\cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded prime ideals $P_1, P_2, \ldots, P_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$. Then $Gr(I) = P_1 \cap P_2 \cap \cdots \cap P_i = M$ which proves that I is a power of M, a contradiction. Hence, there are no graded ideals properly between M^2 and M. Consequently R is a graded Dedekind domain by [12, Lemma 1.1(6)]. $\square$

The following result is a direct corollary of the previous theorem.

Corollary 3.4. *Let G be a group and R be a gr-PID and I be a nonzero proper graded ideal of R. Then I is a graded n-absorbing primary ideal of R if and only if $I = R(p_1^{t_1}p_2^{t_2}\cdots p_i^{t_i})$, where p_j's are prime homogeneous elements of R, $1 \leq i \leq n$ and t_j's are some integers.*

Now, we are able to give the main result of this Section 3.

Theorem 3.5. *Let G be a group and R be a gr-PID and I be a nonzero proper graded ideal of R. The following conditions are equivalent:*

(1) *I is graded n-irreducible;*
(2) *I is graded n-absorbing primary;*
(3) *$I = R\left(p_1^{l_1}\ldots p_m^{l_m}\right)$ for some distinct homogeneous prime elements $p_1, \ldots, p_m$ of R and some natural numbers $l_1, \ldots, l_m$ such that $m \leq n$.*

Proof. (1) $\Rightarrow$ (3) Suppose that $I = Ra$, where a is a nonzero homogeneous element of R and let $a = p_1^{l_1}\ldots p_m^{l_m}$ be a homogeneous prime decomposition for a. We have to claim that $m \leq n$. By contradiction, assume that $m > n$. Since R is a gr-PID and $p_1, \ldots, p_m$ are homogeneous prime elements of R, $I = Rp_1^{l_1} \cap \cdots \cap Rp_m^{l_m}$. Now, since I is graded n-irreducible, there are n of the $Rp_i^{l_i}$'s whose intersection is I. Without loss of generality, suppose that $I = Rp_1^{l_1} \cap \cdots \cap Rp_n^{l_n}$. It follows that $Rp_1^{l_1} \cap \cdots \cap Rp_n^{l_n} \subseteq Rp_{n+1}^{l_{n+1}}$, which is a contradiction.

(3) $\Rightarrow$ (1) Suppose that $I = R\left(p_1^{l_1}\ldots p_n^{l_n}\right)$ for some distinct homogeneous prime elements $p_1, \ldots, p_n$ of R and some natural numbers $l_1, \ldots, l_n$. Let $I = Ra_1 \cap \cdots \cap Ra_{n+1}$ for some homogeneous elements $a_1, \ldots, a_{n+1}$ of R. Hence, $a_1, a_2, \ldots, a_{n+1}$ divides $p_1^{l_1}\cdots p_n^{l_n}$. So $a_i = p_1^{l_{i,1}}\cdots p_n^{l_{i,n}}$, where $l_{i,1}, \ldots, l_{i,n}$ are some nonnegative integers for each $1 \leq i \leq n+1$. Let $\alpha_i = \max\{l_{1,i}, \ldots l_{n+1,i}\}$

for each $1 \leq i \leq n$. Hence, $I = Ra_1 \cap \cdots \cap Ra_{n+1} = R\,[a_1, \ldots, a_{n+1}] = R\,(p_1^{\alpha_1} \cdots p_n^{\alpha_n})$. Without loss of generality, suppose that $\alpha_1 = l_{1,1}, \ldots, \alpha_n = l_{n,n}$. Hence, $I = R\left(p_1^{l_{1,1}} \cdots p_n^{l_{n,n}}\right) = Ra_1 \cap \cdots \cap Ra_n$. As a result, I is graded n-irreducible.

(2) $\Leftrightarrow$ (3) It follows directly from Corollary 3.4. $\square$

Proposition 3.6. *Let I be a proper graded ideal of a graded arithmetical ring R. The following conditions are equivalent*:

(1) *I is a graded n-irreducible ideal of R*;

(2) *I is a strongly graded n-irreducible ideal of R.*

Proof. (1) $\Rightarrow$ (2) Assume that $I_1, \ldots, I_{n+1}$ are graded ideals such that $\bigcap_{i=1}^{n+1} I_i \subseteq I$. Therefore, $I = I + \left(\bigcap_{i=1}^{n+1} I_i\right) = (I + I_1) \cap (I + I_2) \cap \cdots \cap (I + I_{n+1})$ since R is a graded arithmetical ring. By hypothesis, I is graded n-irreducible, then there are n of the $(I + I_i)$'s whose intersection is I. Therefore, there are n of the $(I + I_i)$'s whose intersection is in I which implies that I is a strongly graded n-irreducible ideal of R.

(2) $\Rightarrow$ (1) By Proposition 2.3(1). $\square$

The following corollary is an immediate consequence of Theorem 3.5 and Proposition 3.6.

Corollary 3.7. *Let G be a group and R be a gr-PID and I be a nonzero proper graded ideal of R. The following conditions are equivalent*:

(1) *I is strongly graded n-irreducible*;

(2) *I is graded n-irreducible*;

(3) *I is graded n-absorbing primary*;

(4) *$I = R\left(p_1^{l_1} \ldots p_m^{l_m}\right)$ for some distinct homogeneous prime elements $p_1, \ldots, p_m$ of R and some natural numbers $l_1, \ldots, l_m$ such that $m \leq n$.*

Example 3.8. Let $n \geq 1$ be a positive integer. Then there is a graded n-irreducible, but not a graded $(n-1)$-irreducible ideal of a graded ring R. Let $R = \mathbb{Z}[i]$ be the Gaussian integers ring with $G = \mathbb{Z}_2$ and pick $I = \langle 30 \rangle = \langle (2.3.5) \rangle$. By Corollary 3.7, I is a strongly graded 3-irreducible (graded 3-irreducible) ideal of R. On the other hand, since $Gr(I) = \langle 2 \rangle \cap \langle 3 \rangle \cap \langle 5 \rangle = I$, I is a graded radical ideal and so, by Theorem 2.15, I is not a graded strongly 2-irreducible (not a graded 2-irreducible) ideal of R. More generally, let $p_1, \ldots, p_n \in \mathbb{Z}[i]$ be distinct positive homogeneous primes elements of $\mathbb{Z}[i]$. Then $I = (p_1 \ldots p_n)\,\mathbb{Z}[i]$ is a graded n-irreducible but not graded $(n-1)$-irreducible ideal of $\mathbb{Z}[i]$.

Let G be a group. Recall that a graded ring R is said to be a graded von Neumann regular ring (gr-von Neumann regular for short) if for each $a \in h(R)$, there exists $b \in h(R)$ such that $a = a^2 b$. In this case, the graded principal ideal (a) of R is generated by a homogeneous idempotent element $e \in R$. It is known by [3, Proposition 2] that a ring R is a gr-von Neumann regular ring if and only if its each graded ideal I of R is idempotent, that is, $I = I^2$ if and only if each graded ideal I of R is graded radical, that is, $I = Gr(I)$ if and only if any finitely generated graded ideal of R is a projective module.

Proposition 3.9. *Let G be a group, R be a gr-von Neumann regular ring and I be a nonzero graded ideal of R. Then I is graded n-irreducible if and only if I is graded n-absorbing.*

Proof. It is clear that a commutative graded ring R is a graded von Neumann regular ring if and only if $I_1 \cdots I_{n+1} = I_1 \cap \cdots \cap I_{n+1}$ for any graded ideals $I_1, \ldots, I_{n+1}$ of R. Hence the notions of graded n-irreducible and graded n-absorbing ideals coincide. □

Theorem 3.10. *Let R be a graded Noetherian ring. If I is a graded n-irreducible ideal of R, then either I is graded irreducible or I is the intersection of exactly n graded irreducible ideals.*

Proof. Assume that I is graded n-irreducible. Since R is graded Noetherian, then I can be expressed as an intersection of finitely many graded irreducible ideals of R by [9, Proposition 2.14], say $I = I_1 \cap I_2 \cap \cdots \cap I_k$. It remains to prove that $k \leq n$. Assume that $k \geq n+1$. Then since I is graded n-irreducible, there are n of the I_i 's whose intersection is I. Without loss of generality, suppose that $I_1 \cap I_2 \cap \cdots \cap I_n = I$, hence $I_1 \cap I_2 \cap \cdots \cap I_n \subseteq I_{n+1}$, which is a contradiction. □

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