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ON GRADED (m,n)-CLOSED SUBMODULES

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Abstract. Let A be a G-graded commutative ring with identity and M a graded A-module. Let m,n be positive integers with $m > n$. A proper graded submodule L of M is said to be graded (m,n)-closed if $a_g^m \cdot x_t \in L$ implies that $a_g^n \cdot x_t \in L$, where $a_g \in h(A)$ and $x_t \in h(M)$. The aim of this paper is to explore some basic properties of these class of submodules which are a generalization of graded (m,n)-closed ideals. Also, we investigate $GC_n^m - rad$ property for graded submodules.

1. Introduction

Throughout this paper G is a group, A is a G-graded commutative ring with identity and M is a graded A-module. If $a \in A$, then a can be written uniquely as $\Sigma_{g\in G} a_g$, where a_g is the component of a in A_g. Also, we write $h(A) = \bigcup_{g\in G} A_g$. Let I be an ideal of A. For $g \in G$, let $I_g = I \cap A_g$. Then I is a graded ideal of graded ring A if $I = \oplus_{g\in G} I_g$.

An A-module M is a G-graded A-module (or graded A-module) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ (as abelian groups) and $A_g M_h \subseteq M_{gh}$ for all $g,h \in G$. Also, $A_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $a_g x_h$ with $a_g \in A_g$ and $x_h \in M_h$. Also, we write $h(M) = \bigcup_{g\in G} M_g$ and the elements of $h(M)$ are called homogeneous.

A submodule $N \subseteq M$ is called graded if N is generated by homogeneous elements. We follow [6, 7] for definitions and information on graded rings and graded modules. A proper graded submodule P of a graded A-module M is called graded prime if whenever $a_g x_r \in P$, where $a_g \in h(A)$ and $x_r \in h(M)$, either $x_r \in P$ or $a_g \in (P :_A M)$. For more results of graded prime submodules one may refer to [3] and [9].

A proper graded submodule L of a graded A-module M is called graded semiprime if $a_g^m \cdot x_t \in L$ implies that $a_g \cdot x_t \in L$, where $a_g \in h(A)$ and $x_t \in h(M)$. The concept of semiprime submodules and graded semiprime submodules has been studied in many papers (see, [1], [5], [8] and [10]). It

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was shown that every graded prime submodule is graded semiprime, but the converse is not true in general (see [4]).

(m,n)-closed ideals in a commutative ring with nonzero identity have been introduced and studied in [2].

Let m, n be positive integers with $m > n$. A proper graded ideal I of A is called a graded (m,n)-closed if $a_g^m \cdot b_w \in L$ implies that $a_g^n \cdot b_w \in L$, where $a_g, b_w \in h(A)$. We define L to be a graded (m,n)-closed submodule of a graded A-module M if $a_g^m \cdot x_t \in L$ implies that $a_g^n \cdot x_t \in L$, where $a_g \in h(A)$ and $x_t \in h(M)$. On the other hand, L is called a graded (m,n)-closed submodule of M if $I^m K \subseteq L$ implies that $I^n K \subseteq L$, where $I \subseteq h(A)$ and $K \subseteq h(M)$. Clearly, every graded $(m,1)$-closed submodule is graded semiprime.

In this paper, we characterize graded (m,n)-closed submodules, which are a generalization of graded (m,n)-closed ideals.

A graded (m,n)-closed submodule L of a graded A-module M is said to be graded minimal if whenever $K \subseteq L$ and K is a graded (m,n)-closed submodule of M, $K = L$.

Two graded submodules L, K of a graded A-module M are called graded coprime whenever $K + L = M$, see [11].

A graded A-module M is called a graded multiplication module if $L = (L :_A M)M$ for every graded submodule L of M, see [4].

2. Graded (m,n)-closed submodules

By definition, every graded semiprime submodule of a graded module is a graded (m,n)-closed submodule, but the converse is not true in general. Let $n > 2$ be a positive integer. Consider the set $\mathbb{Z}$ of all integers. Take $G = (\mathbb{Z}, +)$ and $A = (\mathbb{Z}, +, \cdot)$. Define $A_g = \mathbb{Z}$, where $g = 0$, and $A_g = 0$ otherwise. Then each A_g is an additive subgroup of A and A is their internal direct sum. In fact, $1 \in A_0$ and $A_g A_h \subseteq A_{g+h}$. Hence A is a G-graded ring. Now, consider the graded A-module $M = \mathbb{Z}_{2^n}$. Then the graded submodule $L = 0$ of M is not graded semiprime, because $2^t \cdot \bar{2}^{n-t} = 0$ but $2 \cdot \bar{2}^{n-t} \neq 0$. But L is a graded (m,n)-closed submodule of M for every $m > n$.

Proposition 2.1. *Let L be a graded (m,n)-closed submodule of a graded A-module M. Then $(L :_A M)$ is a graded (m,n)-closed ideal of A.*

Proof. Let $a_g^m \cdot b_w \in (L :_A M)$, where $a_g, b_w \in h(A)$. Therefore $a_g^m(b_w M) = (a_g^m b_w)M \subseteq L$ and L graded (m,n)-closed implies that $a_g^n(b_w M) \subseteq L$. Thus $a_g^n b_w \in (L :_A M)$, as needed. □

The converse of Proposition 2.1 is not true in general. Consider the graded $(\mathbb{Z}, +, \cdot)$-module $M = \mathbb{Z} \times \mathbb{Z}$ with $M_0 = \mathbb{Z} \times 0$ and $M_1 = 0 \times \mathbb{Z}$. Let $m > 2$. Then $L = (5^m\mathbb{Z}, 0)$ is not a graded (m,n)-closed submodule of M for every $n < m$. But $(L :_A M) = 0$ is graded prime and so a graded (m,n)-closed ideal of A.

Theorem 2.2. *Let M be a graded multiplication A-module and L be a proper graded submodule of M. Then, L is a graded (m,n)-closed submodule of M if and only if $(L :_A M)$ is a graded (m,n)-closed ideal of A.*

Proof. This is enough to show that if $(L :_A M)$ is a graded (m,n)-closed ideal of A, then L is a graded (m,n)-closed submodule of M. Let $I^m N \subseteq L$, where $I \subseteq h(A)$ and $N \subseteq h(M)$. Then $I^m(N :_A M) \subseteq (I^m N :_A M) \subseteq (L :_A M)$ and $(L :_A M)$ graded (m,n)-closed implies that $I^n(N :_A M) \subseteq (L :_A M)$. Hence $I^n(N :_A M)M \subseteq (L :_A M)M$, that is, $I^n N \subseteq L$. □

Proposition 2.3. *Let L be a proper graded submodule of a graded A-module M and m,n be positive integers with $m > n$. Then the following statements hold.*

(i) *If L is a graded (m,n)-closed submodule, then L is a graded (p,n)-closed submodule for every $p \geq m$.*

(ii) *If L is a graded (m,n)-closed submodule, then L is a graded (m,p)-closed submodule for every $p \geq n$.*

(iii) *If L is a graded (m,n)-closed submodule, then L is a graded (p,q)-closed submodule for each $p \geq m$ and $q \geq n$.*

Proof. (i) Let L be a graded (m,n)-closed submodule of M and $p \geq m$. Let $a_g^p \cdot x_t \in L$, where $a_g \in h(A)$ and $x_t \in h(M)$. Hence $a_g^m(a_g^{p-m} \cdot x_t) \in L$, and L graded (m,n)-closed implies that $a_g^n(a_g^{p-m} \cdot x_t) \in L$. If $n+p-m \leq m$, then $a_g^m \cdot x_t \in L$ and L graded (m,n)-closed implies that $a_g^n \cdot x_t \in L$. Assume that $n+p-m > m$. So, $a_g^n(a_g^{p-m} \cdot x_t) = a_g^m(a_g^{p+n-2m} \cdot x_t) \in L$ and L graded (m,n)-closed implies that $a_g^n(a_g^{p+n-2m} \cdot x_t) \in L$. By this way we get $a_g^v \cdot x_t \in L$, where $v \leq m$. Therefore $a_g^m \cdot x_t \in L$ and L graded (m,n)-closed implies that $a_g^n \cdot x_t \in L$, as needed.

(ii), (iii) The proofs are similar to the proof of (i). □

By Proposition 2.3(i), it is clear that graded (m,n)-closedness is equivalent to graded $(n+1,n)$-closedness.

Proposition 2.4. *Let $L_1, L_2, \ldots, L_t$ be graded (m,n)-closed submodules of a graded A-module M. Then*

(i) *$L_1 \cap L_2 \cap \cdots \cap L_t$ is a graded (m,n)-closed submodule of M.*

(ii) *If $\{L_i\}_{i\in\mathbb{Z}_t}$ is totally ordered (by inclusion) and $L_1 \cup L_2 \cup \cdots \cup L_t \neq M$, then $L_1 \cup L_2 \cup \cdots \cup L_t$ is a graded (m,n)-closed submodule of M.*

Proof. (i) Let $a_g^m \cdot x_q \in L_1 \cap L_2 \cap \cdots \cap L_t$, where $a_g \in h(A)$ and $x_q \in h(M)$. Then for every $i \in \mathbb{Z}_t$, $a_g^m \cdot x_q \in L_i$. Now, L_i graded (m,n)-closed implies that $a_g^n \cdot x_q \in L_i$. Therefore, $a_g^n \cdot x_q \in L_1 \cap L_2 \cap \cdots \cap L_t$, as needed.

(ii) $\cup L_i \neq M$ and $\{L_i\}_{i\in\mathbb{Z}_t}$ totally ordered by inclusion implies that $\cup L_i$ is a proper graded submodule of M. Let $b_g^m \cdot u_p \in L_1 \cup L_2 \cup \cdots \cup L_t$, where $b_g \in h(A)$ and $u_p \in h(M)$. Hence $b_g^m \cdot u_p \in L_i$ for some $i \in \mathbb{Z}_t$. Thus $b_g^n \cdot u_p \in L_i$, and so $b_g^n \cdot u_p \in L_1 \cup L_2 \cup \cdots \cup L_t$, as needed. □

Note that, if M is a G-graded A-module and N is a graded A-submodule of M, then M/N is a G-graded A-module by $(M/N)_g = (M_g + N)/N$ for all $g \in G$.

Lemma 2.5. *Let L_1, L_2 be graded submodules of a graded A-module M with $L_1 \subseteq L_2$. Then L_2 is a graded (m, n)-closed submodule of M if and only if L_2/L_1 is a graded (m, n)-closed submodule of M/L_1.*

Proof. Let L_2 be a graded (m, n)-closed submodule of M and $a_g^m(x_q + L_1) \in L_2/L_1$, where $a_g \in h(A)$ and $x_q \in h(M)$. Then, $a_g^m \cdot x_q \in L_2$ and L_2 graded (m, n)-closed implies that $a_g^n \cdot x_q \in L_2$. Therefore, $a_g^n(x_q + L_1) \in L_2/L_1$, as needed.

Conversely, let $a_g^m \cdot x_q \in L_2$, where $a_g \in h(A)$ and $x_q \in h(M)$. Then $a_g^m(x_q+L_2) \in L_2/L_1$ and L_2/L_1 graded (m, n)-closed implies that $a_g^n(x_q+L_1) \in L_2/L_1$. Hence $a_g^n \cdot x_q \in L_2$ as needed. □

Corollary 2.6. *Let L_1, L_2 be graded submodules of a graded A-module M with $L_1 + L_2 \neq M$ and at least one of L_i is graded (m, n)-closed. Then $L_1 + L_2$ is a graded (m, n)-closed submodule of M.*

Proof. Let L_2 be graded (m, n)-closed. By the second isomorphism theorem for modules $(L_1 + L_2)/L_1 \cong L_2/(L_1 \cap L_2)$. The result holds by Lemma 2.5. □

Proposition 2.7. *Every graded (m, n)-closed submodule of a graded A-module M contains a minimal graded (m, n)-closed submodule.*

Proof. Let L be a graded (m, n)-closed submodule of M and Ω be the set of all graded (m, n)-closed submodules K of M such that $K \subseteq L$. Hence $L \in \Omega$ and $\subseteq$ is a partial order on Ω. Let ω be a non-empty subset of Ω which is totally ordered by $\subseteq$. Therefore by Proposition 2.4(i), $\cap_{K\in\omega} K$ is a graded (m, n)-closed submodule of M. Thus, the result holds by using the Zorn's lemma. □

Note that, if M_1, M_2 are G-graded A-modules, then $M = M_1 \times M_2$ is a G-graded A-module by $M_g = (M_1)_g \times (M_2)_g$ for all $g \in G$.

Proposition 2.8. *Let $A = A_1 \times A_2$ be a G-graded ring and $M = M_1 \times M_2$ be a graded A-module, where M_i is a graded A_i-module for $i \in \mathbb{Z}_2$. Let L_i be a proper graded submodule of M_i. If L_i is a graded (m, n)-closed submodule of M_i for $i \in \mathbb{Z}_2$, then $L_1 \times L_2$ is a graded (m, n)-closed submodule of M.*

Proof. Let L_i's be graded (m, n)-closed submodules of M_i and $(a_p, b_q)^m(x_v, y_u) \in L_1 \times L_2$, where $a_p \in h(A_1)$, $b_q \in h(A_2)$, $x_v \in h(M_1)$ and $y_u \in h(M_2)$. So $(a_p^m x_v, b_q^m y_u) \in L_1 \times L_2$, that is, $a_p^m x_v \in L_1$ and $b_q^m y_u \in L_2$. Thus by hypothesis, $a_p^n x_v \in L_1$ and $b_q^n y_u \in L_2$. Therefore, $(a_p, b_q)^n(x_v, y_u) = (a_p^n x_v, b_q^n y_u) \in L_1 \times L_2$, as needed. □

The following result is a generalization of Proposition 2.8.

Corollary 2.9. *Let $A = A_1 \times A_2 \times \cdots \times A_s$ be a G-graded ring and $M = M_1 \times M_2 \times \cdots \times M_s$ be a graded A-module, where M_i is a graded A_i-module for $i \in \mathbb{Z}_s$. Let L_i be a proper graded submodule of M_i. If L_i is a graded (m,n)-closed submodule of M_i for $i \in \mathbb{Z}_s$, then $L_1 \times L_2 \times \cdots \times L_s$ is a graded (m,n)-closed submodule of M.*

3. $GC_n^m - rad$ property

Let M be a graded A-module and K be a graded submodule of M. The intersection of all graded (m,n)-closed submodules containing K is called the graded closed-radical of K and is denoted by $GC_n^m - rad_M(K)$. If there is no graded (m,n)-closed submodule containing K, then we write $GC_n^m - rad_M(K) = M$. It is clear by Proposition 2.4(i) that for every graded submodule K of M, $GC_n^m - rad_M(K)$ is graded (m,n)-closed, so that $GC_n^m - rad_M(GC_n^m - rad_M(K)) = GC_n^m - rad_M(K)$; therefore

$$K \subseteq GC_n^m - rad_M(K) = GC_n^m - rad_M(GC_n^m - rad_M(K)) = \cdots \subset M.$$

The class of graded (m,n)-closed submodules of a graded A-module M is denoted by $C_n^m(M)$. For a graded submodule L of M, we write:

$$GC_n^m(L)_M = \{N \in C_n^m(M) \mid L \subseteq N\}.$$

Also, we put

$$G_{M_n^m}(L) = \cup_{a_g \in h(A)}(a_g^n M \cap (L :_M a_g^{m-n}))$$

and the graded submodule of M generated by $G_{M_n^m}(L)$ is denoted by $E_{M_n^m}(L)$ which is called graded (m,n)-envelope of L in M. Let $y_g \in E_{M_n^m}(L)$. Then $y_g = a_{g_1}^n x_{r_1} + \cdots + a_{g_t}^n x_{r_t}$, where $a_{g_i} \in h(A)$, $x_{r_i} \in h(M)$ and $a_{g_i}^n x_{r_i} \in (L :_M a_{g_i}^{m-n})$ for $i \in \mathbb{Z}_t$. Let $N \in GC_n^m(L)_M$. Then, $a_{g_i}^m x_{r_i} \in L \subseteq N$ and N graded (m,n)-closed implies that $a_{g_i}^n x_{r_i} \in N$. Therefore, $y_g \in N$ and we get $E_{M_n^m}(L) \subseteq \cap_{N \in GC_n^m(L)_M} N = GC_n^m - rad_M(L)$.

We say that M has $GC_n^m - rad$ property if for every graded submodule L of M,

$$E_{M_n^m}(L) = GC_n^m - rad_M(L).$$

Proposition 3.1. *Let L_1 and L_2 be graded submodules of a graded A-module M with $L_1 \subseteq L_2$. Then*
(i) $E_{(M/L_1)_n^m}(L_2/L_1) = E_{M_n^m}(L_2)/L_1$.
(ii) $GC_n^m - rad_{M/L_1}(L_2/L_1) = GC_n^m - rad_M(L_2)/L_1$.

Proof. (i) Let $a_g \in E_{(M/L_1)_n^m}(L_2/L_1)$. Then $a_g = \Sigma_{i=1}^t a_{g_i}^n(x_{q_i} + L_1)$, where $a_{g_i} \in h(A)$, $x_{q_i} \in h(M)$, $a_{g_i}^n(x_{q_i} + L_1) \in (L_2/L_1 :_{M/L_1} a_{g_i}^{m-n})$ for $i \in \mathbb{Z}_t$. So $a_{g_i}^m(x_{q_i} + L_1) \in L_2/L_1$, that is, $a_{g_i}^m \cdot x_{q_i} \in L_2$. Thus $\Sigma_{i=1}^t a_{g_i}^n \cdot x_{q_i} \in E_{(M)_n^m}(L_2)$, that is, $a_g = \Sigma_{i=1}^t a_{g_i}^n(x_{q_i} + L_1) = \Sigma_{i=1}^t a_{g_i}^n \cdot x_{q_i} + L_1 \in E_{(M)_n^m}(L_2)/L_1$, as needed.

Now, let $b_g = \Sigma_{i=1}^r(b_{g_i}^n \cdot y_{q_i}) + L_1 \in E_{M_n^m}(L_2)/L_1$, where $b_{g_i} \in h(A)$, $y_{q_i} \in h(M)$, $b_{g_i}^n \cdot y_{q_i} \in (L_2 :_M b_{g_i}^{m-n})$ for $i \in \mathbb{Z}_r$. Hence $b_{g_i}^m \cdot y_{q_i} \in L_2$ implies that

$b^m_{g_i} \cdot y_{q_i} + L_1 \in L_2/L_1$. So $b_g = \Sigma^r_{i=1}(b^n_{g_i} \cdot y_{q_i}) + L_1 = \Sigma^r_{i=1} b^n_{g_i}(y_{q_i} + L_1) \in E_{(M/L_1)^m_n}(L_2/L_1)$.

(ii) It is clear by Lemma 2.5. □

Proposition 3.2. *Let L_1 and L_2 be graded coprime submodules of a graded A-module M. Then, $GC^m_n - rad_M(L_2) = E_{(M)^m_n}(L_2)$ if and only if $GC^m_n - rad_{L_1}(L_1 \cap L_2) = E_{(L_1)^m_n}(L_1 \cap L_2)$.*

Proof. Since $L_1 + L_2 = M$ we have $M/L_2 \cong L_1/L_1 \cap L_2$. So, $GC^m_n - rad_M(L_2) = E_{(M)^m_n}(L_2)$ if and only if $GC^m_n - rad_M(L_2)/L_2 = E_{(M)^m_n}(L_2)/L_2$ if and only if $GC^m_n - rad_{M/L_2}(L_2/L_2) = E_{(M/L_2)^m_n}(L_2/L_2)$ if and only if $GC^m_n - rad_{M/L_2}(0) = E_{(M/L_2)^m_n}(0)$ if and only if $GC^m_n - rad_{L_1/L_1\cap L_2}(0) = E_{(L_1/L_1\cap L_2)^m_n}(0)$ if and only if

$$GC^m_n - rad_{L_1/L_1\cap L_2}(L_1 \cap L_2/L_1 \cap L_2) = E_{(L_1/L_1\cap L_2)^m_n}(L_1 \cap L_2/L_1 \cap L_2)$$

if and only if $GC^m_n - rad_{L_1}(L_1 \cap L_2)/L_1 \cap L_2 = E_{(L_1)^m_n}(L_1 \cap L_2)/L_1 \cap L_2$ if and only if $GC^m_n - rad_{L_1}(L_1 \cap L_2) = E_{(L_1)^m_n}(L_1 \cap L_2)$. □

The following result holds by the proof of Proposition 3.2.

Corollary 3.3. *Let L_1 and L_2 be graded coprime submodules of a graded A-module M. Then, $GC^m_n - rad_M(L_1) = E_{(M)^m_n}(L_1)$ if and only if $GC^m_n - rad_{L_2}(L_1 \cap L_2) = E_{(L_2)^m_n}(L_1 \cap L_2)$.*

Proposition 3.4. *Let M be a graded A-module. If every graded submodule of the graded factor A-module $M/E_{(M)^m_n}(0)$ is graded (m, n)-closed, then M has $GC^m_n - rad$ property.*

Proof. Let L be a proper graded submodule of M. Since $E_{(M)^m_n}(0) \subseteq E_{(M)^m_n}(L)$, we get $E_{(M)^m_n}(L)/E_{(M)^m_n}(0)$ is a graded submodule of $M/E_{(M)^m_n}(0)$. So $E_{(M)^m_n}(L)/E_{(M)^m_n}(0)$ is graded (m, n)-closed. Thus

$$\begin{aligned} E_{(M)^m_n}(L)/E_{(M)^m_n}(0) &= GC^m_n - rad_{M/E_{(M)^m_n}(0)}(E_{(M)^m_n}(L)/E_{(M)^m_n}(0)) \\ &= GC^m_n - rad_M(E_{(M)^m_n}(L))/E_{(M)^m_n}(0). \end{aligned}$$

Therefore $E_{(M)^m_n}(L) = GC^m_n - rad_M(E_{(M)^m_n}(L))$. Also, $L \subseteq E_{(M)^m_n}(L)$ implies that $GC^m_n - rad_M(L) \subseteq GC^m_n - rad_M(E_{(M)^m_n}(L))$, that is, $GC^m_n - rad_M(L) \subseteq E_{M^m_n}(L)$. Now by the statement just prior to Proposition 3.1, we have $E_{M^m_n}(L) \subseteq GC^m_n - rad_M(L)$. Therefore, $E_{M^m_n}(L) = GC^m_n - rad_M(L)$. □

Proposition 3.5. *Let M be a graded A-module. If M has $GC^m_n - rad$ property, then $E_{(M)^m_n}(E_{(M)^m_n}(L)) = E_{(M)^m_n}(L)$ for every graded submodule L of M.*

Proof. Let L be a graded submodule of M. Hence $E_{M^m_n}(L) = GC^m_n - rad_M(L)$. Also, $E_{M^m_n}(E_{M^m_n}(L)) = GC^m_n - rad_M(E_{M^m_n}(L))$ for graded submodule $E_{M^m_n}(L)$ of M. Therefore

$$E_{M^m_n}(E_{M^m_n}(L)) = GC^m_n - rad_M(E_{M^m_n}(L))$$

$$= GC_n^m - rad_M(GC_n^m - rad_M(L))$$
$$= GC_n^m - rad_M(L) = E_{M_n^m}(L). \qquad \square$$

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