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THE BONGARTZ'S THEOREM OF GORENSTEIN COSILTING COMPLEXES

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Abstract. We describe the Gorenstein derived categories of Gorenstein rings via the homotopy categories of Gorenstein injective modules. We also introduce the concept of Gorenstein cosilting complexes and study its basic properties. This concept is generalized by cosilting complexes in relative homological methods. Furthermore, we investigate the existence of the relative version of the Bongartz's theorem and construct a Bongartz's complement for a Gorenstein precosilting complex.

1. Introduction

Homological algebra is at the root of modern techniques in many areas of mathematics. Gorenstein homological algebra is the relative version of homological algebra that uses Gorenstein projective, Gorenstein injective and Gorenstein flat resolutions instead of the classical projective, injective, flat resolutions. The Gorenstein methods are great use in investigating commutative and non-commutative algebras, as well as the representation theory and module category theory. Enochs and Jenda [16] introduced Gorenstein modules as a generalization of finitely generated modules of G-dimension zero over a two-sided noetherian ring, in the sense of Auslander and Bridger [3]. The papers [4, 5, 17] represent the subject, which has been developed to an advanced level. For example, the Gorenstein derived category makes Gorenstein quasi-isomorphisms become isomorphisms and have some advantages in relative settings.

Brenner, Butler [9] and Happel, Ringel [19] started considering the classic tilting theory in the context of finitely generated modules over artin algebras. Colpi, Trlifaj [15] and Angeleri-Hügel, Coelho [1] generalized it to the case of infinitely generated modules over arbitrary associative rings. Auslander and

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Solberg [6–8] introduced the concepts of finitely generated tilting and cotilting modules over relative homological algebras. Wei [29] gave an important characterization for relative tilting modules over artin algebras. However, the scope of the relative (co)tilting theory developed by Auslander and Solberg was limited to finitely generated modules over artin algebras. So some authors have attempted to extend the scope to the context of infinitely generated modules in recent years. Especially the relative (co)tilting theory was discussed in the context of Gorenstein homological algebras. For instance, Yan, Li and Ouyang [31] generalized Auslander-Solberg relative notions by giving the definitions of infinitely generated Gorenstein cotilting and tilting modules over Gorenstein rings. In [25], Moradifar and Yassemi established the theory of infinitely generated Gorenstein tilting modules by developing Gorenstein tilting approximations. Furthermore, Rickard [26] introduced the concept of tilting complexes, as a generalization of tilting modules, to study the triangulated equivalence between two bounded derived categories of module categories. And he gave a Morita theory for derived categories. Miyachi [24] studied the tilting complexes over the ring extensions. Keller and Vossieck introduced the notion of silting complexes, which is a generalization of tilting complexes in [22]. It is well known that a module is tilting if and only if it is quasi-isomorphic to a silting complex. In [30], Wei described the semi-tilting (silting) complexes and gave the Bazzoni's characterization. Moreover, the 2-term silting complexes have attracted many scholars' great interest. Hoshino, Kato and Miyachi [20] studied the relation between the 2-term silting complexes and torsion pairs in the category of modules. They characterized the 2-term presilting complexes as a direct summand completion of the 2-term silting complexes, which is the analogue of the Bongartz completion of a classical tilting module. Later, Koga [23] provided a generalization version for finite length presilting complexes in the bounded homotopy category of finitely generated projectives. Buan and Zhou [11] showed that it is reasonable to see the silting theory as the relative version of tilting theory in the level of derived category, and gave a generalization of the classical tilting theorem of Brenner and Butler in term of 2-term silting complexes, which is said to be the silting theorem. Cao and Wei [12] gave that a partial Gorenstein silting complex have a complement in CM-finite algebra. Dual to the silting theory, Zhang and Wei [32] concentrated on the cosilting complexes, and proved them coincides with AIR-cotilting and quasi-cotilting modules.

Evidently, it is vital for cosilting theory that the cosilting complexes are studied in relative homological methods. According to Enochs and Jenda [16] Gorenstein injective modules are rarely finitely generated. This conclusion shows that the construction of Bongartz's theorem in relative cosilting theory is more difficult and important. Inspired by the fruitful results of silting complexes and cosilting complexes, the main purpose of this paper is to concentrate on the cosilting complexes in the context of relative homological algebras.

We characterize the one-to-one correspondence between certain specially contravariantly finite subcategories of derived category and Gorenstein cosilting complexes, and prove that a module is Gorenstein cotilting if and only if it is isomorphic in the Gorenstein derived category to a Gorenstein cosilting complex. More importantly, we give the Bazzoni's characterization of n-Gorenstein cosilting complexes and the Bongartz's theorem corresponding to the Gorenstein precosilting complexes.

The paper is organized as follows. In Section 2, we review some fundamental notions and results. We devote Section 3 to investigating the Gorenstein cosilting complexes and showing that there is a one-to-one correspondence between certain specially contravariantly finite subcategories of derived category and the isomorphism classes of Gorenstein cosilting complexes. We prove the Bazzoni's characterization of n-Gorenstein cosilting complexes. The Bongartz's theorem corresponding to the Gorenstein precosilting complexes is given in Section 4.

2. Preliminaries

2.1. A ring Λ is said to be a Gorenstein ring if Λ is two-sided Noetherian and Λ has finite injective dimension, both as left and right Λ-modules. A Gorenstein ring Λ is l-Gorenstein if the injective dimension of Λ as a left Λ-module is at most l. In this case, the injective dimension of Λ as a right Λ-module is also at most l. Throughout this paper, we fix that Λ is an l-Gorenstein ring.

We denote by Λ-Mod (resp. Λ-mod) the category of (finitely generated) left Λ-modules.

2.2. A Λ-module M is called Gorenstein injective if there exists an exact sequence

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

in Λ-Mod with all terms injective, such that $M = \mathrm{Im}(I_0 \to I^0)$ and the sequence is still exact after applying the functor $\mathrm{Hom}_\Lambda(I, -)$ for any injective left Λ-module I. Let Λ-$\mathcal{GI}$ (resp. Λ-$\mathcal{G}inj$) denote the full subcategory of Λ-modules consisting of Gorenstein injective modules in Λ-Mod (resp. Λ-mod).

2.3. A proper Gorenstein injective resolution of an object M is an exact sequence

$$E^\bullet = 0 \to M \to G_0 \to G_1 \to \cdots$$

such that all $G_i \in \mathcal{GI}$ and $\mathrm{Hom}_\Lambda(E^\bullet, G)$ stays exact for each $G \in \mathcal{GI}$. The second requirement guarantees the uniqueness of such a resolution in the homotopy category.

A Λ-module M has a proper Gorenstein injective resolution, if there is a proper exact sequence

$$0 \to M \to G_0 \to G_1 \to G_2 \to \cdots \to G_i \to \cdots$$

with each $G_i \in \mathcal{GI}$. The Gorenstein injective dimension $\mathcal{GI}$-res.dim$M$ of M is defined to be the smallest integer $s \geqslant 0$, such that there is an exact sequence

$0 \to M \to G_0 \to G_1 \to G_2 \to \cdots \to G_s \to 0$ with all $G_i \in \mathcal{GI}$, and $\mathcal{GI}$-res.dim$M$=∞ if there is no such an exact sequence of finite length.

If Λ is an l-Gorenstein ring, then each Λ-module M admits a proper Gorenstein injective resolution $0 \to M \to G_0 \to G_1 \to \cdots \to G_l$.

2.4. If Λ is a finite-dimensional algebra, then each module M (not necessarily finitely generated) admits a proper Gorenstein injective resolution $0 \to M \to G^\bullet$. In other words, Λ-$\mathcal{GI}$ is covariantly finite in Λ-Mod.

2.5. For an abelian category $\mathcal{A}$ with enough injective objects $\mathcal{I}_{\mathcal{A}}$, or simply $\mathcal{I}$, is the full subcategory of injective objects. A complex $C^\bullet$ is $\mathcal{GI}$-coacyclic, if $\mathrm{Hom}_{\mathcal{A}}(C^\bullet, G)$ is coacyclic for each $G \in \mathcal{GI}$. The complex $C^\bullet$ is also called proper exact. Since $C^\bullet$ is coacyclic if and only if $\mathrm{Hom}_{\mathcal{A}}(C^\bullet, I)$ is coacyclic for each $I \in \mathcal{I}$, we have that a $\mathcal{GI}$-coacyclic complex is coacyclic. By Lemma 2.5 in [13], a complex $C^\bullet$ is $\mathcal{GI}$-coacyclic if and only if $\mathrm{Hom}_{\mathcal{A}}(C^\bullet, G)$ is coacyclic for each $G \in K^+(\mathcal{GI})$.

2.6. We call a chain map $f^\bullet\colon X^\bullet \to Y^\bullet$ a $\mathcal{GI}$-quasi-isomorphism if the $\mathrm{Hom}_{\mathcal{A}}(f^\bullet, G)$ is a quasi-isomorphism for each $G \in \mathcal{GI}$, i.e., there are isomorphisms of abelian groups

$$\mathrm{H}^n\mathrm{Hom}_{\mathcal{A}}(f^\bullet, G)\colon \mathrm{H}^n\mathrm{Hom}_{\mathcal{A}}(Y^\bullet, G) \cong \mathrm{H}^n\mathrm{Hom}_{\mathcal{A}}(X^\bullet, G).$$

Since $f^\bullet\colon X^\bullet \to Y^\bullet$ is a quasi-isomorphism if and only if $\mathrm{H}^n\mathrm{Hom}_{\mathcal{A}}(f^\bullet, I)$ is an isomorphism for each $I \in \mathcal{I}$, it follows that a $\mathcal{GI}$-quasi-isomorphism is a quasi-isomorphism.

2.7. For $* \in \{blank, +, b\}$, $C^*(\mathcal{A})$, $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ represent the corresponding cochain complexes category, homotopy category and derived category of $\mathcal{A}$, respectively. Let $K^*_{coac}(\mathcal{A})\colon = \{X^\bullet \in K^*(\mathcal{A}) \mid X^\bullet \text{ is coacyclic}\}$ and $K^*_{gicoac}(\mathcal{A}) := \{X^\bullet \in K^*(\mathcal{A}) \mid X^\bullet \text{ is } \mathcal{GI}\text{-coacyclic}\}$ denote the homotopy of coacyclic complexes of $\mathcal{A}$ and the (corresponding) homotopy of $\mathcal{GI}$-coacyclic complexes of $\mathcal{A}$, respectively, which are thick triangulated subcategories. For $K^*_{gicoac}(\mathcal{A})$, the corresponding compatible saturated multiplicative system is the collection of all the $\mathcal{GI}$-quasi-isomorphism in $K^*(\mathcal{A})$. Then we have the following triangulated category

$$D^*_{gi}(\mathcal{A})\colon = K^*(\mathcal{A})/K^*_{gicoac}(\mathcal{A}),$$

which is called the Gorenstein injective derived category.

Note that if every object of $\mathcal{A}$ has finite injective dimension, then $\mathcal{A}$-$\mathcal{GI} = \mathcal{I}$ by [17, Proposition 10.1.2], hence $D^*_{gi}(\mathcal{A}) = D^*(\mathcal{A})$.

The following lemma can be straightforward to see by [28, Corollaire 4-3].

Lemma 2.1. *For $* \in \{blank, +, -\}$, there is an isomorphism of triangulated categories*

$$D^*(\mathcal{A}) \cong D^*_{gi}(\mathcal{A})/(K^*_{coac}(\mathcal{A})/K^*_{gicoac}(\mathcal{A})).$$

Note that from the above lemma, we can easily obtain that the quotient functor $D^*_{gi}(\mathcal{A}) \to D^*(\mathcal{A})$ is an equivalence if and only if each quasi-isomorphism in $K^*(\mathcal{A})$ is a $\mathcal{GI}$-quasi-isomorphism. Then the following result holds.

Proposition 2.2. *The quotient functor $D^*_{gi}(\mathcal{A}) \to D^*(\mathcal{A})$ is an equivalence if and only if every Gorenstein injective object is injective.*

Proof. Suppose that each quasi-isomorphism in $K^*(\mathcal{A})$ is also a $\mathcal{GI}$-quasi-isomorphism, and G is a Gorenstein injective object. If $0 \to X \xrightarrow{f} Y \to Z \to 0$ is a short exact sequence, then f induces a quasi-isomorphism $f^\bullet$, which is also a $\mathcal{GI}$-quasi-isomorphism. So, we can obtain the induced sequence $0 \to \operatorname{Hom}_{\mathcal{A}}(Z,G) \to \operatorname{Hom}_{\mathcal{A}}(Y,G) \to \operatorname{Hom}_{\mathcal{A}}(X,G) \to 0$ is exact. Therefore, G is injective. The converse is obvious. □

2.8. For each class $\mathcal{C}$ of objects in a triangulated category $\mathcal{T}$, the full subcategory given by

$$^{\perp}\mathcal{C} = \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(X, Y[n]) = 0, \forall Y \in \mathcal{C}, \forall n \in \mathbb{Z}\}$$

is clearly a triangulated subcategory closed under direct summands, and hence thick by Rickard's criterion [27].

The following facts were stated by Jiaqun Wei in [30], which are frequently used in our papers. We list them as following for convenience.

2.9. Let $\mathcal{C}$ be an idempotent complete triangulated category with [1] the shift functor. Assume that $\mathcal{B}$ is a full subcategory of $\mathcal{C}$. Recall that $\mathcal{B}$ is closed under extension if for any triangle $X \to Y \to Z \to$ in $\mathcal{C}$ with $X, Z \in \mathcal{B}$, we have $Y \in \mathcal{B}$. The subcategory $\mathcal{B}$ is cosuspended (resp. suspended) if it is closed under extension and under functor $[-1]$ (resp. $[1]$). It is easy to see that $\mathcal{B}$ is cosuspended (resp. suspended) if and only if for any triangle $X \to Y \to Z \to$ (resp. $Z \to Y \to X \to$) in $\mathcal{C}$ with $Z \in \mathcal{B}$, one has that $X \in \mathcal{B} \Leftrightarrow Y \in \mathcal{B}$.

2.10. An object $M \in \mathcal{C}$ has a $\mathcal{B}$-resolution (resp. $\mathcal{B}$-coresolution) with the length at most m ($m \geqslant 0$) if there are triangles $M_{i+1} \to X_i \to M_i$ (resp. $M_i \to X_i \to M_{i+1}$) with $0 \leqslant i \leqslant m$ such that $M_0 = M, M_{m+1} = 0$ and each $X_i \in \mathcal{B}$. In this case, we denote by $\mathcal{B}$-res.dim$(M) \leqslant m$ (resp. $\mathcal{B}$-cores.dim$(M) \leqslant m$). One may compare such notions with usual finite resolutions and coresolutions, respectively, in the module category.

2.11. Associated with a subcategory $\mathcal{B}$, we have the following notations, where $n \geqslant 0$ and m is an integer.

$$(\hat{\mathcal{B}})_n = \{M \in \mathcal{C} \mid \mathcal{B}\text{-res.dim}(M) \leqslant n\}.$$

$$(\check{\mathcal{B}})_n = \{M \in \mathcal{C} \mid \mathcal{B}\text{-cores.dim}(M) \leqslant n\}.$$

$$\hat{\mathcal{B}} = \{M \in \mathcal{C} \mid M \in (\hat{\mathcal{B}})_n \text{ for some } n\}.$$

$$\check{\mathcal{B}} = \{M \in \mathcal{C} \mid M \in (\check{\mathcal{B}})_n \text{ for some } n\}.$$

$$\mathcal{B}^{\perp_{\neq 0}} = \{N \in \mathcal{C} \mid \operatorname{Hom}(M, N[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i \neq 0\}.$$

$$^{\perp_{\neq 0}}\mathcal{B} = \{N \in \mathcal{C} \mid \operatorname{Hom}(N, M[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i \neq 0\}.$$

$$\mathcal{B}^{\perp_{>m}} = \{N \in \mathcal{C} \mid \operatorname{Hom}(M, N[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i > m\}.$$

$$^{\perp_{>m}}\mathcal{B} = \{N \in \mathcal{C} \mid \operatorname{Hom}(N, M[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i > m\}.$$

$$\mathcal{B}^{\perp_{\gg 0}} = \{N \in \mathcal{C} \mid N \in \mathcal{B}^{\perp_{>m}} \text{ for some } m\}.$$

Note that $\mathcal{B}^{\perp_{>m}}$ (resp. ${}^{\perp_{>m}}\mathcal{B}$) is suspended (resp. cosuspended) and closed under direct summands and that $\mathcal{B}^{\perp_{\gg 0}}$ is a triangulated subcategory of $\mathcal{C}$.

2.12. The subcategory $\mathcal{B}$ is said to be semi-selforthogonal (resp. selforthogonal) if $\mathcal{B} \subseteq \mathcal{B}^{\perp_{>0}}$ (resp. $\mathcal{B} \subseteq \mathcal{B}^{\perp_{\neq 0}}$). For instance, the subcategories of all projective (resp. all injective Λ-modules) are self-orthogonal in Λ-modules, for a ring Λ.

In the following results in this section, we always assume that $\mathcal{B}$ is additively closed and $\mathcal{B}$ is semi-selforthogonal.

2.13. Associated with the subcategory $\mathcal{B}$, we also have the following two useful subcategories.

$$\mathcal{X}_{\mathcal{B}} = \{N \in {}^{\perp_{\neq 0}}\mathcal{B} \mid \text{there are triangles } N_i \to B_i \to N_{i+1} \to \text{ such that } N_0 = N,$$
$$N_i \in {}^{\perp_{\neq 0}}\mathcal{B}, \text{ and } B_i \in \mathcal{B} \text{ for all } i \leqslant 0\},$$

$${}_{\mathcal{B}}\mathcal{X} = \{N \in \mathcal{B}^{\perp_{\neq 0}} \mid \text{there are triangles } N_{i+1} \to B_i \to N_i \to \text{ such that } N_0 = N,$$
$$N_i \in \mathcal{B}^{\perp_{\neq 0}}, \text{ and } B_i \in \mathcal{B} \text{ for all } i \leqslant 0\}.$$

Since $\mathcal{B}$ is closed under finite direct sums and direct summands, we could summarize some results on the subcategories associated with $\mathcal{B}$.

Lemma 2.3 ([30])**.** *Let $\mathcal{B}$ be a semi-self-orthogonal subcategory of triangulated category $\mathcal{C}$ such that $\mathcal{B}$ is additively closed. Then*

(1) *The three subcategories $\check{\mathcal{B}} \subseteq \mathcal{X}_{\mathcal{B}} \subseteq {}^{\perp_{>0}}\mathcal{B}$ are cosuspended and closed under direct summands.*

(2) *The three subcategories $\widehat{\mathcal{B}} \subseteq {}_{\mathcal{B}}\mathcal{X} \subseteq \mathcal{B}^{\perp_{>0}}$ are suspended and closed under direct summands.*

(3) $\mathcal{B} = \check{\mathcal{B}} \bigcap \mathcal{B}^{\perp_{>0}} = \widehat{\mathcal{B}} \bigcap {}^{\perp_{>0}}\mathcal{B}$.

(4) $(\check{\mathcal{B}})_n = \mathcal{X}_{\mathcal{B}} \bigcap (\mathcal{X}_{\mathcal{B}})^{\perp_{>n}} = \mathcal{X}_{\mathcal{B}} \bigcap ({}^{\perp_{>0}}\mathcal{B})^{\perp_{>n}}$. *In particular, it is closed under extensions and direct summands.*

(5) $(\widehat{\mathcal{B}})_n = {}_{\mathcal{B}}\mathcal{X} \bigcap {}^{\perp_{>n}}({}_{\mathcal{B}}\mathcal{X}) = {}_{\mathcal{B}}\mathcal{X} \bigcap {}^{\perp_{>n}}(\mathcal{B}^{\perp_{>0}})$. *In particular, it is closed under extensions and direct summands.*

(6) *The following three subcategories coincide with each other.*

(i) $\langle \mathcal{B} \rangle$: *the smallest triangulated subcategories containing $\mathcal{B}$.*

(ii) $(\widehat{\mathcal{B}})_- := \{X \in \mathcal{C} \mid$ *there is some* $Y \in \widehat{\mathcal{B}}$ *and some* $i \leqslant 0$ *such that* $X = Y[i]\}$.

(iii) $(\check{\mathcal{B}})_+ := \{X \in \mathcal{C} \mid$ *there is some* $Y \in \check{\mathcal{B}}$ *and some* $i \geqslant 0$ *such that* $X = Y[i]\}$.

(7) $\widehat{\mathcal{B}} = \mathcal{B}^{\perp_{>0}} \bigcap \langle \mathcal{B} \rangle$.

(8) $\check{\mathcal{B}} = {}^{\perp_{>0}}\mathcal{B} \bigcap \langle \mathcal{B} \rangle$.

Let $\mathbf{G} = \mathrm{Gext}^n(-,-) = \mathrm{Ext}^n_{\mathcal{A}\text{-}\mathcal{GI}}(-,-)$ (see [31]). Then $\mathbf{G}$ is an additive subfunctor of $\mathrm{Ext}^1(-,-)$. A short exact sequence $0 \to M \to N \to L \to 0$ is called $\mathbf{G}$-exact if it is in $\mathrm{Gext}^1(L, M)$. For a Λ-module T and some $n > 0$, we give the following notations

$${}^{\perp_{>0}}T^{G} = \{M \in \Lambda\text{-Mod} \mid \mathrm{Gext}^i(M, T) = 0 \text{ for all } i > 0\},$$

$$\mathrm{Copres}^n_{\mathbf{G}}(T) = \{M \in \Lambda\text{-Mod} \mid \text{there is a } \mathbf{G}\text{-exact sequence } T_n \to T_{n-1} \to \cdots \to T_1 \to M \to 0, \text{with each } T_i \in \mathrm{Adp}_\Lambda T\}.$$

Lemma 2.4 ([31])**.** *The following statements are equivalent for an exact sequence* $0 \to M \to N \to L \to 0$.

(1) *The sequence is* $\mathbf{G}$*-exact;*

(2) $0 \to \mathrm{Hom}_\Lambda(P, M) \to \mathrm{Hom}_\Lambda(P, N) \to \mathrm{Hom}_\Lambda(P, L) \to 0$ *is an exact sequence for all* $P \in \mathcal{GP}$;

(3) $0 \to \mathrm{Hom}_\Lambda(L, G) \to \mathrm{Hom}_\Lambda(N, G) \to \mathrm{Hom}_\Lambda(L, G) \to 0$ *is an exact sequence for all* $G \in \mathcal{GI}$.

Lemma 2.5. *Assume that the exact sequence* $0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ *is* $\mathbf{G}$*-exact. Then* $M \xrightarrow{f} N \xrightarrow{g} L \to$ *is a triangle in* $D_{gi}(\Lambda)$.

3. Gorenstein cosilting complexes

In this section, we introduce and study some basic properties of Gorenstein cosilting complexes. We establish the one-to-one correspondence between isomorphism classes of Gorenstein cosilting complexes and certain contravariantly finite cosuspended subcategories. In order to show some results, we generalize the statements in Gorenstein derived categories [18] to Gorenstein injective derived categories. Let $\mathcal{A} = \Lambda$-Mod, we can use them to obtain the desired consequences in this section.

Lemma 3.1 ([13])**.** *A chain map* $f^\bullet \colon X^\bullet \to Y^\bullet$ *is a* $\mathcal{GI}$*-quasi-isomorphism if and only if* $\mathrm{Hom}_{\mathcal{A}}(f^\bullet, G^\bullet)$ *is a quasi-isomorphism for each* $G^\bullet \in K^+(\mathcal{GI})$*, or equivalently, there are isomorphisms of abelian groups for each* $G^\bullet \in K^+(\mathcal{GI})$

$$\mathrm{Hom}_{K(\mathcal{A})}(f^\bullet, G^\bullet[n]) \colon \mathrm{Hom}_{K(\mathcal{A})}(Y^\bullet, G^\bullet[n]) \cong \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, G^\bullet[n]) \ \forall n \in \mathbb{Z}.$$

Lemma 3.2. (1) *Let* $f^\bullet \colon G^\bullet \to X^\bullet$ *be a* $\mathcal{GI}$*-quasi-isomorphism, where* $G^\bullet \in K^+(\mathcal{GI})$*. Then there is a chain map* $g^\bullet \colon X^\bullet \to G^\bullet$ *such that* $g^\bullet f^\bullet$ *is homotopic to* $\mathrm{Id}_{G^\bullet}$.

(2) *Let* $f^\bullet \colon X^\bullet \to G^\bullet$ *be a* $\mathcal{GI}$*-quasi-isomorphism with* $X^\bullet, G^\bullet \in K^+(\mathcal{GI})$*. Then* $f^\bullet$ *is a homotopy equivalence.*

Proof. (1) It is easy to obtain by Lemma 3.1 that

$$\mathrm{Hom}_{K(\mathcal{A})}(f^\bullet, G^\bullet) \colon \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, G^\bullet) \cong \mathrm{Hom}_{K(\mathcal{A})}(G^\bullet, G^\bullet).$$

Then the statement holds.

(2) It follows from (1). □

Proposition 3.3. *The functor* $Q \colon f^\bullet \mapsto \frac{\mathrm{Id}_{G^\bullet}}{f^\bullet}$ *induces an abelian group isomorphism* $\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, G^\bullet) \cong \mathrm{Hom}_{D_{gi}(\mathcal{A})}(X^\bullet, G^\bullet)$*, where* $G^\bullet \in K^+(\mathcal{GI})$ *and* $X^\bullet$ *is an arbitrary complex.*

Proof. Let $\frac{\mathrm{Id}_{\mathrm{G}^\bullet}}{f^\bullet} = 0$, we can get by the calculation of left fractions that there is a $\mathcal{GI}$-quasi-isomorphism $t^\bullet\colon G^\bullet \to Y^\bullet$ such that the homotopy $t^\bullet f^\bullet \sim 0$. Then, it yields from Lemma 3.2 that there is a $\mathcal{GI}$-quasi-isomorphism $g^\bullet\colon Y^\bullet \to G^\bullet$ such that $g^\bullet t^\bullet \sim \mathrm{Id}_{\mathrm{G}^\bullet}$. Hence, $f^\bullet \sim 0$.

On the other hand, assume $\frac{s^\bullet}{f^\bullet} \in \mathrm{Hom}_{D_{gi}(\mathcal{A})}(X^\bullet, G^\bullet)$, and use Lemma 3.2 again, then there is a $\mathcal{GI}$-quasi-isomorphism $g^\bullet\colon Y^\bullet \to G^\bullet$ such that $g^\bullet s^\bullet \sim \mathrm{Id}_{\mathrm{G}^\bullet}$. Thus, $\frac{s^\bullet}{f^\bullet} = \frac{\mathrm{Id}_{\mathrm{G}^\bullet}}{g^\bullet f^\bullet} = Q(g^\bullet f^\bullet)$. The conclusion holds. □

Note that from Proposition 3.3 $K^b(\mathcal{GI})$ and $K^+(\mathcal{GI})$ can be viewed as triangulated subcategories of $D^b_{gi}(\mathcal{A})$ and $D^+_{gi}(\mathcal{A})$, respectively.

In order to give an equivalent characterization of Gorenstein cosilting complexes, we need the following notations and results

$$D^{\geqslant 0}_{gi} := \{X^\bullet \in D_{gi}(\Lambda) \mid \mathrm{Hom}_{D_{gi}}(X^\bullet, G[i]) = 0 \text{ for any } G \in \mathcal{GI} \text{ and all } i > 0\}.$$

It is obvious that $D^{\geqslant 0}_{gi} = {}^{\perp_{>0}}(\mathcal{GI})$. For any $X^\bullet \in D^{\geqslant 0}_{gi}$, we learn from Proposition 3.3 that $X^\bullet$ satisfies that

$$\mathrm{H}^i\mathrm{Hom}_\Lambda(X^\bullet, G) = \mathrm{Hom}_{K(\Lambda)}(X^\bullet, G[i]) \cong \mathrm{Hom}_{D_{gi}}(X^\bullet, G[i]) = 0$$

for any $G \in \mathcal{GI}$ and $i > 0$. We assert $D^+_{gi}(\Lambda) = {}^{\perp_{\gg 0}}(\mathcal{GI})$. Indeed, $X^\bullet \in {}^{\perp_{\gg 0}}(\mathcal{GI})$ if and only if there is an integer n such that $\mathrm{Hom}_{D_{gi}}(X^\bullet, G[i]) \cong \mathrm{Hom}_{K(\Lambda)}(X^\bullet, G) = \mathrm{H}^i\mathrm{Hom}_\Lambda(X^\bullet, G) = 0$ for $G \in \mathcal{GI}$ and $i > n$ if and only if $X^\bullet \in D^+_{gi}(\Lambda)$.

Proposition 3.4. *The following statements hold.*

(1) *$D^+_{gi}(\mathcal{A})$ is a triangulated subcategory of $D_{gi}(\mathcal{A})$;*
(2) *$D^b_{gi}(\mathcal{A})$ is a triangulated subcategory of $D^+_{gi}(\mathcal{A})$.*

Proof. (1) Let $f^\bullet\colon B^\bullet \to Y^\bullet$ be a chain map with $B^\bullet \in K_{gicoac}(\mathcal{A})$ and $Y^\bullet \in K^+(\mathcal{A})$. Without loss of generality, assume that $Y^i = 0$ for $i < 0$, then there is the following natural factorization:

$$\begin{array}{cccccccccccc}
B^\bullet\colon & \cdots & \longrightarrow & B^{-2} & \longrightarrow & B^{-1} & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
B'^\bullet\colon & \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Ker}d^0 & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Y^\bullet\colon & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Y^0 & \longrightarrow & Y^1 & \longrightarrow & \cdots
\end{array}$$

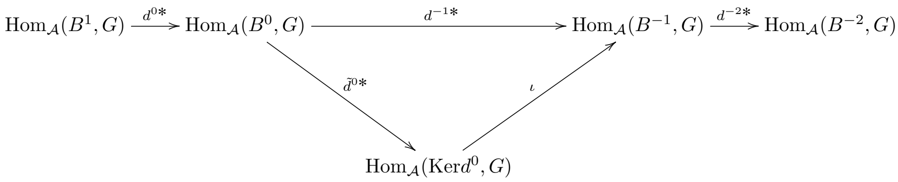
It is enough to prove that $B'^\bullet$ is $\mathcal{GI}$-coacyclic by [21, Lemma 10.3]. We only need to show that the following sequence

$$(*) \qquad \mathrm{Hom}_\mathcal{A}(B^1, G) \xrightarrow{d^{0*}} \mathrm{Hom}_\mathcal{A}(B^0, G) \xrightarrow{\tilde{d}^{0*}} \mathrm{Hom}_\mathcal{A}(\mathrm{Ker}d^0, G)$$

is exact for any $G \in \mathcal{GI}$, where $\tilde{d}^0 \colon \mathrm{Ker}d^0 \to B^0$ is induced by d^0. It is obvious by $B^\bullet$ coacyclic that the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Ker}d^0, G) \xrightarrow{\iota} \mathrm{Hom}_{\mathcal{A}}(B^{-1}, G) \to \mathrm{Hom}_{\mathcal{A}}(B^{-2}, G)$$

is exact. Consider the following commutative diagram:



where the first row is exact. One can easily check that

$$\mathrm{Ker}\tilde{d}^{0*} = \mathrm{Ker}d^{0*} = \mathrm{Im}d^{1*} \text{ and } \mathrm{Im}\tilde{d}^{0*} \cong \mathrm{Im}d^{0*} = \mathrm{Ker}\tilde{d}^{-1*} \cong \mathrm{Hom}_{\mathcal{A}}(\mathrm{Ker}d^0, G).$$

Hence, the sequence $(*)$ is exact. This completes the proof.

(2) It is similar to (1). □

Let $\mathcal{A}$ be an abelian category with enough injective objects and $n \in \mathbb{Z}$. We define the Gorenstein extension functor $\mathrm{Ext}^n_{\mathcal{A}\text{-}\mathcal{GI}}(-,-)$ to be $\mathrm{Hom}_{D^b_{gi}}(-,-[n])$. Then we obtain the following result.

Theorem 3.5. *Let $\mathcal{A}$ be an abelian category with enough injective objects, M an object in $\mathcal{A}$ admitting a proper Gorenstein injective resolution, and N an arbitrary object in $\mathcal{A}$. Then* $\mathrm{Ext}^n_{\mathcal{A}\text{-}\mathcal{GI}}(N, M) = \mathrm{Ext}^n_{\mathcal{G}}(N, M)$.

Proof. Let $0 \to M \to G^\bullet$ be a proper Gorenstein injective resolution of M. Then $M \cong G^\bullet$ in $D^+_{gi}(\mathcal{A})$. It follows from Propositions 3.4 and 3.3 that

$$\begin{aligned}
\mathrm{Ext}^n_{\mathcal{A}\text{-}\mathcal{GI}}(N, M) &= \mathrm{Hom}_{D^b_{gi}(\mathcal{A})}(N, M[n]) \\
&= \mathrm{Hom}_{D^+_{gi}(\mathcal{A})}(N, G[n]) \\
&= \mathrm{Hom}_{K^+(\mathcal{A})}(N, G[n]) \\
&\cong \mathrm{H}^n\mathrm{Hom}_{\mathcal{A}}(N, G) \\
&= \mathrm{Ext}^n_{\mathcal{G}}(N, M).
\end{aligned}$$

□

Now we give the definition of Gorenstein (pre)cosilting complexes and show an equivalent characterization.

Definition. A complex $T^\bullet$ is said to be

(1) Gorenstein precosilting if
 (S1) $T^\bullet \in K^b(\mathcal{GI})$;
 (S2) $\mathrm{Adp}_{D_{gi}} T^\bullet \subseteq {}^{\perp_{i>0}} T^\bullet$.

(2) Gorenstein cosilting if it satisfies (S1), (S2), and
 (S3) $K^b(\mathcal{GI}) = \langle \mathrm{Adp}_{D_{gi}} T^\bullet \rangle$, that is, $K^b(\mathcal{GI})$ coincides with the smallest triangulated subcategory containing $\mathrm{Adp}_{D_{gi}} T^\bullet$.

We also say that a complex $T^\bullet \in D_{gi}(\Lambda)$ is n-Gorenstein cosilting if it is a Gorenstein cosilting complex such that $\mathcal{GI} \in (\widehat{\mathrm{Adp}_{D_{gi}}T^\bullet})_n$.

Theorem 3.6. *Assume that $T^\bullet \in D_{gi}(\Lambda)$ and $\mathrm{Adp}_{D_{gi}}T^\bullet \subseteq D_{gi}^{\geqslant 0}$. Then $T^\bullet$ is Gorenstein cosilting if and only if it satisfies the following three conditions.*
$(S1')$ $T^\bullet \in \breve{\mathcal{GI}}$;
$(S2')$ $\mathrm{Adp}_{D_{gi}}T^\bullet \subseteq {}^{\perp_{i>0}}T^\bullet$;
$(S3')$ $\mathcal{GI} \subseteq \widehat{\mathrm{Adp}_{D_{gi}}T^\bullet}$.

Proof. Sufficiency. It is obvious by $(S1')$ that $\mathrm{Adp}_{D_{gi}}T^\bullet \subseteq \breve{\mathcal{GI}}$. From Lemma 2.3 that $\breve{\mathcal{GI}} \subseteq \langle \mathcal{GI} \rangle = K^b(\mathcal{GI})$, one can get that $T^\bullet \in K^b(\mathcal{GI})$ and $\mathrm{Adp}_{D_{gi}}T^\bullet \subseteq \langle \mathcal{GI} \rangle = K^b(\mathcal{GI})$. We have from $(S3')$ and Lemma 2.3 that $\mathcal{GI} \subseteq \widehat{\mathrm{Adp}_{D_{gi}}T^\bullet} \subseteq \langle \mathrm{Adp}_{D_{gi}}T^\bullet \rangle$. Consequently, $K^b(\mathcal{GI}) = \langle \mathrm{Adp}_{D_{gi}}T^\bullet \rangle$.

Necessity. Since $\mathrm{Adp}_{D_{gi}}T^\bullet \subseteq D_{gi}^{\geqslant 0} = {}^{\perp_{>0}}(\mathcal{GI})$, we have $\mathcal{GI} \subseteq (\mathrm{Adp}_{D_{gi}}T^\bullet)^{\perp_{>0}}$. According to (S3), then $\mathcal{GI} \subseteq \langle \mathrm{Adp}_{D_{gi}}T^\bullet \rangle$. It is easy to see by Lemma 2.3 that $\mathcal{GI} \subseteq \widehat{\mathrm{Adp}_{D_{gi}}T^\bullet}$. Note that $T^\bullet \in K^b(\mathcal{GI}) = \langle \mathcal{GI} \rangle$, so we have that $T^\bullet \in \langle \mathcal{GI} \rangle \bigcap {}^{\perp_{>0}}(\mathcal{GI}) = \breve{\mathcal{GI}}$. The proof is completed. □

The following result is well known in triangulated categories, pure derived categories, and Gorenstein derived categories with respect to Gorenstein projective modules. Now, we give the version of Gorenstein injective derived categories.

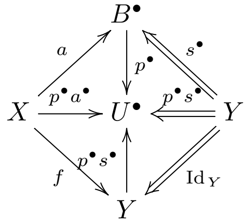
Proposition 3.7. *Let $F\colon \mathcal{A} \to D_{gi}^b(\mathcal{A})$ be the composition of the embedding $\mathcal{A} \to K^b(\mathcal{A})$ and the localization functor $K^b(\mathcal{A}) \to D_{gi}^b(\mathcal{A})$. Then the functor $F\colon \mathcal{A} \to D_{gi}^b(\mathcal{A})$ is fully faithful.*

Proof. Let $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$. If $F(f) = 0$, it is easy to see that there is a $\mathcal{GI}$-quasi-isomorphism $s^\bullet\colon Y \to B^\bullet$ such that there is a homotopy $s^\bullet f \sim 0$. Then we have $\mathrm{H}^0(s^\bullet)\mathrm{H}^0(f) = 0$. It yields that $f = 0$ since $\mathrm{H}^0(s^\bullet)$ is an isomorphism.

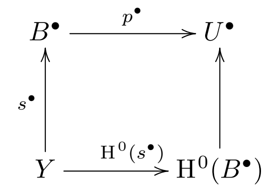
Assume that $\frac{s^\bullet}{a^\bullet} \in \mathrm{Hom}_{D_{gi}(\mathcal{A})}(X^\bullet, G^\bullet)$, which can be represented as the following diagram

$$X \xrightarrow{a^\bullet} B^\bullet \overset{s^\bullet}{\Longleftarrow} Y$$

where $s^\bullet$ is a $\mathcal{GI}$-quasi-isomorphism, $a^\bullet$ is a morphism in $K(\mathcal{A})$. We can obtain $\mathrm{H}^0(s^\bullet)\colon \mathrm{H}^0(B^\bullet) \cong Y$ in $\mathcal{A}$. Consider the truncation $U^\bullet := 0 \to \mathrm{Im}d^1 \to B^1 \to B^2 \to \cdots$ of $B^\bullet$ and the canonical map $p^\bullet\colon \mathcal{B}^\bullet \twoheadrightarrow U^\bullet$. Let $f\colon = \mathrm{H}^0(s^\bullet)^{-1}\mathrm{H}^0(a^\bullet) \in \mathrm{Hom}_{D_{gi}(\mathcal{A})}(X, Y)$. Since $s^\bullet$ is a $\mathcal{GI}$-quasi-isomorphism, we have that $p^\bullet s^\bullet$ is also a $\mathcal{GI}$-quasi-isomorphism. We can get the following diagram of complexes:



Obviously, we only need to prove $p^\bullet s^\bullet f = p^\bullet a^\bullet$. Consider the following commutative diagram:



Then we can learn that $p^\bullet s^\bullet f = p^\bullet s^\bullet \mathrm{H}^0(s^\bullet)^{-1}\mathrm{H}^0(a^\bullet) = p^\bullet a^\bullet$. It follows that $F(f) = \frac{\mathrm{Id}_Y}{f^\bullet} = \frac{s^\bullet}{a^\bullet}$. ☐

Definition ([31]). A Λ-module T is called an n-Gorenstein cotilting module if it satisfies the following three conditions.

(C1) $\mathrm{Id}_G T \leqslant n$;

(C2) $\mathrm{Gext}^i(T^I, T) = 0$ for each $i > 0$ and all sets I;

(C3) There exists a **G**-exact sequence

$$0 \to T^r \to T^{r-1} \to \cdots \to T^0 \to E \to 0$$

with each $T^i \in \mathrm{Adp}\, T$ for all $E \in \mathcal{GI}$.

We can now use the above statements to prove the following results.

Proposition 3.8. *A left Λ-module T is a Gorenstein cotilting module if and only if T is* (*Gorenstein quasi-isomorphic to*) *a Gorenstein cosilting complex.*

Proof. Necessity. From (C3), there is a **G**-exact sequence

$$0 \to T \xrightarrow{f_0} G_0 \xrightarrow{f_1} G_1 \to \cdots \xrightarrow{f_n} G_n \to 0$$

with each $G_i \in \mathcal{GI}$. It follows by Lemma 2.5 that there exists a series of triangles $T_i \to G_i \to T_{i+1} \to$ in $D_{gi}(\Lambda)$, $0 \leqslant i \leqslant n$, such that $T_{n+1} = 0$, $T_0 = T$ and $T_j = \mathrm{Ker} f_{j+1}$ for $0 \leqslant j \leqslant n-1$, namely, $T \in \check{\mathcal{GI}}$. One can get from Theorem 3.5 that $0 = \mathrm{Gext}^i_\Lambda(T^I, T) \cong \mathrm{Hom}_{D_{gi}}(T^I, T[i])$ for any set I and $i > 0$. For any $X \in \mathrm{Adp}_{D_{gi}} T$, we can easily obtain $\mathrm{Hom}_{D_{gi}}(X, T[i]) = 0$. By the assumption, for any $G \in \mathcal{GI}$, there is a **G**-exact sequence

$$0 \to T^r \to T^{r-1} \to \cdots \to T^0 \to G \to 0.$$

Hence, there is a series of triangles $G_{i+1} \to T_i \to G_i \to$ in $D_{gi}(\Lambda)$, where $G_0 = G, G_{r+1} = 0$ for $0 \leqslant i \leqslant r$. Then, $\mathcal{GI} \subseteq \widehat{\mathrm{Adp}_{D_{gi}}} T$.

Sufficiency. Suppose that $G^\bullet \in K^b(\mathcal{GI})$ is a Gorenstein cosilting complex, which is Gorenstein quasi-isomorphic to T. Then we have by Theorem 3.5 that $0 = \mathrm{Hom}_{D_{gi}}(G^{\bullet I}, G^\bullet[i]) \cong \mathrm{Hom}_{D_{gi}}(T^I, T[i]) \cong \mathrm{Gext}^i_\Lambda(T^I, T)$ for any set I and $i > 0$. So the condition (C2) is satisfied.

Since T is isomorphic to a Gorenstein cosilting complex in a derived category and $T \in D^{\geqslant 0}_{gi}$, we can learn from Theorem 3.6 that $T \in \widecheck{\mathcal{GI}}$. It follows by **2.11** that there is a series of triangles $T^\bullet_i \to G_i \xrightarrow{f_i} T^\bullet_{i+1} \to$ in $D_{gi}(\Lambda)$ with $G_i \in \mathcal{GI}$, $T^\bullet_{t+1} = T$, $T^\bullet_0 = 0$, where $0 \leqslant i \leqslant t$. One can easily get that $T^\bullet_i \in {}^{\perp_{>0}}(\mathcal{GI})$ from $T^\bullet_t \cong G_t \in \mathcal{GI}$ and $\mathcal{GI} \subseteq {}^{\perp_{>0}}(\mathcal{GI})$ for each i. Consider the first triangle $T^\bullet_0 \xrightarrow{f_0} G_0 \to T^\bullet_1 \to$, where $T^\bullet_0 = T$ is already a Λ-module. For any $G \in \mathcal{GI}$, applying $\mathrm{Hom}_{D_{gi}}(-, G)$ to the above triangle, we have that $\mathrm{Hom}_{D_{gi}}(G_0, G) \to \mathrm{Hom}_{D_{gi}}(T, G)$ is surjective. We can also observe from Proposition 3.3 that $\mathrm{Hom}_\Lambda(G_0, G) \to \mathrm{Hom}_\Lambda(T, G)$ is surjective and $f_0 \in \mathrm{Hom}_{D_{gi}}(T, G_0) \cong \mathrm{Hom}_\Lambda(T, G_0)$. So, f_0 is a homomorphism between modules. It is well known that all injective modules are Gorenstein injective. We claim that f_0 is injective. Indeed, taking any homomorphism $g : T_0 \to Q$ with Q injective, we can obtain the following commutative diagram in Λ-Mod for some homomorphism h:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(f_0) & \xrightarrow{i} & T_0 & \xrightarrow{f_0} & G_0 \\ & & & & {\scriptstyle g}\downarrow & \swarrow {\scriptstyle h} & \\ & & & & Q & & \end{array}$$

This shows that $gi = hf_0 i = 0$. Note that g is injective, so $i = 0$ and consequently, f_0 is injective. Then there is a **G**-exact sequence $0 \to T_0 \xrightarrow{f_0} G_0 \to \mathrm{Coker} f_0 \to 0$, which induces a triangle $T_0 \to G_0 \to \mathrm{Coker} f_0 \to$ in $D_{gi}(\Lambda)$. So, $T^\bullet_1 \cong \mathrm{Coker} f_0$ in $D_{gi}(\Lambda)$. Therefore, $T^\bullet_1$ is Gorenstein quasi-isomorphic to a Λ-module $\mathrm{Coker} f_0$ in $K(\Lambda)$. Repeating the above discussion for all i, we can get that each f_i is injective and each $T^\bullet_i$ is (Gorenstein quasi-isomorphic to) a Λ-module. Note that $T^\bullet_t = G_t$, we have by the above argument that there is a long exact sequence $0 \to T \to G_1 \to G_2 \to \cdots \to G_t \to 0$. Therefore, $\mathrm{Id}_G(T) \leqslant t$, namely, the condition (C1) holds.

For any $G \in \mathcal{GI}$, it is easy to obtain by Theorem 3.6 that $G \in \mathcal{GI} \subseteq \widehat{\mathrm{Adp}_{D_{gi}}} T$. Then there is a series of triangles $G^\bullet_{i+1} \to T_i \xrightarrow{\alpha_i} G^\bullet_i \to$ in $D_{gi}(\Lambda)$ with $G^\bullet_0 = G, T_i \in \mathrm{Adp}_{D_{gi}} T$ and $G^\bullet_{s+1} = 0$ for $0 \leqslant i \leqslant s$. We have by Theorem 3.5 that $0 = \mathrm{Gext}^j_\Lambda(P, T) \cong \mathrm{Hom}_{D_{gi}}(P, T[j])$ for any $P \in \mathcal{GP}$ and $j > 0$. It follows that $G^\bullet_i \in \mathcal{GP}^{\perp_{>0}}$ for any $0 \leqslant i \leqslant s$. Applying $\mathrm{Hom}_{D_{gi}}(P, -)$ to the triangle $G^\bullet_1 \to T_0 \xrightarrow{\alpha_0} G \to$, we get that $0 \to \mathrm{Hom}_{D_{gi}}(P, T_0) \to \mathrm{Hom}_{D_{gi}}(P, G)$ is injective. It follows from Proposition 3.7 that $0 \to \mathrm{Hom}_\Lambda(P, T_0) \to \mathrm{Hom}_\Lambda(P, G)$ is injective. Since a projective generator in Λ-Mod is a Gorenstein projective module, we obtain $\mathrm{Hom}_\Lambda(P, \mathrm{Coker}\alpha_0) = 0$. Therefore, $\mathrm{Coker}\alpha_0 = 0$, namely, α_0 is surjective. Then there is a $\mathbf{G}$-exact sequence $0 \to \mathrm{Ker}\alpha_0 \to T_0 \xrightarrow{\alpha_0} G \to 0$. It induces a triangle $\mathrm{Ker}\alpha_0 \to T_0 \xrightarrow{\alpha_0} G \to$ in $D_{gi}(\Lambda)$. Hence, $G_1^\bullet \cong \mathrm{Ker}\alpha_0$ in $D_{gi}(\Lambda)$. Moreover, $G_1^\bullet$ is Gorenstein quasi-isomorphic to $\mathrm{Ker}\alpha_0 \in \Lambda$-Mod in $K(\Lambda)$. Repeating the process, there is a $\mathbf{G}$-exact sequence for $G \in \mathcal{GI}$, $0 \to T_s \to T_{s-1} \to \cdots \to T_0 \to G \to 0$ with $T_i \in \mathrm{Adp}_{D_{gi}}(T)$, $0 \leqslant i \leqslant s$. So we get the condition (C3). □

Proposition 3.9. *Let $T^\bullet$ and $T^\bullet \oplus M^\bullet$ be Gorenstein cosilting complexes. Then $M^\bullet \in \mathrm{Adp}_{D_{gi}}T^\bullet$.*

Proof. It is immediate by the assumption that $\langle \mathrm{Adp}_{D_{gi}}(T^\bullet \oplus M^\bullet)\rangle = K^b(\mathcal{GI}) = \langle \mathrm{Adp}_{D_{gi}}T^\bullet\rangle$, and $T^\bullet \oplus M^\bullet \in \mathrm{Adp}_{D_{gi}}(T^\bullet \oplus M^\bullet) \subseteq {}^{\perp_{>0}}(T^\bullet \oplus M^\bullet)$. The latter one yields $M^\bullet \in {}^{\perp_{>0}}T^\bullet$. Then, $T^\bullet \oplus M^\bullet \in \mathrm{Adp}_{D_{gi}}(T^\bullet \oplus M^\bullet) \subseteq (\mathrm{Adp}_{D_{gi}}(T^\bullet \oplus M^\bullet))^{\perp_{>0}} \subseteq (\mathrm{Adp}_{D_{gi}}T^\bullet)^{\perp_{>0}}$ from Lemma 2.3. Furthermore, we get $M^\bullet \in {}^{\perp_{>0}}T^\bullet \bigcap \langle \mathrm{Adp}_{D_{gi}}(T^\bullet \oplus M^\bullet)\rangle = {}^{\perp_{>0}}T^\bullet \bigcap \langle \mathrm{Adp}_{D_{gi}}T^\bullet\rangle = \widetilde{\mathrm{Adp}_{D_{gi}}}T^\bullet$. Using Lemma 2.3 again, we can obtain $M^\bullet \in \utilde{\mathrm{Adp}_{D_{gi}}}T^\bullet \bigcap (\mathrm{Adp}_{D_{gi}}T^\bullet)^{\perp_{>0}} = \mathrm{Adp}_{D_{gi}}T^\bullet$. □

Proposition 3.10. *Assume that $T^\bullet$ is Gorenstein cosilting with $\mathrm{Adp}_{D_{gi}}T^\bullet \subseteq D_{gi}^{\geqslant 0}$. Then the following statements hold.*

(1) *If there are triangles $T_i^\bullet \to G_i \to T_{i+1}^\bullet \to$ with $G_i \in \mathcal{GI}$ for all $0 \leqslant i \leqslant n$ and $T_0^\bullet = T^\bullet$, $T_{n+1}^\bullet = 0$, then $\mathrm{Adp}_{D_{gi}}(\bigoplus_{i=0}^n G_i) = \mathcal{GI}$.*

(2) *If there are triangles $G_{i+1} \to T_i^\bullet \to G_i \to$ with $T_i^\bullet \in \mathrm{Adp}_{D_{gi}}T^\bullet$ for all $0 \leqslant i \leqslant m$, where $G_0 = G$, $G_{m+1} = 0$, then $\oplus_{i=0}^m T_i^\bullet$ is a Gorenstein cosilting complex. Moreover, $\mathrm{Adp}_{D_{gi}}(\oplus_{i=0}^m T_i^\bullet) = \mathrm{Adp}_{D_{gi}}T^\bullet$.*

(3) *$T^\bullet \in (\widecheck{\mathcal{GI}})_n$ if and only if $\mathcal{GI} \subseteq (\widehat{\mathrm{Adp}_{D_{gi}}T^\bullet})_n$ for any $n \geqslant 0$.*

Proof. (1) Since $\bigoplus_{i=0}^n G_i \in \mathcal{GI}$ is prod-semi-selforthogonal, we can obtain that $\bigoplus_{i=0}^n G_i$ satisfies (S2). One can easily check from Lemma 2.3 that $\mathrm{Adp}_{D_{gi}}T^\bullet \subseteq \langle \bigoplus_{i=0}^n G_i\rangle$. So $\langle \mathcal{GI}\rangle = K^b(\mathcal{GI}) = \langle \mathrm{Adp}_{D_{gi}}T^\bullet\rangle \subseteq \langle \mathrm{Adp}_{D_{gi}}(\bigoplus_{i=0}^n G_i)\rangle \subseteq \langle \mathcal{GI}\rangle$, i.e., $\langle \mathrm{Adp}_{D_{gi}}(\bigoplus_{i=0}^n G_i)\rangle = \langle \mathcal{GI}\rangle = K^b(\mathcal{GI})$. Then $\bigoplus_{i=0}^n G_i$ satisfies (S3). It is obvious that $\bigoplus_{i=0}^n G_i$ and $\bigoplus_{i=0}^n G_i \oplus G$ are Gorenstein cosilting, for any $G \in \mathcal{GI}$. Then, $\mathrm{Adp}_{D_{gi}}(\bigoplus_{i=0}^n G_i) = \mathcal{GI}$ by Proposition 3.9.

(2) It is not difficult to verify from Theorem 3.6 that both $(\bigoplus_{i=0}^m T_i^\bullet) \oplus T^\bullet$ and $\bigoplus_{i=0}^m T_i^\bullet$ are Gorenstein cosilting. Then we can learn by Proposition 3.9 that $\mathrm{Adp}_{D_{gi}}(\oplus_{i=0}^m T_i^\bullet) = \mathrm{Adp}_{D_{gi}}T^\bullet$.

(3) Let $G \in \mathcal{GI}$, we can get from Theorem 3.6 that $G \in (\widehat{\mathrm{Adp}_{D_{gi}}T^\bullet})_m$ for some $m > 0$. If $m \leqslant n$, then the conclusion obviously holds. Assume that $m > n$, there is a series of triangles $G_{i+1}^\bullet \to T_i^\bullet \to G_i^\bullet \to$ in $D_{gi}(\Lambda)$ with $T_i^\bullet \in \mathrm{Adp}_{D_{gi}}T^\bullet$, where $G_0^\bullet = G$, $G_{m+1}^\bullet = 0$ for $0 \leqslant i \leqslant m$. Note that

$G^\bullet_m \cong T^\bullet_m \in \mathrm{Adp}_{D_{gi}} T^\bullet$. It is straightforward to get

$$\begin{aligned}\mathrm{Hom}_{D_{gi}}(G^\bullet_{m-1}, G^\bullet_m[1]) &\cong \mathrm{Hom}_{D_{gi}}(G^\bullet_{m-2}, G^\bullet_m[2]) \\ &\cong \cdots \\ &\cong \mathrm{Hom}_{D_{gi}}(G^\bullet_0, G^\bullet_m[m]) = \mathrm{Hom}_{D_{gi}}(G, G^\bullet_m[m]).\end{aligned}$$

One can observe by $T^\bullet \in (\widetilde{\mathcal{GI}})_n$ and Lemma 2.3 that $\mathrm{Adp}_{D_{gi}} T^\bullet \subseteq ({}^{\perp_{>0}}\mathcal{GI})^{\perp_{>n}}$. It follows from $G \in {}^{\perp_{>0}}\mathcal{GI}$ that $\mathrm{Hom}_{D_{gi}}(G, G^\bullet_m[m]) = 0$ for $m > n$. Therefore, $\mathrm{Hom}_{D_{gi}}(G^\bullet_{m-1}, G^\bullet_m[1]) = 0$, moreover, the triangle $T^\bullet_m = G^\bullet_m \to T^\bullet_{m-1} \to G^\bullet_{m-1} \to$ splits. Then, $G^\bullet_{m-1} \in \mathrm{Adp}_{D_{gi}} T^\bullet$ and $G \in (\widehat{\mathrm{Adp}_{D_{gi}} T^\bullet})_{m-1}$. Repeating this process, the result holds.

On the other hand, it is obvious that $T^\bullet \in K^b(\mathcal{GI}) = \langle \mathcal{GI} \rangle$. Then, $T^\bullet \in \langle \mathcal{GI} \rangle \bigcap {}^{\perp_{>0}}\mathcal{GI} = \widetilde{\mathcal{GI}}$. Moreover, there is a series of triangles $T^\bullet_i \to G^\bullet_i \to T^\bullet_{i+1} \to$ in $D_{gi}(\Lambda)$ with $T^\bullet_0 = T^\bullet, G_i \in \mathcal{GI}$, and $T^\bullet_{n+1} = 0$ for $0 \leqslant i \leqslant n$. Hence, $T^\bullet_n \cong G_n \in \mathcal{GI}$. We conclude $T^\bullet \in (\widetilde{\mathcal{GI}})_n$. □

Lemma 3.11. *Suppose that* $T^\bullet \in D_{gi}(\Lambda)$ *with* $\mathrm{Adp}_{D_{gi}} T^\bullet \in {}^{\perp_{>0}}T^\bullet$. *Then* ${}^{\perp_{>0}}T^\bullet = \mathcal{X}_{\mathrm{Adp}_{D_{gi}} T^\bullet}$.

Proof. Clearly, $\mathcal{X}_{\mathrm{Adp}_{D_{gi}} T^\bullet} \subseteq {}^{\perp_{>0}}T^\bullet$. It is enough to show the inverse inclusion.

Let $M^\bullet \in {}^{\perp_{>0}}T^\bullet$ and consider the triangle $M^\bullet \xrightarrow{f} T^{\bullet \mathrm{Hom}_{D_{gi}}(M^\bullet, T^\bullet)} \to M^\bullet_1 \to$, where f is the canonical evaluation map. Then we can easily get that $\mathrm{Hom}_{D_{gi}}(M^\bullet_1, T^\bullet[i]) = 0$ for all $i > 0$, i.e., $M^\bullet_1 \in {}^{\perp_{>0}}T^\bullet$. Continuing the process, there are triangles $M^\bullet_j \to T^\bullet_j \to M^\bullet_{j+1} \to$ with each $T^\bullet_j \in \mathrm{Adp}_{D_{gi}} T^\bullet$ and $M^\bullet_j \in {}^{\perp_{>0}}T^\bullet$, $M^\bullet_0 = M^\bullet$ for all $j \geqslant 0$. Therefore, $M^\bullet \in \mathcal{X}_{\mathrm{Adp}_{D_{gi}} T^\bullet}$. Consequently, ${}^{\perp_{>0}}T^\bullet \subseteq \mathcal{X}_{\mathrm{Adp}_{D_{gi}} T^\bullet}$. This completes the proof. □

Proposition 3.12. $T^\bullet$ *is a Gorenstein cosilting complex with* $\mathrm{Adp}_{D_{gi}} T^\bullet \subseteq D^{\geqslant 0}_{gi}$ *if and only if* $T^\bullet$ *is Gorenstein precosilting and* ${}^{\perp_{>0}}T^\bullet \subseteq D^{\geqslant 0}_{gi}$.

Proof. Necessity. By the assumption, we have that $G \in (\widehat{\mathrm{Adp}_{D_{gi}} T^\bullet})_n$ for any $G \in \mathcal{GI}$ and some n. Hence, there is a series of triangles $G^\bullet_{i+1} \to T^\bullet_i \to G^\bullet_i \to$ with $G^\bullet_0 = G$, $T^\bullet_i \in \mathrm{Adp}_{D_{gi}} T^\bullet$ and $G^\bullet_{n+1} = 0$ for $0 \leqslant i \leqslant n$. For any $M^\bullet \in {}^{\perp_{>0}}T^\bullet$, applying the functor $\mathrm{Hom}_{D_{gi}}(M^\bullet, -)$ to these triangles, we can get $M^\bullet \in {}^{\perp_{>0}}G$. Then, ${}^{\perp_{>0}}T^\bullet \subseteq D^{\geqslant 0}_{gi}$.

Sufficiency. Since $T^\bullet \in D^{\geqslant 0}_{gi}$ is a Gorenstein precosilting complex, we can learn from $T^\bullet \in K^b(\mathcal{GI}) = \langle \mathcal{GI} \rangle$ that $T^\bullet \in \langle \mathcal{GI} \rangle \bigcap {}^{\perp_{>0}}\mathcal{GI} = \widetilde{\mathcal{GI}}$. Moreover, there is a series of triangles $T^\bullet_i \to G_i \to T^\bullet_{i+1} \to$ in $D_{gi}(\Lambda)$ with $T^\bullet_0 = T^\bullet$, $G_i \in \mathcal{GI}$, and $T^\bullet_{n+1} = 0$ for $0 \leqslant i \leqslant n$. Then, $T^\bullet_n \cong G_n \in \mathcal{GI}$. Applying the functor $\mathrm{Hom}_{D_{gi}}(G, -)$ to these triangles for any $G \in \mathcal{GI}$, we have isomorphisms

$$\mathrm{Hom}_{D_{gi}}(G, T^\bullet[j]) \cong \mathrm{Hom}_{D_{gi}}(G, T^\bullet_1[j-1]) \cong \cdots \cong \mathrm{Hom}_{D_{gi}}(G, T^\bullet_n[j-n]) = 0$$

for $j > n$. Then, it is easy to see $\mathcal{GI} \subseteq {}^{\perp_{>n}}T^\bullet$, namely $\mathcal{GI}[-n] \subseteq {}^{\perp_{>0}}T^\bullet$. As $0 \in {}^{\perp_{>0}}T^\bullet$, we get $\mathcal{GI} \in (\widehat{{}^{\perp_{>0}}T^\bullet})_n$. By Lemma 3.11 and [30, Corallary 2.8], there is a triangle $G \to X^\bullet \to Y^\bullet \to$ in $D_{gi}(\Lambda)$ with $X^\bullet \in (\widehat{\mathrm{Adp}_{D_{gi}}T^\bullet})_n$ and $Y^\bullet \in {}^{\perp_{>0}}T^\bullet$. We can immediately obtain from ${}^{\perp_{>0}}T^\bullet \subseteq D_{gi}^{\geqslant 0}$ that the triangle is split. Since $(\widehat{\mathrm{Adp}_{D_{gi}}T^\bullet})_n$ is closed under direct summands, we can obtain $G \in (\widehat{\mathrm{Adp}_{D_{gi}}T^\bullet})_n$ for any $G \in \mathcal{GI}$. The proof is completed. □

The Bazzoni's characterization of n-Gorenstein cotilting modules was given in [31] that a Λ-module T is n-Gorenstein cotilting if and only if ${}^{\perp_{>0}}T^G = \mathrm{Copres}^n_G(T)$.

In order to give the Bazzoni's characterization of n-Gorenstein cosilting complexes, we need the following subcategory of $D_{gi}(\Lambda)$. Let $T^\bullet \in D_{gi}(\Lambda)$ and $n > 0$, we denote

$$\begin{aligned}\mathrm{Copres}^n_{D_{gi}^{\geqslant 0}}(T^\bullet) = \{M^\bullet \in D_{gi}(\Lambda) \mid & \text{ there exist some triangles} \\ & M_i^\bullet \to T_i^\bullet \to M_{i+1}^\bullet \to \text{ with } T_i^\bullet \in \mathrm{Adp}_{D_{gi}}T^\bullet \text{ for all } 0 \leqslant i < n, \\ & \text{where } M_n^\bullet \in D_{gi}^{\geqslant 0} \text{ and } M_0^\bullet = M^\bullet\}.\end{aligned}$$

Obviously, $\mathrm{Copres}^n_{D_{gi}^{\geqslant 0}}(T^\bullet)$ is closed under products. The following result gives more properties about this subcategory.

Lemma 3.13. (1) $D_{gi}^{\geqslant 0}[-n] \subseteq \mathrm{Copres}^n_{D_{gi}^{\geqslant 0}}(T^\bullet)$.

(2) *If* $\mathrm{Adp}_{D_{gi}}T^\bullet \subseteq D_{gi}^{\geqslant 0}$, *then* $\mathrm{Copres}^n_{D_{gi}^{\geqslant 0}}(T^\bullet) \subseteq D_{gi}^{\geqslant 0}$.

Proof. Since $0 \in \mathrm{Adp}_{D_{gi}}T^\bullet$ and $D_{gi}^{\geqslant 0}$ is cosuspended, one can obtain by the definitions that the conclusions hold. □

Proposition 3.14. *If* $T^\bullet$ *with* $\mathrm{Adp}_{D_{gi}}T^\bullet \subseteq D_{gi}^{\geqslant 0}$ *is an* n*-Gorenstein cosilting complex, then* ${}^{\perp_{>0}}T^\bullet = \mathrm{Copres}^n_{D_{gi}^{\geqslant 0}}(T^\bullet)$.

Proof. The ${}^{\perp_{>0}}T^\bullet \subseteq D_{gi}^{\geqslant 0}$ directly follows from Proposition 3.12. It implies by Lemma 3.11 that ${}^{\perp_{>0}}T^\bullet = \mathcal{X}_{\mathrm{Adp}_{D_{gi}}T^\bullet}$. Take any $M^\bullet \in {}^{\perp_{>0}}T^\bullet$, it is obvious that there is a series of triangles $M_i^\bullet \to T_i^\bullet \to M_{i+1}^\bullet \to$ in $D_{gi}(\Lambda)$ with $T_i^\bullet \in \mathrm{Adp}_{D_{gi}}T^\bullet$ for all $0 \leqslant i < n$, where $M_n^\bullet \in \mathcal{X}_{\mathrm{Adp}_{D_{gi}}T^\bullet} \subseteq D_{gi}^{\geqslant 0}$ and $M_0^\bullet = M^\bullet$. Hence, ${}^{\perp_{>0}}T^\bullet \subseteq \mathrm{Copres}^n_{D_{gi}^{\geqslant 0}}(T^\bullet)$.

Conversely, for any $M^\bullet \in \mathrm{Copres}^n_{D_{gi}^{\geqslant 0}}(T^\bullet)$, there are triangles $M_i^\bullet \to T_i^\bullet \to M_{i+1}^\bullet \to$ in $D_{gi}(\Lambda)$ with $M_n^\bullet \in D_{gi}^{\geqslant 0}$, $T_i^\bullet \in \mathrm{Adp}_{D_{gi}}T^\bullet$ and $M_0^\bullet = M^\bullet$ for all $0 \leqslant i < n$. Therefore, $X^\bullet \in {}^{\perp_{>n}}T^\bullet$ for any $X^\bullet \in D_{gi}^{\geqslant 0} = {}^{\perp_{>0}}\mathcal{GI}$.

It is apparent from Proposition 3.10 that $T^\bullet \in (\widetilde{\mathcal{GI}})_n$. Then there is a series of triangles $T_i^\bullet \to G_i \to T_{i+1}^\bullet \to$ in $D_{gi}(\Lambda)$ with $T_{n+1}^\bullet = 0$, $G_i \in \mathcal{GI}$ and $T_0^\bullet = T^\bullet$ for $0 \leqslant i \leqslant n$. Applying the functor $\mathrm{Hom}_{D_{gi}}(X^\bullet, -)$ to these triangles, we have by $X^\bullet \in D_{gi}^{\geqslant 0} = {}^{\perp_{>0}}\mathcal{GI}$ and $T_n^\bullet \cong G_n^\bullet$ that there are isomorphisms

$$\begin{aligned}\operatorname{Hom}_{D_{gi}}(X^\bullet, T^\bullet[j]) &\cong \operatorname{Hom}_{D_{gi}}(X^\bullet, T_1^\bullet[j-1]) \cong \cdots \\ &\cong \operatorname{Hom}_{D_{gi}}(X^\bullet, T_n^\bullet[j-n]) = \operatorname{Hom}_{D_{gi}}(X, G_n^\bullet[j-n]) = 0\end{aligned}$$

for $j > n$.

Applying $\operatorname{Hom}_{D_{gi}}(-, T^\bullet)$ to the triangles $M_i^\bullet \to T_i^\bullet \to M_{i+1}^\bullet \to$ for $0 \leqslant i < n$, it yields by $M_n^\bullet \in D_{gi}^{\geqslant 0} = {}^{\perp_{>0}}\mathcal{GI}$ that

$$\operatorname{Hom}_{D_{gi}}(M^\bullet, T^\bullet[i]) \cong \cdots \cong \operatorname{Hom}_{D_{gi}}(M_n^\bullet, T^\bullet[i+n]) = 0.$$

Therefore, $\operatorname{Hom}_{D_{gi}}(M^\bullet, T^\bullet[i]) = 0$, i.e., $M^\bullet \in {}^{\perp_{>0}}T^\bullet$. Hence, $\operatorname{Copres}^n_{D_{gi}^{\geqslant 0}}(T^\bullet) \subseteq {}^{\perp_{>0}}T^\bullet$. □

In order to show the sufficient condition of n-Gorenstein cosilting complexes, we should give the following statements as a foundation.

Lemma 3.15. *Let $\mathcal{X}$ be an additive full subcategory of $\mathcal{A}$. Assume that for every $X^\bullet \in K^b(\mathcal{X})$ there is a $\mathcal{GI}$-quasi-isomorphism $X^\bullet \to G_{X^\bullet}$ with $G_{X^\bullet} \in K^+(\mathcal{GI})$. Then there is a functor $\psi : K^b(\mathcal{X}) \to K^+(\mathcal{GI})$ such that $\psi(X^\bullet) = G_{X^\bullet}$, and a $\mathcal{GI}$-quasi-isomorphism $\phi_{X^\bullet} \colon X^\bullet \to \psi(X^\bullet)$ for each $X^\bullet \in K^b(\mathcal{X})$ such that $\phi_{X^\bullet}$ is functorial in $X^\bullet$.*

Proof. Let $X^\bullet, Y^\bullet \in K^b(\mathcal{X})$. We can obtain by the assumption that there are $\mathcal{GI}$-quasi-isomorphisms $\phi_{X^\bullet} \colon X^\bullet \to G_{X^\bullet}$ and $\phi_{Y^\bullet} \colon Y^\bullet \to G_{Y^\bullet}$. It follows by Lemma 3.1 that $\phi_{X^\bullet}$ induces an isomorphism

$$\operatorname{Hom}_{K^+(\mathcal{A})}(G_{X^\bullet}, G_{Y^\bullet}) \cong \operatorname{Hom}_{K^+(\mathcal{A})}(X^\bullet, G_{Y^\bullet}).$$

For each chain map $f^\bullet \colon X^\bullet \to Y^\bullet$, it is clear that there is a unique $g^\bullet \colon G_{X^\bullet} \to G_{Y^\bullet}$ such that there is the following commutative diagram:

$$\begin{array}{ccc} X^\bullet & \xrightarrow{\phi_{X^\bullet}} & G_{X^\bullet} \\ \downarrow{\scriptstyle f^\bullet} & & \downarrow{\scriptstyle g^\bullet} \\ Y^\bullet & \xrightarrow{\phi_{Y^\bullet}} & G_{Y^\bullet} \end{array}$$

Let $Y^\bullet = X^\bullet$ and $f^\bullet = \operatorname{Id}_{X^\bullet}$, we can get that, up to a homotopy equivalence, $G_{X^\bullet}$ is uniquely determined by $X^\bullet$. Moreover, it yields that there is a functor ψ, such that $\psi(X^\bullet) = G_{X^\bullet}$ and $\psi(f^\bullet) = g^\bullet$. Then we can easily see from the above commutative diagram that $\phi_{X^\bullet}$ is functorial in $X^\bullet$. □

Let $f\mathcal{GI}$ be the full subcategory of $\mathcal{A}$ consisting of objects with finite Gorenstein injective dimension. Then $f\mathcal{GI}$ is an additive category. It follows that $K^b(f\mathcal{GI})$ is a triangulated category.

Proposition 3.16. *There exist a functor* $\psi\colon K^b(f\mathcal{GI}) \to K^b(\mathcal{GI})$*, and a* $\mathcal{GI}$*-quasi-isomorphism* $\phi_{X^\bullet}\colon X^\bullet \to \psi(X^\bullet)$ *for each* $X^\bullet \in K^b(f\mathcal{GI})$*, which is functorial in* $X^\bullet$*. Moreover, the inclusion* $K^b(\mathcal{GI}) \to K^b(f\mathcal{GI})$ *has a left adjoint* ψ*.*

Proof. In order to get the functor ψ, taking $\mathcal{X}$ be $f\mathcal{GI}$ in Lemma 3.15, it suffices to show that for $X^\bullet \in K^b(\mathcal{X})$ there is a $\mathcal{GI}$-quasi-isomorphism $X^\bullet \to G_{X^\bullet}$ with $G_{X^\bullet} \in K^b(\mathcal{GI})$. Let $w(X^\bullet)$ denote the width of $X^\bullet$, which is the number of non-zero components of $X^\bullet$. We will prove the statements by induction on $w(X^\bullet)$.

In case of $w(X^\bullet) = 1$, then it is a direct consequence of **2.3**. So, there is a $\mathcal{GI}$-quasi-isomorphism $\phi_{X^\bullet} : X^\bullet \to G_{X^\bullet}$ with $\mathcal{G}_{X^\bullet} \in K^b(\mathcal{GI})$.

Assume $w(X^\bullet) \geqslant 2$ with $X^j \neq 0$ and $X^i = 0$ for $i > j$. Then we have the distinguished triangle $X_1^\bullet \xrightarrow{u} X_2^\bullet \to X^\bullet \to X_1^\bullet[1]$ in $K^b(\mathcal{X})$, where $X_1^\bullet = X^{\bullet < j}$,$X_2^\bullet := X^j[j+1]$. By the induction hypothesis, there exist $\mathcal{GI}$-quasi-isomorphisms

$$\phi_1 : X_1^\bullet \to G_{X_1^\bullet}, \qquad \phi_2 : X_2^\bullet \to G_{X_2^\bullet}$$

with $G_{X_1^\bullet}, G_{X_2^\bullet} \in K^b(\mathcal{GI})$. We can obtain from Lemma 3.1 that ϕ_1 induces an isomorphism

$$\mathrm{Hom}_{K^+(\mathcal{X})}(G_{X_1^\bullet}, G_{X_2^\bullet}) \cong \mathrm{Hom}_{K^+(\mathcal{X})}(X_1^\bullet, G_{X_2^\bullet}).$$

Moreover, there is a unique morphism $f^\bullet : G_{X_1^\bullet} \to G_{X_2^\bullet}$ such that $f^\bullet\phi_1 = \phi_2 u$, up to homotopy. Let $G_{X^\bullet}$ be the mapping cone of $f^\bullet$, we can get from the distinguished triangle $G_{X_1^\bullet} \xrightarrow{f^\bullet} G_{X_2^\bullet} \to G_{X^\bullet} \to G_{X_1^\bullet}[1]$ that $G_{X^\bullet} \in K^b(\mathcal{GI})$. One can easily see that there is $\phi_{X^\bullet} : X^\bullet \to G_{X^\bullet}$, such that there is a commutative diagram:

$$\begin{array}{ccccccc}
X_1^\bullet & \xrightarrow{u} & X_2^\bullet & \longrightarrow & X^\bullet & \longrightarrow & X_1^\bullet[1] \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_{X^\bullet}} & & \downarrow{\scriptstyle \phi_1[1]} \\
G_{X_1^\bullet} & \xrightarrow{f} & G_{X_2^\bullet} & \longrightarrow & G_{X^\bullet} & \longrightarrow & G_{X_1^\bullet}[1]
\end{array}$$

Applying the $(-, Q) := \mathrm{Hom}_{K^+(\mathcal{A})}(-[n], Q)$ to the above diagram, where Q is a Gorenstein injective object, then it induces the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
(G_{X_2^\bullet}[1], Q) & \xrightarrow{u} & (G_{X_1^\bullet}[1], Q) & \longrightarrow & (G_{X^\bullet}, Q) & \longrightarrow & (G_{X_2^\bullet}, Q) & \longrightarrow & (G_{X_1^\bullet}, Q) \\
\downarrow{\scriptstyle (\phi_2[1])^*} & & \downarrow{\scriptstyle (\phi_1[1])^*} & & \downarrow{\scriptstyle (\phi_{X^\bullet})^*} & & \downarrow{\scriptstyle (\phi_2)^*} & & \downarrow{\scriptstyle (\phi_1)^*} \\
(X_2^\bullet[1], Q) & \xrightarrow{f} & (X_1^\bullet[1], Q) & \longrightarrow & (X^\bullet, Q) & \longrightarrow & (X_2^\bullet, Q) & \longrightarrow & (X_1^\bullet, Q)
\end{array}$$

Since ϕ_1 and ϕ_2 are $\mathcal{GI}$-quasi-isomorphisms, we can obtain that $(\phi_1)^*$, $(\phi_2)^*$, $(\phi_1[1])^*$, $(\phi_2[1])^*$ are isomorphisms. Thus, $(\phi_{X^\bullet})^*$ is an isomorphism for each n, that is, $(\phi_{X^\bullet})^*$ is a $\mathcal{GI}$-quasi-isomorphism.

The $\mathcal{GI}$-quasi-isomorphism $\phi_{X^\bullet} : X^\bullet \to G_{X^\bullet}$ induces an isomorphism from Lemma 3.15

$$\mathrm{Hom}_{K^b(f\mathcal{GI})}(X^\bullet, Q^\bullet) \cong \mathrm{Hom}_{K^b(\mathcal{GI})}(G_{X^\bullet}, Q^\bullet),$$

which is functorial both in $Q^\bullet \in K^b(\mathcal{GI})$ and $X^\bullet \in K^b(f\mathcal{GI})$, i.e., ψ is a left adjoint of the inclusion $K^b(\mathcal{GI}) \to K^b(f\mathcal{GI})$. □

Let $K^b_{gicoac}(f\mathcal{GI})$ denote the bounded homotopy category of $\mathcal{GI}$-coacyclic complexes of objects in $f\mathcal{GI}$. Then $K^b_{gicoac}(f\mathcal{GI})$ is a thick triangulated subcategory of $K^b(f\mathcal{GI})$ by **2.7**. Note that from **2.7**, if $\mathcal{X}$ is an additive full subcategory of $\mathcal{A}$, then $D^*_{gi}(\mathcal{X}) := K^*(\mathcal{X})/K^*_{gicoac}(\mathcal{X})$ is well-defined, so is $D^b_{gi}(f\mathcal{GI})$ by Proposition 3.16. It follows from **2.7** that the saturated multiplicative system determined by $K^b_{gicoac}(f\mathcal{GI})$ is the class of $\mathcal{GI}$-quasi-isomorphisms in $K^b(f\mathcal{GI})$.

We observe that $D^b_{gi}(f\mathcal{GI})$ is not a full subcategory of $D_{gi}(\mathcal{A})$ in general. However, we can get from a similar statements to Proposition 3.3 that the following lemma.

Lemma 3.17. *Let $G^\bullet \in K^b(\mathcal{GI})$. Then $Q : f^\bullet \to \frac{\mathrm{Id}_{G^\bullet}}{f^\bullet}$ induces an isomorphism $\mathrm{Hom}_{K^b(f\mathcal{GI})}(X^\bullet, G^\bullet) \cong \mathrm{Hom}_{D^b_{gi}(f\mathcal{GI})}(X^\bullet, G^\bullet)$ of abelian groups. In particular, $K^b(\mathcal{GI})$ can be viewed as a triangulated subcategory of $D^b_{gi}(f\mathcal{GI})$.*

Theorem 3.18. *Let $\mathcal{A}$ be an abelian category with enough injective objects. Then $D^b_{gi}(f\mathcal{GI}) \cong K^b(\mathcal{GI})$.*

Proof. Let $F : K^b(\mathcal{GI}) \to D^b_{gi}(f\mathcal{GI})$ be the composition of embedding map $K^b(\mathcal{GI}) \to K^b(f\mathcal{GI})$ and the localization functor $Q : K^b(f\mathcal{GI}) \to D^b_{gi}(f\mathcal{GI})$. It follows by Proposition 3.16(1) that F is dense. It can be obtained from Lemma 3.17 that F is fully faithful. □

The following proposition is a direct consequence of Theorem 3.18.

Proposition 3.19. *Assume that Λ is a Gorenstein ring. Then we have a triangle-equivalence $D^b_{gi}(\Lambda\text{-Mod}) \simeq K^b(\mathcal{GI})$.*

Proposition 3.20. *Let $T^\bullet$ with $\mathrm{Adp}_{D_{gi}} T^\bullet \subseteq D^b_{gi}(\Lambda) \bigcap D^{\geqslant 0}_{gi}$ satisfy ${}^{\perp_{>0}}T^\bullet = \mathrm{Copres}^n_{D^{\geqslant 0}_{gi}}(T^\bullet)$. Then $T^\bullet$ is an n-Gorenstein cosilting complex.*

Proof. Since $T^\bullet \in \mathrm{Copres}^n_{D^{\geqslant 0}_{gi}}(T^\bullet) = {}^{\perp_{>0}}T^\bullet$ and $\mathrm{Copres}^n_{D^{\geqslant 0}_{gi}}(T^\bullet)$ is closed under products, we obtain $\mathrm{Adp}_{D_{gi}} T^\bullet \subseteq {}^{\perp_{>0}}T^\bullet$. It follows by the assumption and Proposition 3.19 that $T^\bullet \in D^b_{gi}(\Lambda)$. Then, $T^\bullet \in K^b(\mathcal{GI}) = \langle \mathcal{GI} \rangle$. Consequently, $T^\bullet$ is Gorenstein precosilting.

Clearly, ${}^{\perp_{>0}}T^\bullet = \mathrm{Copres}^n_{D^{\geqslant 0}_{gi}}(T^\bullet) \subseteq D^{\geqslant 0}_{gi}$ is an immediate consequence of the hypothesis and Lemma 3.13. Then, we have by Proposition 3.12 that $T^\bullet$ is Gorenstein cosilting.

Note that $T^\bullet \in \langle \mathcal{GI} \rangle \bigcap {}^{\perp_{>0}}\mathcal{GI} = \breve{\mathcal{GI}}$ follows from $T^\bullet \in D_{gi}^{\geqslant 0} = {}^{\perp_{>0}}\mathcal{GI}$ and Lemma 2.3. Then there is an integer m such that $T^\bullet \in (\breve{\mathcal{GI}})_m$. Moreover, $T^\bullet$ is m-Gorenstein cosilting by Proposition 3.10. So, there is a series of triangles $T_i^\bullet \to G_i \to T_{i+1}^\bullet \to$ in $D_{gi}(\Lambda)$ with $G_i \in \mathcal{GI}$ for all $0 \leqslant i \leqslant m$, where $T_{m+1}^\bullet = 0$ and $T_0^\bullet = T^\bullet$. Applying $\mathrm{Hom}_{D_{gi}}(G, -)$ to the triangles, for any $G \in \mathcal{GI}$, we have that

$$\begin{aligned}\mathrm{Hom}_{D_{gi}}(G, T_m^\bullet) &\cong \mathrm{Hom}_{D_{gi}}(G, T_{m-1}^\bullet[1]) \cong \cdots \\ &\cong \mathrm{Hom}_{D_{gi}}(G, T_0^\bullet[m]) = \mathrm{Hom}_{D_{gi}}(G, T^\bullet[m]) = 0.\end{aligned}$$

Since $G[-n] \in D_{gi}^{\geqslant 0}[-n] \subseteq \mathrm{Copres}^n_{D_{gi}^{\geqslant 0}}(T^\bullet) = {}^{\perp_{>0}}T^\bullet$, we can obtain $\mathrm{Hom}_{D_{gi}}(G, T^\bullet[i+n]) = 0$ for any $i > 0$. If $m \leqslant n$, then T is n-Gorenstein cosilting. If $m > n$, then the triangle $T_{m-1}^\bullet \to G_{m-1} \to T_m^\bullet \to$ is split, i.e., $T^\bullet \in (\breve{\mathcal{GI}})_{m-1}$. Repeating the process, one can obtain $T^\bullet \in (\breve{\mathcal{GI}})_n$. Then, $T^\bullet$ is n-Gorenstein cosilting. □

Denote that

$D_{gi}^{[a,b]} = \{M^\bullet \in D_{gi}(\Lambda) \mid \mathrm{H}^i\mathrm{Hom}_{D_{gi}}(M^\bullet, G[i]) = 0$, where $i > b$ and for $G \in \mathcal{GI}\}$.

The following Bazzoni's characterization of n-Gorenstein cosilting complexes can be obtained from Proposition 3.14 in combination with Proposition 3.20.

Theorem 3.21. *Suppose that* $\mathrm{Adp}_{D_{gi}} T^\bullet \subseteq D_{gi}^{[0,t]}$ *for* $t \geqslant 0$. *Then the following statements are equivalent.*

(1) $T^\bullet$ *is* n*-Gorenstein cosilting*;

(2) ${}^{\perp_{>0}}T^\bullet = \mathrm{Copres}^n_{D_{gi}^{\geqslant 0}}(T^\bullet)$.

Auslander and Reiten [5] showed that there is a one-to-one correspondence between isomorphism classes of basic cotilting modules and certain contravariantly finite cosuspended subcategories. Later, Buan [10] showed that there is a one-to-one correspondence between basic cotilting complexes and certain contravariantly finite subcategories of the bounded derived category of an artin algebra. In the following, we aim to extend such a result to the case of Gorenstein cosilting complexes.

We list the following definitions and a useful lemma at first.

Definition. Let $\mathcal{X} \subseteq \mathcal{Y}$ be two subcategories of $\mathcal{D}$.

(1) $\mathcal{X}$ is said to be contravariantly finite in $\mathcal{Y}$ if for any $Y \in \mathcal{Y}$, there is a homomorphism $f : X \to Y$ for some $X \in \mathcal{X}$ such that $\mathrm{Hom}_{\mathcal{D}}(X', f)$ is surjective for any $X' \in \mathcal{X}$.

(2) $\mathcal{X}$ is said to be specially contravariantly finite in $\mathcal{Y}$ if for any $Y \in \mathcal{Y}$, there is a triangle $U \to X \to Y \to$ with some $X \in \mathcal{X}$ such that $\mathrm{Hom}_{\mathcal{D}}(X', U[1]) = 0$ for any $X' \in \mathcal{X}$.

Note that in the latter case, one has that $U \in \mathcal{X}^{\perp_{i>0}}$ if $\mathcal{X}$ is closed under $[-1]$.

Proposition 3.22. *Assume that* $\mathrm{Adp}_{D_{gi}}T^\bullet \in D_{gi}^{\geqslant 0}$ *and* $T^\bullet$ *is Gorenstein cosilting. Then* $\widehat{{}^{\perp_{>0}}T^\bullet} = D_{gi}^+$ *and* ${}^{\perp_{>0}}T^\bullet \subseteq D_{gi}^{\geqslant 0}$ *is specially contravariantly finite in* D_{gi}^+.

Proof. It follows by Proposition 3.12 that ${}^{\perp_{>0}}T^\bullet \subseteq D_{gi}^{\geqslant 0}$. Therefore, we get that $\widehat{{}^{\perp_{>0}}T^\bullet} = D_{gi}^+$. Take any $X^\bullet \in D_{gi}^+$, it is obvious by $T^\bullet \in K^b(\mathcal{GI})$ that $X^\bullet \in {}^{\perp_{\geqslant m}}T^\bullet$ for some m, that is, $X^\bullet[-m] \in {}^{\perp_{>0}}T^\bullet$. As $0 \in \mathrm{Adp}_{D_{gi}}T^\bullet$, we can see that $X^\bullet \in \widehat{{}^{\perp_{>0}}T^\bullet}$ from the definition. Hence, $\widehat{{}^{\perp_{>0}}T^\bullet} = D_{gi}^+$.

Note that ${}^{\perp_{>0}}T^\bullet = \mathcal{X}_{\mathrm{Adp}_{D_{gi}}T^\bullet}$ from Lemma 3.11. By the dual of [30, Corollary 2.8], we obtain for any $X^\bullet \in D_{gi}^+ = \widehat{{}^{\perp_{>0}}T^\bullet} = \mathcal{X}_{\mathrm{Adp}_{D_{gi}}T^\bullet}$ there is a triangle $U^\bullet \to Y^\bullet \to X^\bullet \to$ with $U^\bullet \in \widehat{\mathrm{Adp}_{D_{gi}}}T^\bullet$ and $Y^\bullet \in {}^{\perp_{>0}}T^\bullet$. Since $\widehat{\mathrm{Adp}_{D_{gi}}}T^\bullet \subseteq ({}^{\perp_{>0}}T^\bullet)^{\perp_{>0}}$, we specially get that $\mathrm{Hom}_{D_{gi}}(M^\bullet, U^\bullet[1]) = 0$ for any $M^\bullet \in {}^{\perp_{>0}}T^\bullet$. It follows that ${}^{\perp_{>0}}T^\bullet$ is specially contravariantly finite in $\mathcal{D}_{gi}^+$. □

Proposition 3.23. *Assume that* $\mathcal{T} \subseteq D_{gi}^{\geqslant 0}$ *is specially contravariantly finite in* D_{gi}^+ *and cosuspended such that* $\widehat{\mathcal{T}} = D_{gi}^+$. *If* $\mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}}$ *is closed under products, then there is a Gorenstein cosilting complex* T *such that* $\mathcal{T} = {}^{\perp_{>0}}T^\bullet$.

Proof. It is easy to verify $D_{gi}^+ = \widehat{\mathcal{T}} \subseteq {}^{\perp_{\gg 0}}(\mathcal{T}^{\perp_{>0}})$. Hence, we can obtain $\mathcal{T}^{\perp_{>0}} \subseteq K^b(\mathcal{GI})$.

Take any $M^\bullet \in D_{gi}^+$, there is a series of triangles $M_{j+1}^\bullet \to T_j^\bullet \to M_j^\bullet \to$ with $T_j^\bullet \in \mathcal{T}$ for all $j \geqslant 0$, where $M_0^\bullet := M^\bullet$ and each $M_j^\bullet \in \mathcal{T}^{\perp_{>0}}$ for $j \geqslant 1$. It follows that $T_j^\bullet \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}}$ for all $j \geqslant 1$. Then we can check by $\mathcal{T}^{\perp_{>0}} \subseteq K^b(\mathcal{GI})$ and $M^\bullet \in D_{gi}^+$ that $M^\bullet \in {}^{\perp_{>n}}(\mathcal{T}^{\perp_{>0}})$ for some n depending on $M^\bullet$. Applying the functor $\mathrm{Hom}_{D_{gi}}(-, M_{n+1}^\bullet)$ to the above triangles, we obtain that

$$\mathrm{Hom}_{D_{gi}}(M_n^\bullet, M_{n+1}^\bullet[1]) \simeq \cdots \simeq \mathrm{Hom}_{D_{gi}}(M^\bullet, M_{n+1}^\bullet[n+1]) = 0.$$

Thus, the triangle $M_{n+1}^\bullet \to T_n^\bullet \to M_n^\bullet \to$ is split. Moreover, $M_n^\bullet$ is a direct summand of T_n. Since $\mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}}$ is closed under products and $\mathcal{T}$ is cosuspended, it implies that $\mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}}$ is closed under direct summands. Then, $M_n^\bullet \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}}$.

Since $\mathcal{T} \subseteq D_{gi}^{\geqslant 0} = {}^{\perp_{>0}}\mathcal{GI}$, we have $\mathcal{GI} \subseteq \mathcal{T}^{\perp_{>0}}$. In particular, let the object $M^\bullet$ in the above be $G \in \mathcal{GI}$, then we can obtain the triangles $G_{j+1} \to T_j^{\bullet\prime} \to G_j \to$ with $G_j \in \mathcal{T}^{\perp_{>0}}$ and $T_j^{\bullet\prime} \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}}$ for all $0 \leqslant j \leqslant n$, where $G_0 = G$ and $G_{n+1} = 0$. Take $T^\bullet = \bigoplus_{j=0}^n T_j^{\bullet\prime}$ and then we show that $T^\bullet$ is Gorenstein cosilting. Indeed, as $\mathcal{T}^{\perp_{>0}} \subseteq K^b(\mathcal{GI})$ and $\mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}}$ is closed under products, one can easily verify that $T^\bullet$ is Gorenstein precosilting. Moreover, it is easy to see by the above argument that $\mathcal{GI} \in \widehat{\mathrm{Adp}_{D_{gi}}}T^\bullet$. Consequently, $T^\bullet$ is Gorenstein cosilting.

It remains to prove that $\mathcal{T} = {}^{\perp_{>0}}T^\bullet$. We can easily check by $T^\bullet = \bigoplus_{j=0}^{n} T_j^{\bullet\prime}$ and $T_j^{\bullet\prime} \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}}$ that $\mathcal{T} \subseteq {}^{\perp_{>0}}T^\bullet$ for all $0 \leqslant j \leqslant n$. Take any $N \in {}^{\perp_{>0}}T^\bullet$, similar to the above argument, we have a series of triangles $N_{j+1} \to T_j^{\bullet\prime\prime} \to N_j \to$ with $N_j \in \mathcal{T}^{\perp_{>0}}$, $T_0^{\bullet\prime\prime} \in \mathcal{T}$ and $T_j^{\bullet\prime\prime} \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}}$ for all $1 \leqslant j \leqslant m$, where $N_0 = N$ and $N_{m+1} = 0$. Observe that all objects in these triangles are in ${}^{\perp_{>0}}T^\bullet$. For any $L^\bullet \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}}$, one can get that $T^\bullet \oplus L^\bullet$ is also Gorenstein cosilting. Then we have from Proposition 3.9 that $L^\bullet \in \mathrm{Adp}_{D_{gi}} T^\bullet$. It follows that $\mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}} \subseteq \mathrm{Adp}_{D_{gi}} T^\bullet$. Moreover, $\mathcal{T} \bigcap \mathcal{T}^{\perp_{>0}} = \mathrm{Adp}_{D_{gi}} T^\bullet$. So, the above triangles imply that $N_1^\bullet \in \widehat{\mathrm{Adp}_{D_{gi}}} T^\bullet$ and consequently, $N_1^\bullet \in {}^{\perp_{>0}}T^\bullet \bigcap \widehat{\mathrm{Adp}_{D_{gi}}} T^\bullet = \mathrm{Adp}_{\mathcal{D}_{gi}} T^\bullet$. It follows by $\mathcal{T}$ cosuspended that $N^\bullet \in \mathcal{T}$. Then the triangle $N_1^\bullet \to T_0^\bullet \to N^\bullet \to$ is split. So, we can obtain that ${}^{\perp_{>0}}T^\bullet \subseteq \mathcal{T}$. This completes the proof. □

Combining Propositions 3.22 with 3.23, we obtain the following desired result. Here, we say two Gorenstein complexes $M^\bullet$ and $N^\bullet$ are equivalent if $\mathrm{Adp}_{D_{gi}} M^\bullet = \mathrm{Adp}_{D_{gi}} N^\bullet$.

Theorem 3.24. *There is a one-to-one correspondence, given by* $u : T^\bullet \mapsto {}^{\perp_{>0}}T^\bullet$, *between equivalence class of Gorenstein cosilting complexes with* $\mathrm{Adp}_{D_{gi}} T^\bullet \subseteq D_{gi}^{\geqslant 0}$ *and subcategories* $\mathcal{T} \subseteq D_{gi}^{\geqslant 0}$ *which is specially contravariantly finite in* D_{gi}^{+}, *cosuspended and closed under products such that* $\widehat{\mathcal{T}} = D_{gi}^{+}$.

Proof. It is not hard to check from Propositions 3.22 and 3.23 that the correspondence is well-defined. Note that u is surjective by Proposition 3.23. If both $T_1^\bullet$ and $T_2^\bullet$ are Gorenstein cosilting with ${}^{\perp_{>0}}T_1^\bullet = {}^{\perp_{>0}}T_2^\bullet$, then we can verify that $T_1^\bullet \oplus T_2^\bullet$ is also Gorenstein cosilting. It follows from Proposition 3.9 that $\mathrm{Adp}_{D_{gi}} T_1^\bullet = \mathrm{Adp}_{D_{gi}} T_2^\bullet$. So, $T_1^\bullet$ and $T_2^\bullet$ are equivalent. Consequently, u is bijective. □

4. The Bongartz's theorem of Gorenstein precosilting complexes

According to [2], given an artin algebra Λ, recall that the Nakayama functor on mod-Λ is defined as $\nu := D\mathrm{Hom}_\Lambda(-, \Lambda) : \text{mod-}\Lambda \to \text{mod-}\Lambda$. Restriction of the Nakayama functor to the category *proj*-Λ induces an equivalence *proj*-$\Lambda \to$ *inj*-Λ.

Assume that Λ is an artin algebra of CM-finite type over a field k, there is a finitely generated Gorenstein projective Λ-module G such that $\mathcal{G}proj = \mathrm{add}G$ and $\mathcal{G}inj = \mathrm{add}\nu G$. Clearly, $\mathrm{add}G$ (resp. $\mathrm{add}\nu G$) is a contravariantly (resp. covariantly) finite subcategory of mod-Λ.

In order to show the complement theorem of Gorenstein precosilting complexes, we at first introduce the small version of Gorenstein cosilting complexes.

Definition. (1) A complex T is called a Gorenstein precosilting complex if it satisfies the following two conditions.

(s1) $T^\bullet \in K^b(\mathcal{G}inj)$;

(s2) $T^\bullet \in {}^{\perp_{i>0}}T^\bullet$.

(2) A complex T is called a Gorenstein cosilting complex if it satisfies (s1), (s2) and

(s3) $K^b(\mathcal{G}inj) = \langle \mathrm{adp}_{D_{gi}} T^\bullet \rangle$, that is, $K^b(\mathcal{G}inj)$ coincides with the smallest triangulated subcategory containing $\mathrm{adp}_{D_{gi}} T^\bullet$.

Remark 4.1. All results in Section 3 hold if we replace $\mathrm{Adp}_{D_{gi}} T^\bullet$ and $K^b(\mathcal{GI})$ by $\mathrm{adp}_{D_{gi}} T^\bullet$ and $K^b(\mathcal{G}inj)$, respectively.

Definition. For a complex $X^\bullet$, we define the following notions.

(1) There is a truncation of $X^\bullet$,

$$\tau_{\geqslant n}(X^\bullet) : \cdots \to 0 \to \mathrm{Im} d^{n-1} \to X^n \to X^{n+1} \to \cdots,$$

(2) If $X^\bullet \in K^b(\mathcal{G}inj)$, then

$$\mathcal{R}(X^\bullet) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}^i \mathrm{Hom}_\Lambda(X^\bullet, \nu G) \neq 0\}.$$

Remark 4.2. For any morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $K^b(\mathcal{G}inj)$, it is obvious that there is a triangle $X^\bullet \to Y^\bullet \to \mathrm{Cone} f^\bullet \to$. If $\mathcal{R}(X^\bullet) = m$ and $\mathcal{R}(Y^\bullet) = n$, then $\mathcal{R}(\mathrm{Cone} f^\bullet) = \max\{m, n\}$.

We give a truncation of a complex in $K^b(\mathcal{G}inj)$ in term of $\mathcal{R}(X^\bullet)$.

Lemma 4.3. *Let $X^\bullet \in K^b(\mathcal{G}inj)$. Then $X^\bullet$ is Gorenstein quasi-isomorphic to $\tau_{\geqslant -\mathcal{R}(X^\bullet)}(X^\bullet)$ in $K^b(\mathcal{G}inj)$.*

Proof. Without loss of generality, assume that $X^i = 0$ for $i > 0$ and $i < -s$, then

$$X^\bullet : \cdots \to 0 \to X^{-s} \xrightarrow{d^{-s}} X^{-s+1} \xrightarrow{d^{-s+1}} \cdots \xrightarrow{d^{-1}} X^0 \to 0 \to \cdots.$$

Let $\mathcal{R}(X^\bullet) = m$. If $m = s$ or $m = s-1$, it is easy to verify that $X^\bullet$ is Gorenstein quasi-isomorphic to $\tau_{\geqslant -m}(X^\bullet)$.

Assume $-m \geqslant -s+2$, we have the following commutative diagram:

$$\begin{array}{cccccccccccc}
X^\bullet: & \cdots \to & 0 & \to & X^{-s} & \to & X^{-s+1} & \to & X^{-s+2} & \to & X^{-s+3} & \to \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \| & \\
\tau_{\geqslant -s+2}(X^\bullet): & \cdots \to & 0 & \to & 0 & \to & \mathrm{Im} d^{-s+1} & \to & X^{-s+2} & \to & X^{-s+3} & \to \cdots
\end{array}$$

Applying the functor $\mathrm{Hom}_\Lambda(-, \nu G)$ to the above diagram, we obtain the following commutative diagram:

$$\begin{array}{cccccccccc}
(\tau_{\geqslant -s+2}(X^\bullet), \nu G): & \cdots \to & (X^{-s+2}, \nu G) & \to & (\mathrm{Im} d^{-s+1}, \nu G) & \to & 0 & \to & 0 & \to \cdots \\
\downarrow & & \| & & \downarrow & & \downarrow & & \downarrow & \\
(X^\bullet, \nu G): & \cdots \to & (X^{-s+2}, \nu G) & \to & (X^{-s+1}, \nu G) & \to & (X^{-s}, \nu G) & \to & 0 & \to \cdots
\end{array}$$

where $(-, \nu G)$ denotes the functor $\mathrm{Hom}_\Lambda(-, \nu G)$. Since $0 \to X^{-s} \to X^{-s+1} \to \mathrm{Im} d^{-s+1} \to 0$ is exact, we can obtain by the above diagram the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(\mathrm{Im} d^{-s+1}, \nu G) \to \mathrm{Hom}_\Lambda(X^{-s+1}, \nu G) \to \mathrm{Hom}_\Lambda(X^{-s}, \nu G) \to 0.$$

It is obvious that the sequence $0 \to X^{-s} \to X^{-s+1} \to \mathrm{Im}d^{-s+1} \to 0$ is **G**-exact. As $X^{-s} \in \mathcal{G}inj$, we can easily obtain that the above **G**-exact sequence is split. Then, $\mathrm{Im}d^{-s+1} \in \mathcal{G}inj$. It is not hard to check that $X^\bullet$ is Gorenstein quasi-isomorphic to $\sigma_{\geqslant -\mathcal{R}(X^\bullet)}(X^\bullet)$ in $K^b(\mathcal{G}inj)$. Repeating the process, one can easily get the result. □

Given two complexes $X^\bullet \in K^b(\mathcal{G}inj)$ and $Y^\bullet \in K^b(\mathcal{G}inj)$, there is a morphism $f^\bullet : X^\bullet \to Y^{\bullet n}$, where $n = \mathrm{Hom}_{K(\Lambda)}(X^\bullet, Y^\bullet)$ and $\mathrm{Hom}_{K(\Lambda)}(X^\bullet, Y^\bullet) \in \mathrm{mod}k$, i.e., $\mathrm{Hom}_{K(\Lambda)}(X^\bullet, Y^\bullet)$ is finite dimensional as a k-vector space. Then one can easily obtain that f induces an epimorphism $\mathrm{Hom}_{K(\Lambda)}(f^\bullet, Y^\bullet)$ by [14, Proposition 4.2.2].

Now, we introduce a series of distinguished triangles in $K^b(\mathcal{G}inj)$. For the following part, we fix a complex $U^\bullet \in K^b(\mathcal{G}inj)$ and $V_0^\bullet = \nu G$ as a 0-th stalk complex in $K^b(\mathcal{G}inj)$. Let n_0 be the cardinality of a generating set of $\mathrm{Hom}_{K(\Lambda)}(V_0^\bullet, U^\bullet)$. Then there is the morphism $f_0^\bullet : V_0^\bullet \to U^{\bullet n_0}$ in $K^b(\mathcal{G}inj)$ such that $\mathrm{Hom}_{K(\Lambda)}(f_0^\bullet, U^\bullet)$ is surjective. It induces a triangle in $K^b(\mathcal{G}inj)$

$$\Delta_0 : V_0^\bullet \xrightarrow{f_0^\bullet} U^{\bullet n_0} \xrightarrow{g_0^\bullet} V_1^\bullet \to .$$

Similar to the above argument, there is a series of triangles in $K^b(\mathcal{G}inj)$

$$\Delta_k : V_k^\bullet \xrightarrow{f_k^\bullet} U^{\bullet n_k} \xrightarrow{g_k^\bullet} V_{k+1}^\bullet \to$$

with $\mathrm{Hom}_{K(\Lambda)}(f_k^\bullet, U^\bullet)$ surjective for $k \geqslant 1$, where n_k is the cardinality of a generating set of $\mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, U^\bullet)$.

We denote $E{:} = \nu G$ in the following statements.

Lemma 4.4. *Let $U^\bullet \in K^b(\mathcal{G}inj)$ be a complex with $\mathcal{R}(U^\bullet) = m \geqslant 0$. Then $\mathcal{R}(V_k^\bullet) \leqslant k + \max\{m-1, 0\}$ for any $k \geqslant 0$.*

Proof. We will prove the result by induction on k. The statement is trivial in case $k = 0$, i.e., $\mathrm{H}^i\mathrm{Hom}_\Lambda(V_0^\bullet, E) = 0$ for $i > \max\{m-1, 0\}$. Assume inductively that $\mathrm{H}^i\mathrm{Hom}_\Lambda(V_{k-1}^\bullet, E) = 0$ for $i > k-1+\max\{m-1, 0\}$. Applying the functor $\mathrm{Hom}_\Lambda(-, E)$ to Δ_{k-1}, we have the following long exact sequence

$$\cdots \to \mathrm{H}^i\mathrm{Hom}_\Lambda(V_{k-1}^\bullet, E) \to \mathrm{H}^{i+1}\mathrm{Hom}_\Lambda(V_k^\bullet, E) \to \mathrm{H}^{i+1}\mathrm{Hom}_\Lambda(U^{\bullet n_{k-1}}, E) \to \cdots .$$

Since $\mathrm{H}^{i+1}\mathrm{Hom}_\Lambda(U^{\bullet n_{k-1}}, E) = \prod_{n_{k-1}} \mathrm{H}^{i+1}\mathrm{Hom}_\Lambda(U^\bullet, E) = 0$, it is easy to obtain $\mathrm{H}^i\mathrm{Hom}_\Lambda(V_k^\bullet, E) = 0$ for $i > k + \max\{m-1, 0\}$. □

Lemma 4.5. *Suppose that $U^\bullet \in K^b(\mathcal{G}inj)$ is a Gorenstein precosilting complex with $\mathcal{R}(U^\bullet) = m \geqslant 0$. Then the following statements hold for any $k \geqslant 0$.*

(1) $\mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, U^\bullet[i]) = 0$ *for* $i > 0$.

(2) $\mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, V_k^\bullet[i]) \simeq \mathrm{H}^{i+k}\mathrm{Hom}_\Lambda(V_k^\bullet, E)$ *for* $i > 0$ *and* $k \leqslant 0$. *In particular, we have that* $\mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, V_k^\bullet[i]) = 0$ *for* $i > \max\{m-1, 0\}$.

Proof. (1) If $i > k$, applying $\mathrm{Hom}_{K(\Lambda)}(-, U^\bullet)$ to $\Delta_0, \ldots, \Delta_{k-1}$, we have that

$$\mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, U^\bullet[i]) \simeq \mathrm{Hom}_{K(\Lambda)}(V_{k-1}^\bullet, U^\bullet[i-1])$$

$$\simeq \cdots$$
$$\simeq \mathrm{Hom}_{K(\Lambda)}(V_0^\bullet, U^\bullet[i-k]) = 0.$$

In case of $0 < i \leqslant k$, applying $\mathrm{Hom}_{K(\Lambda)}(-, U^\bullet)$ to Δ_{k-i}, there is a long exact sequence

$$\cdots \to \mathrm{Hom}_{K(\Lambda)}(U^{\bullet n_{k-i}}, U^\bullet) \xrightarrow{\mathrm{Hom}_{K(\Lambda)}(f_{k-i}, U^\bullet)} \mathrm{Hom}_{K(\Lambda)}(V_{k-i+1}^\bullet, U^\bullet)$$
$$\to \mathrm{Hom}_{K(\Lambda)}(V_{k-i+1}^\bullet, U^\bullet[1]) \to \mathrm{Hom}_{K(\Lambda)}(U^{\bullet n_{k-i}}, U^\bullet[1]) \to \cdots.$$

Since $U^\bullet$ is a Gorenstein precosilting complex, we have $\mathrm{Hom}_{K(\Lambda)}(U^{\bullet n_{k-i}}, U^\bullet[1]) = 0$. It follows from the construction of $f_{k-i}^\bullet$ that $\mathrm{Hom}_{K(\Lambda)}(f_{k-i}, U^\bullet)$ is an epimorphism. Then, $\mathrm{Hom}_{K(\Lambda)}(V_{k-i+1}^\bullet, U^\bullet[1]) = 0$. Moreover, it implies that

$$\begin{aligned}\mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, U^\bullet[i]) &\simeq \mathrm{Hom}_{K(\Lambda)}(V_{k-1}^\bullet, U^\bullet[i-1]) \\ &\simeq \cdots \\ &\simeq \mathrm{Hom}_{K(\Lambda)}(V_{k-i+1}^\bullet, U^\bullet[1]) = 0.\end{aligned}$$

(2) Applying $\mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, -)$ to $\Delta_0, \ldots, \Delta_{k-1}$, we have

$$\begin{aligned}\mathrm{Hom}_{K(\Lambda)}(V^\bullet, V_k^\bullet[i]) &\simeq \mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, V_{k-1}^\bullet[i+1]) \\ &\simeq \cdots \\ &\simeq \mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, V_0^\bullet[i+k]) \\ &\simeq \mathrm{H}^{i+k}\mathrm{Hom}_\Lambda(V_k^\bullet, E).\end{aligned}$$

In particular, it is a consequence of (1) that $\mathrm{H}^{i+k}\mathrm{Hom}_\Lambda(V_k^\bullet, E) = 0$ for $i > \max\{m-1, 0\}$. □

Lemma 4.6. *Let $U^\bullet \in K^b(\mathcal{G}inj)$ be a complex with $\mathcal{R}(U^\bullet) = m \geqslant 0$. If $k \geqslant m$, then $\mathrm{Hom}_{K(\Lambda)}(U^\bullet, V_k^\bullet[i]) = 0$ for $i > 0$.*

Proof. Applying $\mathrm{Hom}_{K(\Lambda)}(U^\bullet, -)$ to Δ_j for $0 \leqslant j \leqslant k-1$, we can learn by the assumption that

$$\begin{aligned}\mathrm{Hom}_{K(\Lambda)}(U^\bullet, V_k^\bullet[i]) &\simeq \mathrm{Hom}_{K(\Lambda)}(U^\bullet, V_{k-1}^\bullet[i+1]) \\ &\simeq \mathrm{Hom}_{K(\Lambda)}(U^\bullet, V_{k-2}^\bullet[i+2]) \\ &\simeq \cdots \\ &\simeq \mathrm{Hom}_{K(\Lambda)}(U^\bullet, V_0^\bullet[i+k]).\end{aligned}$$

It follows from $V_0^\bullet = \nu G$ and $\mathcal{R}(U^\bullet) = m \geqslant 0$ that $\mathrm{Hom}_{K(\Lambda)}(U^\bullet, V_0^\bullet[i+k]) = 0$. Then the result holds. □

Lemma 4.7. *Suppose that $U^\bullet \in K^b(\mathcal{G}inj)$ is a Gorenstein precosilting complex with $\mathcal{R}(U^\bullet) = m \geqslant 2$. For any $k \geqslant m$, then the following statements are equivalent.*

(1) $\mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, V_k^\bullet[i]) = 0$ *for* $0 < i < m$;
(2) $\mathrm{Hom}_{K(\Lambda)}(f_i^\bullet, E[m])$ *is surjective for* $0 < i < m$;
(3) $\mathcal{R}(V_m^\bullet) \leqslant m$;

(4) $\mathcal{R}(V_k^\bullet) \leqslant k$.

Proof. (1) $\Rightarrow$ (2) Applying the functor $\mathrm{Hom}_{K(\Lambda)}(-, E)$ to the Δ_i, one can easily obtain an exact sequence

$$\cdots \to \mathrm{Hom}_{K(\Lambda)}(U^{\bullet n_i}, E[m]) \xrightarrow{\mathrm{Hom}_{K(\Lambda)}(f_i, E[m])} \mathrm{Hom}_{K(\Lambda)}(V_i^\bullet, E[m]) \\ \to \mathrm{Hom}_{K(\Lambda)}(V_{i+1}^\bullet, E[m+1]) \to \mathrm{Hom}_{K(\Lambda)}(U^{\bullet n_i}, E[m+1]) \to \cdots .$$

Since $\mathcal{R}(U^\bullet) = m$, we have $\mathrm{Hom}_{K(\Lambda)}(U^{\bullet n_i}, E[m+1]) = 0$. Therefore, one can easily get $\mathrm{Coker}\mathrm{Hom}_{K(\Lambda)}(f_i, E[m]) \simeq \mathrm{Hom}_{K(\Lambda)}(V_{i+1}^\bullet, E[m+1])$. It suffices to show $\mathrm{Hom}_{K(\Lambda)}(V_{i+1}^\bullet, E[m+1]) = 0$. Applying the functor $\mathrm{Hom}_{K(\Lambda)}(-, E)$ to the $\Delta_i, \ldots, \Delta_{k-1}$, we can obtain that

$$\begin{aligned} \mathrm{Coker}\mathrm{Hom}_\Lambda(f_i^\bullet, E[m]) &\simeq \mathrm{Hom}_{K(\Lambda)}(V_{i+1}^\bullet, E[m+1]) \\ &\simeq \mathrm{Hom}_{K(\Lambda)}(V_{i+2}^\bullet, E[m+2]) \\ &\simeq \cdots \\ &\simeq \mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, E[m+k-i]) \\ &\simeq \mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, V_0^\bullet[m+k-i]) \\ &\simeq \mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, V_k^\bullet[m-i]) = 0. \end{aligned}$$

The last isomorphism follows from Lemma 4.5.

(2) $\Rightarrow$ (3) We will show that $\mathcal{R}(V_i^\bullet) \leqslant m$ for $1 \leqslant i \leqslant m$ by induction on i. If $i = 1$, applying $\mathrm{Hom}_{K(\Lambda)}(-, E)$ to Δ_0, we have a long exact sequence

$$\cdots \to \mathrm{Hom}_{K(\Lambda)}(V_0^\bullet, E[j-1]) \to \mathrm{Hom}_{K(\Lambda)}(V_1^\bullet, E[j]) \to \mathrm{Hom}_{K(\Lambda)}(U^{\bullet n_0}, E[j]) \to \cdots .$$

Since $\mathrm{Hom}_{K(\Lambda)}(V_0^\bullet, E[j-1]) = 0$ by Lemma 4.5, and $\mathcal{R}(U^\bullet) = m \geqslant 2$, it is clear that $\mathrm{Hom}_{K(\Lambda)}(V_1^\bullet, E[j]) = 0$ for $j > m$.

Assume that $\mathcal{R}(V_i^\bullet) \leqslant m$ for $0 \leqslant i < m$. Applying $\mathrm{Hom}_{K(\Lambda)}(-, E)$ to the Δ_{m-1}, it is apparent to obtain a long exact sequence

$$\cdots \to \mathrm{Hom}_{K(\Lambda)}(U^{\bullet n_{m-1}}, E[m]) \xrightarrow{\mathrm{Hom}_{K(\Lambda)}(f_{m-1}^\bullet, E[m])} \mathrm{Hom}_{K(\Lambda)}(V_{m-1}^\bullet, E[m]) \\ \to \mathrm{Hom}_{K(\Lambda)}(V_m^\bullet, E[m+1]) \to \cdots .$$

Since $\mathrm{Hom}_{K(\Lambda)}(f_{m-1}^\bullet, E[m])$ is surjective and $\mathcal{R}(U^\bullet) = m$, it is straightforward that $\mathrm{Hom}_{K(\Lambda)}(V_m^\bullet, E[m+1]) = 0$. Therefore, $\mathcal{R}(V_m^\bullet) \leqslant m$.

(3) $\Rightarrow$ (4) Applying $\mathrm{Hom}_{K(\Lambda)}(-, E)$ to Δ_j for $0 \leqslant j \leqslant k-1$, we have

$$\begin{aligned} \mathrm{Hom}_{K(\Lambda)}(V_k^\bullet, E[i]) &\simeq \mathrm{Hom}_{K(\Lambda)}(V_{k-1}^\bullet, E[i-1]) \\ &\simeq \cdots \\ &\simeq \mathrm{Hom}_{K(\Lambda)}(V_m^\bullet, E[i+m-k]) = 0 \end{aligned}$$

for $i > k$.

(4) $\Rightarrow$ (1) It is trivial from Lemma 4.5. □

Proposition 4.8. *Assume that $U^\bullet \in K^b(\mathcal{G}inj)$ is a Gorenstein precosilting complex with $\mathcal{R}(V_m^\bullet) \leqslant m$. Then $U^\bullet \oplus V_k^\bullet$ is a Gorenstein cosilting complex for $k \geqslant m$.*

Proof. It is easy to obtain by the construction that $U^\bullet \oplus V_k^\bullet \in K^b(\mathcal{G}inj)$. Therefore, the condition (s1) holds. One can easily verify from the assumption and Lemmas 4.5, 4.6, 4.7 that condition (s2) $U^\bullet \oplus V_k^\bullet \in (U^\bullet \oplus V_k^\bullet)^{\perp_{>0}}$. It suffices to show that $\langle \mathrm{adp}_{D_{gi}}(U^\bullet \oplus V_k^\bullet)\rangle = K^b(\mathcal{G}inj)$. Indeed, we need to prove that if $\mathrm{Hom}_{K(\Lambda)}(U^\bullet \oplus V_k^\bullet, X[i]) = 0$ for any i, then $X^\bullet$ is zero. We may construct the following triangles

$$\nabla_{j-1} : V_{j-1}^\bullet \to U^{\bullet n_{j-1}} \oplus V_k^\bullet \to V_j^\bullet \oplus V_k^\bullet \to$$

for $j \leqslant 1$. It is clear for $s \geqslant 0$ that

$$\mathrm{Hom}_{K(\Lambda)}(U^{\bullet n_{j-1}} \oplus V_k^\bullet[s+1], X^\bullet[i]) = \mathrm{Hom}_{K(\Lambda)}(U^{\bullet n_{j-1}} \oplus V_k^\bullet[s], X^\bullet[i]) = 0.$$

Applying the functor $\mathrm{Hom}_{K(\Lambda)}(-, X^\bullet[i])$ to the triangles $\nabla_1, \dots, \nabla_{k-1}$, one can easily check that

$$\mathrm{Hom}_{K(\Lambda)}(V_{s+1}^\bullet \oplus V_k^\bullet[s+1], X^\bullet[i]) = \mathrm{Hom}_{K(\Lambda)}(V_s^\bullet[s], X^\bullet[i]) = 0$$

for $0 \leqslant s \leqslant k-1$. It is obvious that $\mathrm{Hom}_{K(\Lambda)}(V_k^\bullet[s+1], X^\bullet[i]) = 0$ for $0 \leqslant s \leqslant k-1$. Then we can obtain that

$$\begin{aligned} \mathrm{Hom}_{K(\Lambda)}(V_0^\bullet, X^\bullet[i]) &\simeq \mathrm{Hom}_{K(\Lambda)}(V_1^\bullet[1], X^\bullet[i]) \\ &\simeq \cdots \\ &\simeq \mathrm{Hom}_{K(\Lambda)}(V_k^\bullet[k], X^\bullet[i]) = 0 \end{aligned}$$

for any i. Therefore, we get $\mathrm{Hom}_\Lambda(\mathrm{add}\nu G, X^\bullet) = 0$. Moreover, we can learn $\mathrm{Hom}_\Lambda(X^j, X^\bullet) = 0$ for any j. It is easy to see from [13, Lemma 2.4] that $\mathrm{H}^i\mathrm{Hom}_\Lambda(X^\bullet, X^\bullet) = 0$ for any i. Then, $\mathrm{H}^0\mathrm{Hom}_\Lambda(X^\bullet, X^\bullet) = \mathrm{Hom}_\Lambda(X^\bullet, X^\bullet) = 0$, that is $X^\bullet = 0$. So, the condition (s3) is true. As a consequence, $U^\bullet \oplus V_k^\bullet$ is Gorenstein cosilting. □

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