J. Korean Math. Soc. **60** (2023), No. 6, pp. 1303–1336
https://doi.org/10.4134/JKMS.j230105
pISSN: 0304-9914 / eISSN: 2234-3008

ON THE η-PARALLELISM IN ALMOST KENMOTSU 3-MANIFOLDS

Jun-ichi Inoguchi and Ji-Eun Lee

Abstract. In this paper, we study the η-parallelism of the Ricci operator of almost Kenmotsu 3-manifolds. First, we prove that an almost Kenmotsu 3-manifold M satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α has dominantly η-parallel Ricci operator if and only if it is locally symmetric. Next, we show that if M is an H-almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for a constant α, then M is a Kenmotsu 3-manifold or it is locally isomorphic to certain non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure. The dominantly η-parallelism of the Ricci operator is equivalent to the local symmetry on homogeneous almost Kenmotsu 3-manifolds.

1. Introduction

As is well known, local symmetry causes a strong restriction for almost contact metric manifolds. Boeckx and Cho ([5]) proved that a locally symmetric contact Riemannian manifold is either normal and of constant curvature 1 or locally isometric to the product space $\mathbb{R}^{n+1} \times \mathbb{S}^n(4)$. In particular, locally symmetric K-contact manifolds are of constant curvature 1 ([22, 31]).

On the other hand, Kenmotsu ([19]) proved that a locally symmetric Kenmotsu manifold is of constant curvature -1 (see also [14] for Kenmotsu 3-manifolds).

In [13], Dileo and Pastore proposed the following question:

Is a locally symmetric almost Kenmotsu manifold either Kenmotsu of constant curvature -1 or locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$?

As a partial affirmative answer, they proved that locally symmetric almost Kenmotsu manifolds of dimension greater than 3 satisfying $R(X, Y)\xi = 0$ for all vector fields X and Y orthogonal to ξ are Kenmotsu of constant curvature -1 or locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Here ξ is the characteristic vector field.

Received February 27, 2023; Revised July 19, 2023; Accepted September 5, 2023.
2020 *Mathematics Subject Classification.* Primary 53D15, 53C25, 53C30.
Key words and phrases. Almost Kenmotsu manifold, local symmetry, η-parallelism.
The first named author is partially supported by Kakenhi JP19K03461, JP23K03081.

Next, Wang and Liu [33] proved that locally symmetric CR-integrable almost Kenmotsu manifolds of dimension greater than 3 are Kenmotsu of constant curvature -1 or locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Since the associated almost CR-structure of an almost Kenmotsu 3-manifold is integrable, one might expect that the Wang-Liu's classification result also holds in dimension 3. As a negative answer to Dileo-Pastor's question, the following result was obtained: Locally symmetric almost Kenmotsu 3-manifolds are locally isomorphic to one of the following model spaces (see [16, 29]):

- the hyperbolic 3-space $\mathbb{H}^3(-1)$ of curvature -1 equipped with a homogeneous Kenmotsu structure.
- the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$ of the hyperbolic plane of curvature -4 and the real line equipped with a homogeneous strictly H-almost Kenmotsu structure.
- the Riemannian product $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ of the hyperbolic plane of curvature $-4-\gamma^2 < -4$ and the real line equipped with a homogeneous strictly almost Kenmotsu structure.

In 3-dimensional Riemannian geometry, we know that the local symmetry (parallelism of the Riemannian curvature R) is equivalent to the parallelism of the Ricci operator S. As a generalization of local symmetry from almost contact geometric viewpoint, we consider almost contact metric 3-manifolds with *η-parallel* Ricci operator, *i.e.*,

$$g((\nabla_X S)Y, Z) = 0$$

for all vector fields X, Y and Z orthogonal to ξ ([20]). For contact metric 3-manifolds, Cho ([9]) classified contact metric 3-manifolds with η-parallel Ricci operator under the assumption the characteristic vector field is an eigenvector field of the Ricci operator (so-called H-contact 3-manifolds). On the other hand, in [11], Cho and the second named author of the present article classified contact metric 3-manifolds with η-parallel Ricci operator under the condition $\nabla_\xi h = 2\alpha h\varphi$ for some constant α (for the definition of the endomorphism field h, see Subsection 2.1). In particular, they classified those 3-manifolds satisfying $\nabla_\xi h = 0$.

The purpose of the present article is to study almost Kenmotsu 3-manifolds with η-parallel Ricci operator. In this direction, the first named author of the present article classified Kenmotsu 3-manifolds with η-parallel Ricci operator [15]. In this article we generalize the classification obtained in [15] to *almost* Kenmotsu 3-manifolds. Moreover, we study almost Kenmotsu 3-manifolds with *dominantly η-parallel* Ricci operator, that is, S satisfies

$$g((\nabla_X S)Y, Z) = 0$$

for all vector fields X and Y on M and any vector field Z on M orthogonal to ξ. Note that the Ricci operator S is said to be *strongly η-parallel* if

$$g((\nabla_X S)Y, Z) = 0$$

holds for all vector field X on M and vector fields Y and Z on M orthogonal to ξ ([15, 21]).

First, we show the existence of almost Kenmotsu 3-manifolds with η-parallel Ricci operator. More precisely we exhibit explicit examples of homogeneous almost Kenmotsu 3-manifolds with η-parallel Ricci operator in Section 4. Next, we give an example of non-homogeneous almost Kenmotsu 3-manifold with η-parallel Ricci operator in Section 5.

We start our investigation with *homogeneous* almost Kenmotsu 3-manifolds. We prove that the only homogeneous almost Kenmotsu 3-manifolds with dominantly η-parallel Ricci operator are locally symmetric ones in Section 4 (Theorem 4.3).

In Section 6, we prove that an almost Kenmotsu 3-manifold M with dominantly η-parallel Ricci operator is locally symmetric under certain condition on the endomorphism field h. In addition, we prove that if an almost Kenmotsu 3-manifold whose characteristic vector field is an eigenvector field of S satisfying $\nabla_\xi h = -2\alpha h\varphi$ (for some constant α) has η-parallel Ricci operator, then it is a Kenmotsu 3-manifold or it is locally isomorphic to certain almost Kenmotsu Lie group.

To close Introduction we should mention the so-called *local φ-symmetry* (η-parallelism of the Riemannian curvature) of almost Kenmotsu 3-manifolds. As is well known the Riemannian curvature R of a Riemannian 3-manifold M is explicitly described by the Ricci operator S. As a consequence, the parallelism of S is equivalent to the parallelism of R (local symmetry). More generally, the semi-parallelism (resp. pseudo-parallelism) of S is equivalent to the semi-parallelism (resp. pseudo-parallelism) of the Riemannian curvature.

However, for almost contact metric 3-manifolds, the η-parallelism of R (local φ-symmetry) is not equivalent to that of S. Indeed, on an almost contact metric 3-manifold M, if the Ricci operator S and scalar curvature r are η-parallel, then so is the Riemannian curvature R (Proposition 2.2). This phenomena means that the η-parallelism of S has own interest.

The authors would like to acknowledge to the anonymous reviewer for pointing out several issues of the first version of this article.

2. Almost contact metric manifolds

In this section, we recall fundamental ingredients of almost contact metric geometry. For general information on almost contact metric geometry, we refer to Blair's monograph [2].

2.1. Almost contact structures

An *almost contact metric structure* of a $(2n+1)$-manifold M is a quartet (φ, ξ, η, g) of structure tensor fields which satisfies:

$$\eta(\xi) = 1, \quad d\eta(\xi, \cdot) = 0,$$
$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \varphi\xi = 0,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

A $(2n+1)$-manifold $M = (M, \varphi, \xi, \eta, g)$ equipped with an almost contact metric structure is called an *almost contact metric manifold*. The vector field ξ is called the *characteristic vector field* of M. The 2-form

$$\Phi(X, Y) = g(X, \varphi Y)$$

is called the *fundamental 2-form* of M. On an almost contact metric manifold M, we introduce an endomorphism field h which plays a prominent role in this study by

$$h = \frac{1}{2}\pounds_\xi \varphi,$$

where $\pounds_\xi$ denotes the Lie differentiation by ξ.

Definition. Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. A tangent plane Π_p at $p \in M$ is said to be *holomorphic* if it is invariant under φ_p.

It is easy to see that a tangent plane Π_p is holomorphic if and only if ξ_p is orthogonal to Π_p. The sectional curvature $K(\Pi_p)$ of a holomorphic plane Π_p is called the *holomorphic sectional curvature* (also called φ-sectional curvature) of Π_p. In case $\dim M = 3$, the holomorphic sectional curvature $K(\Pi_p)$ is denoted by H_p and called the *holomorphic sectional curvature* at p.

Here we introduce the η-parallelism for endomorphism fields and scalar fields:

Definition. An endomorphism field F on an almost contact metric manifold M is said to be

- *η-parallel* if it satisfies $g((\nabla_X F)Y, Z) = 0$ for all vector fields X, Y and Z on M orthogonal to ξ.
- *strongly η-parallel* if it satisfies $g((\nabla_X F)Y, Z) = 0$ for all vector field X on M and any vector fields Y and Z on M orthogonal to ξ.
- *dominantly η-parallel* if it satisfies $g((\nabla_X F)Y, Z) = 0$ for all vector fields X and Y on M and any vector field Z on M orthogonal to ξ.

Definition. A scalar field f on an almost contact metric manifold M is said to be *η-parallel* if it satisfies

$$df(X) = g(\operatorname{grad} f, X) = 0$$

for all vector field X orthogonal to ξ.

Here we mention a notion which is related to the η-parallelism. According to Blair [1], an endomorphism field F on an almost contact metric manifold M is said to be *Killing* if it satisfies $(\nabla_X F)X = 0$ for all vector field on M. More generally F is said to be *transversally Killing* if $(\nabla_X F)Y + (\nabla_Y F)X = 0$ for all vector fields X and Y on M orthogonal to ξ [8]. In [24, Remark 1.2], the authors claimed that the transversal Killing property for the Ricci operator S of an almost contact metric 3-manifold is much weaker than the η-parallelism of S. However we shall see after (Remark 6.11), "transversal-Killing S" is stronger than "η-parallel S".

2.2. The local symmetry and the $\boldsymbol{\eta}$-parallelism

Let us denote by R the Riemannian curvature of a Riemannian m-manifold (M, g) defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M).$$

Here ∇ is the Levi-Civita connection and $\mathfrak{X}(M)$ is the Lie algebra of all smooth vector fields on M. The *Ricci tensor field* ρ is defined as the contraction of R of the type

$$\rho(X, Y) = \mathcal{C}(R)(X, Y) = \sum_{i=1}^{m} g(R(e_i, X)Y, e_i),$$

where $\mathcal{C}$ is the contraction operator and $\{e_1, e_2, \ldots, e_m\}$ is a local orthonormal frame field. The *Ricci operator* S is a self-adjoint endomorphism field defined by

$$\rho(X, Y) = g(SX, Y).$$

The *scalar curvature* r is defined as $r = \operatorname{tr} \rho = \operatorname{tr} S$. One can see that

$$(\nabla_X \rho)(Y, Z) = g((\nabla_X S)Y, Z) = g(Y, (\nabla_X S)Z).$$

The covariant derivative $\nabla\rho$ defined by

$$(\nabla\rho)(Y, Z; X) := (\nabla_X \rho)(Y, Z)$$

satisfies

$$(\nabla\rho)(Y, Z; X) = (\nabla\rho)(Z, Y; X),$$

but not totally symmetric, in general. A Riemannian manifold (M, g) is said to be a *space of harmonic curvature* if $\nabla\rho$ is totally symmetric.

Since the contraction operator $\mathcal{C}$ commutes with the covariant differentiation ∇_X, we get

$$\begin{aligned} g((\nabla_W S)Y, Z) = (\nabla_W \rho)(Y, Z) &= \sum_{i=1}^{m} g((\nabla_W R)(e_i, Y)Z, e_i) \\ &= g(\operatorname{tr}_g(\nabla_W R)(\cdot, Y)Z, \cdot). \end{aligned}$$

Obviously, the local symmetry ($\nabla R = 0$) implies the parallelism of S ($\nabla S = 0$).

Now let us concentrate our attention to Riemannian 3-manifolds. On a Riemannian 3-manifold $M = (M, g)$, the Riemannian curvature R is described as

$$R(X, Y)Z = \rho(Y, Z)X - \rho(Z, X)Y + g(Y, Z)SX - g(Z, X)Y - \frac{r}{2}R_1(X, Y)Z.$$

Here the curvature-like tensor field R_1 is defined by

$$R_1(X, Y)Z = (X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y.$$

The covariant derivative ∇R is computed as

$$\begin{aligned} (\nabla_W R)(X, Y)Z = {} & (\nabla_W \rho)(Y, Z)X - (\nabla_W \rho)(Z, X)Y \\ & + g(Y, Z)(\nabla_W S)X - g(Z, X)(\nabla_W S)Y - \frac{dr}{2}(W)R_1(X, Y)Z. \end{aligned}$$

Hence the covariant derivative ∇R satisfies the following formula:

$$\begin{aligned} g((\nabla_W R)(X,Y)Z,V) = g((\nabla_W S)Y,Z)g(X,V) - g((\nabla_W S)Z,X)g(Y,V) \\ + g(Y,Z)g((\nabla_W S)X,V) - g(Z,X)g((\nabla_W S)Y,V) \\ - \frac{dr}{2}(W)g(R_1(X,Y)Z,V). \end{aligned} \tag{1}$$

We know that the local symmetry ($\nabla R = 0$) implies the constancy of the scalar curvature, thus we confirm the following well-known fact:

Proposition 2.1. *A Riemannian* 3*-manifold* M *is locally symmetric if and only if its Ricci operator is parallel.*

Now let us consider almost contact metric manifold M of arbitrary odd-dimension. We introduce the η-parallelism of the Riemannian curvature in the following manner:

Definition. The Riemannian curvature R of an almost contact metric manifold M is said to be *η-parallel* if R satisfies

$$g((\nabla_W R)(X,Y)Z,V) = 0$$

for all vector fields X, Y, Z, W and V orthogonal to ξ.

Definition. The Riemannian curvature R of an almost contact metric manifold M is said to be *strongly η-parallel* if R satisfies

$$g((\nabla_W R)(X,Y)Z,V) = 0$$

for all vector fields X, Y, Z and V orthogonal to ξ and any vector field W on M.

Definition. The Riemannian curvature R of an almost contact metric manifold M is said to be *dominantly η-parallel* if R satisfies

$$g((\nabla_W R)(X,Y)Z,V) = 0$$

for all vector fields X, Y, Z and W on M and V orthogonal to ξ.

According to these definitions we deduce the following fact:

Proposition 2.2. *On an almost contact metric* 3*-manifold* M*, if the Ricci operator* S *and scalar curvature* r *are η-parallel, then so is the Riemannian curvature* R*.*

Conversely, if the Riemannian curvature R *is η-parallel, then the Ricci operator* S *is η-parallel if and only if*

$$\eta((\nabla_W R)(\xi,X)Y) = 0 \tag{2}$$

holds for all vector fields W*,* X *and* Y *orthogonal to* ξ*.*

Proof. ($\Rightarrow$) Assume that S and r are η-parallel, then from (1) R is η-parallel.

($\Leftarrow$) Conversely, take a local orthonormal frame field of the form $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$, then under the assumption R is η-parallel, from (1), we obtain

$$g((\nabla_W S)X, Y) = \eta((\nabla_W R)(\xi, X)Y)$$

for all W, X and Y orthogonal to ξ. Thus the η-parallelism of S is equivalent to (2). □

Remark 2.3. Obviously, η-parallelism of R together with the η-parallelism of S implies that of r.

While the local symmetry is equivalent to the parallelism of the Ricci operator on arbitrary Riemannian 3-manifolds, especially almost contact metric 3-manifolds, the η-parallelism of R is not equivalent to that of S on almost contact metric 3-manifolds.

We investigate η-parallelism of R on *almost Kenmotsu* 3-*manifold* in the final section. We shall show that the η-parallelism of R is equivalent to that of S on every Kenmotsu 3-manifold.

The η-parallelism of the Riemannian curvature was investigated first by Takahashi [30] for Sasakian manifolds. Precisely speaking, Takahashi considered Sasakian manifolds satisfying

$$\varphi^2\{(\nabla_W R)(X, Y)Z\} = 0 \tag{3}$$

for all vector fields X, Y, Z and W orthogonal to ξ. Obviously this conditions is equivalent to the η-parallelism of R.

Buken and Vanhecke pointed out that if a K-contact manifold M has η-parallel Riemannian curvature, then M is Sasakian [7]. A K-contact manifold M is said to be a (Sasakian) *φ-symmetric space* (in the sense of Takahashi) if its Riemannian curvature R is η-parallel. A K-contact manifold M is locally φ-symmetric if and only if all the characteristic reflections are isometric [7,30]. There are two directions to generalize the local φ-symmetry to general almost contact metric manifolds. Blair, Koufogiorgos and Sharma [3] introduced the notion of local φ-symmetry for general contact metric manifolds by the η-parallelism of R. On the other hand, Boeckx and Vanhecke [6] defined the local φ-symmetry of contact metric manifolds by the property "all the characteristic reflections are isometric". To distinguish these two classes, Boeckx, Buken and Vanhecke [4] proposed the terminologies "strongly locally φ-symmetric space" and "weakly locally φ-symmetric space". According to [4], a contact metric manifold is said to be a *weakly locally φ-symmetric space* if its Riemannian curvature R is η-parallel. On the other hand, a contact metric manifold is said to be a *strongly locally φ-symmetric space* if all the characteristic reflections are isometric. They showed that strongly locally φ-symmetric spaces are weakly locally φ-symmetric. For more information on weakly locally φ-symmetric contact metric 3-manifolds, we refer to [26].

2.3. Normality

On an almost contact metric manifold M, we define a torsion tensor field N by

$$N(X,Y) := [\varphi,\varphi](X,Y) + 2d\eta(X,Y)\xi, \quad X,Y \in \mathfrak{X}(M).$$

Here $[\varphi,\varphi]$ is the Nijenhuis torsion of φ. An almost contact metric manifold M is said to be *normal* if $N(X,Y) = 0$ for all X, $Y \in \mathfrak{X}(M)$.

2.4. Almost Kenmotsu structure

Definition ([18]). An almost contact metric manifold M is said to be *almost Kenmotsu* if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. An almost Kenmotsu manifold is said to be *Kenmotsu* if it is normal. An almost Kenmotsu manifold is said to be *strictly almost Kenmotsu* if it is non-normal.

It should be remarked that every almost Kenmotsu manifold satisfies $\operatorname{div}\xi = 2n$. Hence almost Kenmotsu manifolds can not be compact.

Proposition 2.4 ([13]). *If an almost Kenmotsu manifold is of constant curvature, then it is a Kenmotsu manifold of constant curvature* -1.

In the class of Sasakian manifolds (resp. cosymplectic manifolds), there is a particularly nice subclass, the class of *Sasakian space forms* (resp. *cosymplectic space forms*). In the class of Kenmotsu manifolds, constancy of holomorphic sectional curvature is a too strong restriction. In fact, Kenmotsu [19] showed:

Proposition 2.5. *Let M be a Kenmotsu manifold of dimension greater than* 3. *Then M is of constant holomorphic sectional curvature if and only if it is of constant curvature* -1.

Three dimensional case will be discussed in the next subsection. To close this subsection we introduce the following notion.

Definition. An almost Kenmotsu manifold whose characteristic vector field ξ is a harmonic unit vector field is called an *H-almost Kenmotsu manifold*.

Perrone showed the following fundamental fact ([27, Theorem 4.1], [28, Proposition 7]).

Proposition 2.6. *An almost Kenmotsu manifold M is H-almost Kenmotsu if and only if ξ is an eigenvector field of S.*

2.5. Kenmotsu 3-manifolds

Here we recall curvature properties of *Kenmotsu* 3-*manifolds*.

Proposition 2.7. *The Riemannian curvature R of a Kenmotsu* 3*-manifold M has the form*

$$R(X,Y)Z = \frac{r+4}{2}(X \wedge Y)Z + \frac{r+6}{2}[\xi \wedge \{(X \wedge Y)\xi\}]Z.$$

The Ricci operator S has the form

$$S = \frac{r+2}{2}\,\mathrm{I} - \frac{r+6}{2}\,\eta \otimes \xi.$$

The principal Ricci curvatures are $(r+2)/2$, $(r+2)/2$ and -2. The Ricci operator S commutes with φ. For a unit vector $X \in TM$ orthogonal to ξ, the sectional curvatures of planes $X \wedge \varphi X$ and $X \wedge \xi$ are given by

$$H = K(X \wedge \varphi X) = \frac{r}{2} + 2, \quad K(X \wedge \xi) = -1.$$

Note that every Kenmotsu 3-manifold is H-almost Kenmotsu. Thus the notion of H-almost Kenmotsu is intermediate notion between Kenmotsu and almost Kenmotsu.

Proposition 2.8 ([14]). *The following three properties for a Kenmotsu 3-manifold M are mutually equivalent.*

- *M has constant holomorphic sectional curvature.*
- *M has constant scalar curvature.*
- *M is of constant curvature -1.*

3. Almost Kenmotsu 3-manifolds

3.1. Fundamental formulas

Let M be an almost Kenmotsu 3-manifold. Denote by $\mathcal{U}_1$ the open subset of M consisting of points p such that $h \neq 0$ around p. Next, let $\mathcal{U}_0$ be the open subset of M consisting of points $p \in M$ such that $h = 0$ around p. Since h is smooth, $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_0$ is an open dense subset of M. So any property satisfied in $\mathcal{U}$ is also satisfied in whole M. For any point $p \in \mathcal{U}$, there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ around p, where e_1 is an eigenvector field of h.

Lemma 3.1 (*cf.* [10]). *Let M be an almost Kenmotsu 3-manifold. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ on $\mathcal{U}$ such that*

$$he_1 = \lambda e_1, \quad e_2 = \varphi e_1, \quad e_3 = \xi$$

for some locally defined smooth function λ. The Levi-Civita connection ∇ is described as

$$\nabla_{e_1} e_1 = -be_2 - \xi, \quad \nabla_{e_1} e_2 = be_1 + \lambda\xi, \quad \nabla_{e_1} e_3 = e_1 - \lambda e_2,$$
$$\nabla_{e_2} e_1 = ce_2 + \lambda\xi, \quad \nabla_{e_2} e_2 = -ce_1 - \xi, \quad \nabla_{e_2} e_3 = -\lambda e_1 + e_2,$$
$$\nabla_{e_3} e_1 = \alpha e_2, \quad \nabla_{e_3} e_2 = -\alpha e_1, \quad \nabla_{e_3} e_3 = 0,$$

where

$$b = -\frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1)), \quad c = -\frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2)),$$

and σ is the 1-form metrically equivalent to $S\xi$, that is,

$$\sigma = g(S\xi, \cdot) = \rho(\xi, \cdot).$$

The covariant derivative $\nabla_\xi h$ *of* h *by* ξ *is given by*

$$\nabla_\xi h = -2\alpha h\varphi + \frac{\xi(\lambda)}{\lambda} h$$

for $h \neq 0$ *on the open subset* $\mathcal{U}$.

The commutation relations are

$$[e_1, e_2] = be_1 - ce_2, \quad [e_2, e_3] = (\alpha - \lambda)e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\alpha + \lambda)e_2.$$

The Jacobi identity is described as

$$e_1(\alpha - \lambda) + \xi(b) + c(\alpha - \lambda) + b = 0, \quad e_2(\alpha + \lambda) - \xi(c) + b(\alpha + \lambda) - c = 0.$$

Remark 3.2. On an almost Kenmotsu 3-manifold M with $h \neq 0$,

$$\alpha = g(\nabla_\xi W, \varphi W)$$

is independent of the choice of unit eigenvector W of h.

The Riemannian curvature R is computed by the table of Levi-Civita connection in Lemma 3.1:

$$\begin{aligned}
&R(e_1, e_2)e_1 = -He_2 - \sigma(e_2)\xi, \quad R(e_1, e_2)e_2 = He_1 + \sigma(e_1)\xi,\\
&R(e_1, e_2)e_3 = \sigma(e_2)e_1 - \sigma(e_1)e_2, \quad R(e_2, e_3)e_1 = \sigma(e_1)e_2 - \{\xi(\lambda) + 2\lambda\}\,\xi,\\
&R(e_2, e_3)e_2 = -\sigma(e_1)e_1 - K_{23}\xi, \quad R(e_2, e_3)e_3 = \{\xi(\lambda) + 2\lambda\}\,e_1 + K_{23}e_2,\\
&R(e_3, e_1)e_1 = \sigma(e_2)e_2 + K_{13}\xi, \quad R(e_3, e_1)e_2 = -\sigma(e_2)e_1 + \{\xi(\lambda) + 2\lambda\}\,\xi,\\
&R(e_3, e_1)e_3 = -K_{13}e_1 - \{\xi(\lambda) + 2\lambda\}\,e_2.
\end{aligned}$$

Here the sectional curvatures $K_{ij} = K(e_i \wedge e_j)$ are given by

$$H = K_{12} = \frac{r}{2} + 2(\lambda^2 + 1), \quad K_{13} = -(\lambda^2 + 2\alpha\lambda + 1), \quad K_{23} = -(\lambda^2 - 2\alpha\lambda + 1).$$

Next, the Ricci operator S of an almost Kenmotsu 3-manifold M is described as:

$$\begin{aligned}
Se_1 &= \left(\frac{r}{2} + \lambda^2 - 2\alpha\lambda + 1\right) e_1 + \{\xi(\lambda) + 2\lambda\}e_2 + \sigma(e_1)\xi,\\
Se_2 &= \{\xi(\lambda) + 2\lambda\}e_1 + \left(\frac{r}{2} + \lambda^2 + 2\alpha\lambda + 1\right) e_2 + \sigma(e_2)\xi,\\
Se_3 &= \sigma(e_1)e_1 + \sigma(e_2)e_2 - 2(\lambda^2 + 1)\xi.
\end{aligned}$$

Note that the scalar curvature r is computed as

$$r = -2\{e_1(c) + e_2(b) + b^2 + c^2 + \lambda^2 + 3\}$$

and

$$\sigma(e_1) = -e_2(\lambda) - 2\lambda b, \quad \sigma(e_2) = -e_1(\lambda) - 2\lambda c.$$

Remark 3.3. Since M is 3-dimensional, we have the relations

$$\rho_{11} = H + K_{13}, \quad \rho_{22} = H + K_{23}, \quad \rho_{33} = K_{13} + K_{23}.$$

4. Homogeneous almost Kenmotsu 3-manifolds

As we proclaimed in Introduction, the principal purpose of this article is to investigate almost Kenmotsu 3-manifolds with η-parallel Ricci operator. To show this purpose is not worthless, we provide explicit examples of those manifolds. In this section, we exhibit homogeneous examples. Non-homogeneous examples will be exhibited in the next section.

4.1. Two classes

Perrone ([28]) proved that every simply connected homogeneous almost Kenmotsu 3-manifold is a 3-dimensional non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure. There are two classes of 3-dimensional almost Kenmotsu Lie groups.

- The characteristic vector field ξ is orthogonal to the unimodular kernel (Type II Lie groups).
- The characteristic vector field ξ is transversal to the unimodular kernel but not orthogonal (Type IV Lie groups).

After performing normalization procedure, those Lie group has the Lie algebra determined by the following commutation relations:

- The type II Lie algebra $\mathfrak{g} = \mathfrak{g}(\lambda, \alpha)$ is generated by

$$[e_1, e_2] = 0, \quad [e_2, e_3] = (\alpha - \lambda)e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\alpha + \lambda)e_2,$$

  where λ, $\alpha \in \mathbb{R}$.
- The type IV Lie algebra $\mathfrak{g} = \mathfrak{g}[\alpha, \gamma]$ is generated by

$$[e_1, e_2] = \gamma e_1, \quad [e_2, e_3] = 2\alpha e_1, \quad [e_3, e_1] = -2e_1,$$

  where α, $\gamma \in \mathbb{R}$ and $\gamma \neq 0$.

In this section, we exhibit these Lie groups in detail. For more information on these examples, we refer to [16, 17].

4.2. Type II Lie groups

Let $G(\lambda, \alpha)$ be a 3-dimensional non-unimodular Lie group with Lie algebra $\mathfrak{g}(\lambda, \alpha)$ generated by the orthonormal basis $\{e_1, e_2, e_3\}$ with commutation relations

$$[e_1, e_2] = 0, \quad [e_2, e_3] = (\alpha - \lambda)e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\alpha + \lambda)e_2.$$

Then a left invariant almost contact structure (φ, ξ, η) compatible to the left invariant metric g is defined by

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \quad \xi = e_3, \quad \eta = g(e_3, \cdot).$$

Then (φ, ξ, η, g) is a left invariant almost Kenmotsu structure. Note that $\{e_1, e_2, e_3\}$ is regarded as a global orthonormal frame field as in Lemma 3.1 under the choice $b = c = 0$. The Levi-Civita connection is described as

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = \lambda e_3, \quad \nabla_{e_1} e_3 = e_1 - \lambda e_2,$$

$$\nabla_{e_2}e_1 = \lambda e_3, \quad \nabla_{e_2}e_2 = -e_3, \quad \nabla_{e_2}e_3 = -\lambda e_1 + e_2,$$
$$\nabla_{e_3}e_1 = \alpha e_2, \quad \nabla_{e_3}e_2 = -\alpha e_1, \quad \nabla_{e_3}e_3 = 0.$$

The Riemannian curvature R of $G(\lambda,\alpha)$ is given by

$$R(e_1,e_2)e_1 = -K_{12}e_2, \quad R(e_1,e_2)e_2 = K_{12}e_1, \quad R(e_1,e_2)e_3 = 0,$$
$$R(e_2,e_3)e_1 = -2\lambda\xi, \quad R(e_2,e_3)e_2 = -K_{23}\xi, \quad R(e_2,e_3)e_3 = 2\lambda e_1 + K_{23}e_2,$$
$$R(e_3,e_1)e_1 = K_{13}\xi, \quad R(e_3,e_1)e_2 = 2\lambda\xi, \quad R(e_3,e_1)e_3 = -K_{13}e_1 - 2\lambda e_2,$$

where

$$K_{12} = -(1-\lambda^2), \quad K_{13} = -(\lambda^2 + 2\lambda\alpha + 1), \quad K_{23} = -(\lambda^2 - 2\lambda\alpha + 1).$$

The Ricci operator S is given by

$$Se_1 = -2(1+\lambda\alpha)e_1 + 2\lambda e_2, \quad Se_2 = 2\lambda e_1 - 2(1-\lambda\alpha)e_2, \quad Se_3 = -2(1+\lambda^2)\xi.$$

Thus every $G(\lambda,\alpha)$ is H-almost Kenmotsu. The scalar curvature r is computed as

$$r = -2(3+\lambda^2).$$

The principal Ricci curvatures are

$$\rho_1 = -2 + 2\lambda\sqrt{1+\alpha^2}, \quad \rho_2 = -2 - 2\lambda\sqrt{1+\alpha^2}, \quad \rho_3 = -2(1+\lambda^2).$$

Direct computation show that

$$(\nabla_{e_1}S)e_1 = 2\alpha\lambda\xi, \quad (\nabla_{e_1}S)e_2 = 2\lambda(\lambda^2 + \alpha\lambda - 1)\xi,$$
$$(\nabla_{e_1}S)e_3 = 2\alpha\lambda e_1 + 2\lambda(\lambda^2+\alpha\lambda-1)e_2, \quad (\nabla_{e_2}S)e_1 = 2\lambda(\lambda^2 - \alpha\lambda - 1)\xi,$$
$$(\nabla_{e_2}S)e_2 = -2\alpha\lambda\xi, \quad (\nabla_{e_2}S)e_3 = 2\lambda(\lambda^2-\alpha\lambda-1)e_1 - 2\alpha\lambda e_2,$$
$$(\nabla_{e_3}S)e_1 = -4\alpha\lambda e_1 - 4\alpha^2\lambda e_2, \ (\nabla_{e_3}S)e_2 = -4\alpha^2\lambda e_1 + 4\alpha\lambda e_2, \ (\nabla_{e_3}S)e_3 = 0.$$

From this table, one can see that the locally symmetric Lie group $G(\lambda,\alpha)$ of type II are

- $G(0,\alpha)$ for any α. The Lie group $G(0,\alpha)$ is isometric to the hyperbolic 3-space $\mathbb{H}^3(-1)$ of constant curvature equipped with a left invariant Kenmotsu structure.
- $G(\pm 1, 0)$ which is isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$ equipped with a left invariant strictly H-almost Kenmotsu structure.

Note that the transversally Killing property for S is equivalent to the local symmetry.

The Lie derivative $\pounds_\xi S$ is given by

$$(\pounds_\xi S)e_1 = -4\lambda\alpha(e_1 + (\lambda+\alpha)e_2), \ (\pounds_\xi S)e_2 = 4\lambda\alpha((\lambda-\alpha)e_1 + e_2), \ (\pounds_\xi S)e_3 = 0.$$

Hence

$$\pounds_\xi S = 0 \Longleftrightarrow \nabla_\xi S = 0 \Longleftrightarrow \lambda = 0 \ \text{or} \ \alpha = 0.$$

Moreover we obtain the following classification:

Proposition 4.1. *Every almost Kenmotsu Lie group $G(\lambda,\alpha)$ has η-parallel Ricci operator. Moreover*

- *The Ricci operator S is strongly η-parallel when and only when $\lambda = 0$ or $\alpha = 0$. In the former case $G(0, \alpha)$ is isometric to the hyperbolic 3-space $\mathbb{H}^3(-1)$ equipped with a left invariant Kenmotsu structure. Among the latter case $G(\pm 1, 0)$ is isometric to $\mathbb{H}^2(-4) \times \mathbb{R}$ equipped with a left invariant strictly H-almost Kenmotsu structure.*
- *The Ricci operator S is dominantly η-parallel when and only when $\lambda = 0$ or $\alpha = \lambda^2 - 1 = 0$. Thus S is dominantly η-parallel if and only if $G(\lambda, \alpha)$ is locally symmetric.*

Remark 4.2. Every $G(\lambda, \alpha)$ has η-parallel Riemannian curvature (see Example 7.2).

4.3. Type IV Lie groups

Let us consider a 3-dimensional non-unimodular Lie group $G = G[\alpha, \gamma]$ of type IV equipped with a left invariant almost Kenmotsu structure. The Lie algebra $\mathfrak{g} = \mathfrak{g}[\alpha, \gamma]$ is determined by the commutation relations:

$$[e_1, e_2] = \gamma e_1, \quad [e_2, e_3] = 2\alpha e_1, \quad [e_3, e_1] = -2e_1, \quad \alpha \in \mathbb{R}, \quad \gamma \neq 0.$$

Then the Levi-Civita connection is described as

$$\nabla_{e_1} e_1 = -\gamma e_2 - 2e_3, \quad \nabla_{e_1} e_2 = \gamma e_1 - \alpha e_3, \quad \nabla_{e_1} e_3 = 2e_1 + \alpha e_2,$$
$$\nabla_{e_2} e_1 = -\alpha e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = \alpha e_1,$$
$$\nabla_{e_3} e_1 = \alpha e_2, \quad \nabla_{e_3} e_2 = -\alpha e_1, \quad \nabla_{e_3} e_3 = 0.$$

The Lie group is strictly almost Kenmotsu. The unimodular kernel is spanned by

$$e_1, \quad \xi + \frac{2}{\gamma}(e_1 - e_2).$$

The operators h and h' are given by

$$he_1 = -\alpha e_1 + e_2, \quad he_2 = e_1 + \alpha e_2, \quad h' e_1 = e_1 + \alpha e_2, \quad h' e_2 = \alpha e_1 - e_2.$$

The eigenvalues of h are 0, λ and $-\lambda$ where

$$\lambda = \sqrt{\alpha^2 + 1}.$$

The covariant derivative $\nabla_\xi h$ is computed as

$$\nabla_\xi h = -2\alpha h\varphi.$$

Hence $\nabla_\xi h = 0$ holds when and only when $\alpha = 0$.

The Riemannian curvature R and the Ricci operator S are described as

$$R(e_1, e_2)e_1 = \left(\gamma^2 - \alpha^2\right) e_2 + 2\gamma e_3, \quad R(e_1, e_2)e_2 = \left(\alpha^2 - \gamma^2\right) e_1 + 2\alpha\gamma e_3,$$
$$R(e_1, e_2)e_3 = -2\gamma(e_1 + \alpha e_2), \quad R(e_1, e_3)e_1 = 2\gamma e_2 + \left(4 - \alpha^2\right) e_3,$$
$$R(e_1, e_3)e_2 = -2\gamma e_1 + 4\alpha e_3, \quad R(e_1, e_3)e_3 = \left(\alpha^2 - 4\right) e_1 - 4\alpha e_2,$$
$$R(e_2, e_3)e_1 = 2\alpha\gamma e_2 + 4\alpha e_3, \quad R(e_2, e_3)e_2 = -2\alpha\gamma e_1 + 3\alpha^2 e_3,$$
$$R(e_2, e_3)e_3 = -4\alpha e_1 - 3\alpha^2 e_2.$$

$$\rho_{11} = 2\alpha^2 - \gamma^2 - 4, \quad \rho_{12} = -4\alpha, \quad \rho_{13} = 2\alpha\gamma,$$
$$\rho_{22} = -2\alpha^2 - \gamma^2, \quad \rho_{23} = -2\gamma, \quad \rho_{33} = -2\alpha^2 - 4.$$

The principal Ricci curvatures are

$$-4 - 2\alpha^2 - \gamma^2, \quad -4 - 2\alpha^2 - \gamma^2, \quad 2\alpha^2.$$

Direct computation show that

$$\begin{aligned}
(\nabla_{e_1}S)e_1 &= -4\alpha^2 \left\{\frac{1}{2}\gamma e_2 + e_3\right\}, \\
(\nabla_{e_1}S)e_2 &= 2\alpha \left\{-\alpha\gamma e_1 + 2\gamma e_2 - (\frac{1}{2}\gamma^2 - 2)e_3\right\}, \\
(\nabla_{e_1}S)e_3 &= 2\alpha \left\{(-2\alpha)e_1 + (2 - \frac{1}{2}\gamma^2)e_2 - 2\gamma e_3\right\}, \\
(\nabla_{e_2}S)e_1 &= -2\alpha \left\{(-2\alpha)\gamma e_1 + \gamma e_2 + \frac{1}{2}(4\alpha^2 - \gamma^2)e_3\right\}, \\
(\nabla_{e_2}S)e_2 &= -2\alpha(\gamma e_2 - 2\alpha e_3), \\
(\nabla_{e_2}S)e_3 &= -2\alpha \left\{\frac{1}{2}(4\alpha^2 - \gamma^2)e_1 - 2\alpha e_2 + 2\alpha\gamma e_3)\right\}, \\
(\nabla_{e_3}S)e_1 &= -2\alpha \left\{-4\alpha e_1 - 2(\alpha^2 - 1)e_2 - \gamma e_3\right\}, \\
(\nabla_{e_3}S)e_2 &= 2\alpha \left\{2(\alpha^2 - 1)e_1 - 4\alpha e_2 + \alpha\gamma e_3\right\}, \\
(\nabla_{e_3}S)e_3 &= 2\alpha\gamma(e_1 + \alpha e_2).
\end{aligned}$$

One can see that $\pounds_\xi S = 0$ holds when and only when $\alpha = 0$. From these we obtain

$$\nabla S = 0 \Longleftrightarrow (\nabla_\xi S)\xi = 0 \Longleftrightarrow \pounds_\xi S = 0 \Longleftrightarrow \alpha = 0.$$

Hence $G[\alpha, \gamma]$ has η-parallel Ricci operator when and only when $\alpha = 0$. Moreover, the η-parallelism of S is equivalent to the local symmetry of $G[\alpha, \gamma]$. The strictly almost Kenmotsu Lie group $G[0, \gamma]$ is not H-almost Kenmotsu. Moreover $G[0, \gamma]$ is isometric to $\mathbb{H}^2(-4 - \gamma^2) \times \mathbb{R}$ and satisfying $\nabla_\xi h = 0$. The Ricci operator S is transversally Killing if and only if $\nabla S = 0$.

Theorem 4.3. *For a homogeneous almost Kenmotsu* 3*-manifold* M*, the dominantly* η*-parallelism of the Ricci operator is equivalent to the local symmetry of* M*.*

5. Generalized almost Kenmotsu (κ, μ, ν)-spaces

Here we give some examples of non-homogeneous almost Kenmotsu 3-manifolds with η-parallel Ricci operator.

Definition. An almost Kenmotsu 3-manifold M is said to be a *generalized almost Kenmotsu* (κ, μ, ν)*-space* if there exist three functions κ, μ and ν such that

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}$$

$$+\nu\{\eta(Y)\varphi hX-\eta(X)\varphi hY\}$$

for all vector fields X and Y on M. In particular, M is said to be an *almost Kenmotsu* (κ,μ,ν)*-space* if it is a generalized almost Kenmotsu (κ,μ,ν)-space all of κ, μ and ν are constant.

Definition. Let M be a generalized almost Kenmotsu (κ,μ,ν)-space. If all the functions κ, μ and ν are constants, then M is called an *almost Kenmotsu* (κ,μ,ν)*-space*. A generalized almost Kenmotsu (κ,μ,ν)-space is said to be *proper* if $|d\kappa|^2+|d\mu|^2+|d\nu|^2\neq 0$.

Theorem 5.1 ([17,23]). *Let M be an almost Kenmotsu 3-manifold. If M is a generalized almost Kenmotsu (κ,μ,ν)-space, then M is an H-almost Kenmotsu manifold. Conversely if M is an H-almost Kenmotsu manifold, then M satisfies the generalized (κ,μ,ν)-condition on an open dense subset. In such a case we have*

$$\kappa=-(\lambda^2+1),\quad \mu=-2\alpha,\quad \lambda\nu=2\lambda+\xi(\lambda).$$

The Ricci operator has the form

$$S=\left(\frac{r}{2}-\kappa\right)\mathrm{I}-\left(\frac{r}{2}-3\kappa\right)\eta\otimes\xi+\mu h+\nu\varphi h.$$

Moreover, we have

$$S\xi=2\kappa\xi,\quad \mathrm{tr}\,(h^2)=-2(\kappa+1).$$

Example 5.2 (The strictly almost Kenmotsu Lie group $G(\lambda,0)$). For any $\lambda\neq 0$, the strictly almost Kenmotsu Lie group $G(\lambda,0)$ is an almost Kenmotsu $(-1-\lambda^2,0,2)$-space. As we saw before $G(\pm 1,0)=\mathbb{H}^2(-4)\times\mathbb{R}$ are locally symmetric.

Example 5.3 ([25]). Let $M=\{(x,y,z)\in\mathbb{R}^3\,|\,z>0\}$ be the upper half space. We introduce an almost contact Riemannian structure on M by

$$\xi=\frac{\partial}{\partial z},\quad \eta=dz,\quad g=ze^{2z}dx^2+\frac{e^{2z}}{z}dy^2+dz^2,$$

$$\varphi\frac{\partial}{\partial x}=z\frac{\partial}{\partial y},\quad \varphi\frac{\partial}{\partial y}=-\frac{1}{z}\frac{\partial}{\partial x},\quad \varphi\frac{\partial}{\partial z}=0.$$

Then $M=(M,\varphi,\xi,\eta,g)$ is a strictly almost Kenmotsu 3-manifold. We can take a global orthonormal frame field

$$e_1=\frac{e^{-z}}{\sqrt{2}}\left(\frac{1}{\sqrt{z}}\frac{\partial}{\partial x}+\sqrt{z}\frac{\partial}{\partial y}\right),\quad e_2=-\frac{e^{-z}}{\sqrt{2}}\left(\frac{1}{\sqrt{z}}\frac{\partial}{\partial x}-\sqrt{z}\frac{\partial}{\partial y}\right),\quad e_3=\xi.$$

Then $\{e_1,e_2,e_3\}$ satisfies

$$[e_1,e_2]=0,\quad [e_2,e_3]=-\lambda\,e_1+e_2,\quad [e_3,e_1]=-e_1+\lambda\,e_2.$$

Thus $\{e_1,e_2,e_3\}$ is a global orthonormal frame field as in Lemma 3.1 satisfying $b=c=0$, $\alpha=0$ and $\lambda=1/(2z)$. Note that the coordinate vector fields ∂_x

and ∂_y are eigenvector fields of $h' = h\varphi$ corresponding to $\lambda = 1/(2z)$ and $-\lambda$, respectively. The sectional curvatures are given by

$$H = -(1-\lambda^2), \quad K_{13} = K_{23} = -(1+\lambda^2).$$

The Ricci tensor field and the scalar curvature are computed as

$$\rho = -\frac{e^{2z}(4z^2+2z-1)}{2z}\,dx^2 - \frac{e^{2z}(4z^2-2z+1)}{2z^3}\,dy^2 - \frac{4z^2+1}{2z^2}\,dz^2,$$

$$r = -\frac{12z^2+1}{2z^2} = -2(\lambda^2+3) = 2(\kappa-2).$$

The components of S relative to $\{e_1, e_2, e_3\}$ are given by

$$S = \begin{pmatrix} -2 & 2\lambda(1-\lambda) & 0 \\ 2\lambda(1-\lambda) & -2 & 0 \\ 0 & 0 & -2(1+\lambda^2) \end{pmatrix} = \begin{pmatrix} -2 & 1/z - 1/(2z^2) & 0 \\ 1/z-1/(2z^2) & -2 & 0 \\ 0 & 0 & -2(1+1/(2z)^2) \end{pmatrix}.$$

One can check that $(M, \varphi, \xi, \eta, g)$ is a generalized almost Kenmotsu $(\kappa, 0, \nu)$-space with

$$\kappa = -1 - \frac{1}{4z^2} < -1, \quad \nu = 2 - \frac{1}{z}.$$

In particular κ and ν satisfy $d\kappa \wedge \eta = d\nu \wedge \eta = 0$. Moreover $\nabla_\xi h$ has the form

$$\nabla_\xi h = (\nu - 2)h = -\frac{1}{z}h.$$

The generalized almost Kenmotsu $(\kappa, 0, \nu)$-space M has principal Ricci curvatures

$$\rho_1 = -2 + \frac{2z-1}{2z^2}, \quad \rho_2 = -2 - \frac{2z-1}{2z^2}, \quad \rho_3 = -2 - \frac{1}{2z^2}.$$

Hence the Ricci operator is η-parallel, but not strongly η-parallel.

6. The η-parallelism

In this section we investigate almost Kenmotsu 3-manifolds with η-parallel Ricci operator. Before investigating the η-parallelism of the Ricci operator, we study η-parallelism of the operator h.

6.1. The η-parallelism of the operator h

The normality of an almost Kenmotsu 3-manifold M is characterized by the vanishing of the operator h. This fact motivates us to characterize H-almost property in terms of the operator h. Here we give the characterization of H-almost Kenmotsu property in terms of h.

Proposition 6.1. *Let M be an almost Kenmotsu 3-manifold.*

(i) *If the operator h is η-parallel, then M is H-almost Kenmotsu and the scalar field α is η-parallel.*

(ii) *The operator h is strongly η-parallel if and only if M is Kenmotsu or locally isomorphic to one of the strictly H-almost Kenmotsu Lie group $G(\lambda, 0)$ with $\lambda \neq 0$.*

(iii) *The operator h is dominantly η-parallel if and only if it vanishes.*

Proof. Let M be an almost Kenmotsu 3-manifold. On an open set $\mathcal{U}_0$, h vanishes. Hereafter we work on the open set $\mathcal{U}_1$. Take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 3.1, then the covariant derivative ∇h is computed as

$$(\nabla_{e_1} h)e_1 = e_1(\lambda)e_1 - 2b\lambda e_2 - \lambda\xi, \quad (\nabla_{e_1} h)e_2 = -2b\lambda e_1 - e_1(\lambda)e_2 - \lambda^2\xi,$$
$$(\nabla_{e_1} h)e_3 = -\lambda e_1 - \lambda^2 e_2,$$
$$(\nabla_{e_2} h)e_1 = e_2(\lambda)e_1 + 2c\lambda e_2 + \lambda^2\xi, \quad (\nabla_{e_2} h)e_2 = 2c\lambda e_1 - e_2(\lambda)e_2 + \lambda\xi,$$
$$(\nabla_{e_2} h)e_3 = \lambda^2 e_1 + \lambda e_2,$$
$$(\nabla_{e_3} h)e_1 = \xi(\lambda)e_1 + 2\alpha\lambda e_2, \quad (\nabla_{e_3} h)e_2 = 2\alpha\lambda e_1 - \xi(\lambda)e_2,$$
$$(\nabla_{e_3} h)e_3 = 0.$$

Thus h is η-parallel if and only if

$$e_1(\lambda) = e_2(\lambda) = 0, \quad b = c = 0.$$

These conditions implies $\rho_{13} = \rho_{23} = 0$. Hence $\mathcal{U}_1$ satisfies H-almost Kenmotsu condition. Since H-almost Kenmotsu condition is satisfied on $\mathcal{U}_0$, we conclude that the whole M is H-almost Kenmotsu. Note that from the Jacobi identity, $e_1(\alpha) = e_2(\alpha) = 0$ holds on $\mathcal{U}_1$. This means that the scalar field α is η-parallel. Moreover, one can see that h is η-parallel then M is H-almost Kenmostu and the scalar field α is η-parallel.

Next, on $\mathcal{U}_1$, h is strongly η-parallel if and only if h is η-parallel and $\xi(\lambda) = \alpha = 0$. Thus if M has strongly η-parallel h, then λ is non-zero constant and the frame field $\{e_1, e_2, e_3\}$ satisfies the commutation relations

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -\lambda\, e_1 + e_2, \quad [e_3, e_1] = -e_1 + \lambda\, e_2.$$

This shows that $\mathcal{U}_1$ is locally isomorphic to the Lie group $G(\lambda, 0)$ of type II. Note that $G(0, 0)$ is isometric to $\mathbb{H}^3(-1)$ and has vanishing h. Conversely if M is Kenmotsu or locally isomorphic to $G(\lambda, 0)$ with $\lambda \neq 0$, then M has strongly η-parallel S.

Finally, h is dominantly η-parallel on $\mathcal{U}_1$ if and only if $\lambda = 0$. □

6.2. The system of local symmetry

Now we start our study on the η-parallelism of the Ricci operator. First of all we recall the table of covariant derivative ∇S over $\mathcal{U}_1$ obtained in [16, 17].

Take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 3.1, then direct computation shows the following results.

$$(\nabla_{e_1} S)e_1 = \{e_1(\rho_{11}) + 2b\rho_{12} + 2\rho_{13}\}e_1$$
$$+ \{e_1(\rho_{12}) - b(\rho_{11} - \rho_{22}) - \lambda\rho_{13} + \rho_{23}\}e_2$$

$$+\{e_1(\rho_{13})-(\rho_{11}-\rho_{33})+\lambda\rho_{12}+b\rho_{23}\}e_3,$$
$$(\nabla_{e_1}S)e_2=\{e_1(\rho_{12})-b(\rho_{11}-\rho_{22})-\lambda\rho_{13}+\rho_{23}\}e_1$$
$$+\{e_1(\rho_{22})-2b\rho_{12}-2\lambda\rho_{23}\}e_2$$
$$+\{e_1(\rho_{23})-\rho_{12}-b\rho_{13}+\lambda(\rho_{22}-\rho_{33})\}e_3,$$
$$(\nabla_{e_1}S)e_3=\{e_1(\rho_{13})-(\rho_{11}-\rho_{33})+\lambda\rho_{12}+b\rho_{23}\}e_1$$
$$+\{e_1(\rho_{23})-\rho_{12}-b\rho_{13}+\lambda(\rho_{22}-\rho_{33})\}e_2$$
$$+\{e_1(\rho_{33})-2\rho_{13}+2\lambda\rho_{23}\}e_3.$$

Hence $\nabla_{e_1}S=0$ if and only if

$$e_1(\rho_{11})+2b\rho_{12}+2\rho_{13}=0, \tag{4}$$
$$e_1(\rho_{12})-b(\rho_{11}-\rho_{22})-\lambda\rho_{13}+\rho_{23}=0, \tag{5}$$
$$e_1(\rho_{13})-(\rho_{11}-\rho_{33})+\lambda\rho_{12}+b\rho_{23}=0, \tag{6}$$
$$e_1(\rho_{22})-2b\rho_{12}-2\lambda\rho_{23}=0, \tag{7}$$
$$e_1(\rho_{23})-\rho_{12}-b\rho_{13}+\lambda(\rho_{22}-\rho_{33})=0, \tag{8}$$
$$e_1(\rho_{33})-2\rho_{13}+2\lambda\rho_{23}=0. \tag{9}$$

$$(\nabla_{e_2}S)e_1=\{e_2(\rho_{11})-2c\rho_{12}-2\lambda\rho_{13}\}e_1$$
$$+\{e_2(\rho_{12})+c(\rho_{11}-\rho_{22})+\rho_{13}-\lambda\rho_{23}\}e_2$$
$$+\{e_2(\rho_{13})+\lambda(\rho_{11}-\rho_{33})-\rho_{12}-c\rho_{23}\}e_3,$$
$$(\nabla_{e_2}S)e_2=\{e_2(\rho_{12})+c(\rho_{11}-\rho_{22})+\rho_{13}-\lambda\rho_{23}\}e_1$$
$$+\{e_2(\rho_{22})+2c\rho_{12}+2\rho_{23}\}e_2$$
$$+\{e_2(\rho_{23})+\lambda\rho_{12}+c\rho_{13}-(\rho_{22}-\rho_{33})\}e_3,$$
$$(\nabla_{e_2}S)e_3=\{e_2(\rho_{13})+\lambda(\rho_{11}-\rho_{33})-\rho_{12}-c\rho_{23}\}e_1$$
$$+\{e_2(\rho_{23})+\lambda\rho_{12}+c\rho_{13}-(\rho_{22}-\rho_{33})\}e_2$$
$$+\{e_2(\rho_{33})+2\lambda\rho_{13}-2\rho_{23}\}e_3.$$

Hence $\nabla_{e_2}S=0$ if and only if

$$e_2(\rho_{11})-2c\rho_{12}-2\lambda\rho_{13}=0, \tag{10}$$
$$e_2(\rho_{12})+c(\rho_{11}-\rho_{22})+\rho_{13}-\lambda\rho_{23}=0, \tag{11}$$
$$e_2(\rho_{13})+\lambda(\rho_{11}-\rho_{33})-\rho_{12}-c\rho_{23}=0, \tag{12}$$
$$e_2(\rho_{22})+2c\rho_{12}+2\rho_{23}=0, \tag{13}$$
$$e_2(\rho_{23})+\lambda\rho_{12}+c\rho_{13}-(\rho_{22}-\rho_{33})=0, \tag{14}$$
$$e_2(\rho_{33})+2\lambda\rho_{13}-2\rho_{23}=0. \tag{15}$$

$$(\nabla_{e_3}S)e_1=\{e_3(\rho_{11})-2\alpha\rho_{12}\}e_1+\{e_3(\rho_{12})+\alpha(\rho_{11}-\rho_{22})\}e_2$$
$$+\{e_3(\rho_{13})-\alpha\rho_{23}\}e_3,$$
$$(\nabla_{e_3}S)e_2=\{e_3(\rho_{12})+\alpha(\rho_{11}-\rho_{22})\}e_1+\{e_3(\rho_{22})+2\alpha\rho_{12})\}e_2$$

$$+ \{e_3(\rho_{23}) + \alpha\rho_{13}\}e_3,$$

$$(\nabla_{e_3} S)e_3 = \{e_3(\rho_{13}) - \alpha\rho_{23}\}e_1 + \{e_3(\rho_{23}) + \alpha\rho_{13}\}e_2 + e_3(\rho_{33})e_3.$$

Thus $\nabla_\xi S = 0$ if and only if

$$e_3(\rho_{11}) - 2\alpha\rho_{12} = 0, \tag{16}$$

$$e_3(\rho_{12}) + \alpha(\rho_{11} - \rho_{22}) = 0, \tag{17}$$

$$e_3(\rho_{13}) - \alpha\rho_{23} = 0, \tag{18}$$

$$e_3(\rho_{22}) + 2\alpha\rho_{12} = 0, \tag{19}$$

$$e_3(\rho_{23}) + \alpha\rho_{13} = 0, \tag{20}$$

$$e_3(\rho_{33}) = 0. \tag{21}$$

6.3. Locally symmetric spaces

Now let us assume that M is a locally symmetric almost Kenmotsu 3-manifold. Then the scalar curvature r is constant on M. If $M = \mathcal{U}_0$, then M is a Kenmotsu manifold of constant curvature -1. Hence M is locally isometric to $\mathbb{H}^3(-1)$. Moreover M is locally isomorphic to $G(0, \alpha)$ for some α as a Kenmotsu 3-manifold.

Hereafter we assume that $\mathcal{U}_1$ is non-empty. On $\mathcal{U}$ we take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 3.1.

From $g((\nabla_{e_3} S)e_3, e_3) = 0$, that is, (21), we get $\xi(\lambda) = 0$. Thus we have $\rho_{12} = 2\lambda$. Thus we obtain

$$g((\nabla_{e_3} S)e_1, e_2) = \alpha(\rho_{11} - \rho_{22}) = 0,$$

and since $\rho_{11} - \rho_{22} = -4\alpha\lambda$, we get $\alpha^2\lambda = 0$. Thus $\alpha = 0$ on $\mathcal{U}_1$. This implies that $\nabla_\xi h = 0$ on $\mathcal{U}_1$. Moreover

$$\rho_{11} = \rho_{22} = \frac{r}{2} + 1 + \lambda^2$$

holds on $\mathcal{U}_1$.

Proposition 6.2. *If an almost Kenmotsu* 3*-manifold M is locally symmetric, then M satisfies $\nabla_\xi h = 0$ holds on M.*

The converse statement of this proposition does not hold. In fact, the Lie group $G(\lambda, 0)$ with $\lambda^2 \neq 1$ in Type ll Lie groups satisfies $\nabla_\xi h = 0$ but it is not locally symmetric.

Locally symmetric almost Kenmotsu 3-manifolds are classified as follows:

Theorem 6.3 ([16, 29])**.** *Let M be an almost Kenmotsu* 3*-manifold. Then M is locally symmetric if and only if M is one of the following spaces*:

- (i) *If M is H-almost Kenmotsu, then M is a Kenmotsu manifold of constant curvature -1 or locally isomorphic to $\mathbb{H}^2(-4) \times \mathbb{R}$ or*
- (ii) *If M is non H-almost Kenmotsu, then M is locally isomorphic to $\mathbb{H}^2(-4 - \gamma^2) \times \mathbb{R}$ for some $\gamma \neq 0$.*

Corollary 6.4. *Every complete locally symmetric almost Kenmotsu 3-manifold is realized as a Lie group equipped with a left invariant almost Kenmotsu structure.*

From Theorem 6.3, the following corollary is deduced.

Corollary 6.5. *Let M be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0$. Then M has harmonic curvature, that is,*

$$(\nabla_X S)Y = (\nabla_Y S)X$$

for all vector fields on X and Y if and only if M is one of the following spaces:

(i) *If M is H-almost Kenmotsu, then M is a Kenmotsu manifold of constant curvature -1 or locally isomorphic to $\mathbb{H}^2(-4) \times \mathbb{R}$ or*
(ii) *If M is non H-almost Kenmotsu, then M is locally isomorphic to $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ for some $\gamma \neq 0$.*

6.4. Dominantly η-parallel Ricci operator

In this subsection, we study almost Kenmotsu 3-manifolds with dominantly or strongly η-parallel Ricci operator. The first named author obtained the following classification.

Theorem 6.6 ([15])**.** *A Kenmotsu 3-manifold M has η-parallel Ricci operator if and only if it is locally isomorphic to the warped product $I \times_{ce^t} \overline{M}$, where I is an open interval with coordinate t, $\overline{M}$ is a Riemannian 2-manifold of constant curvature and c is a positive constant.*

From Theorem 6.6, we know that

$$\mathbb{R} \times_{ce^t} \mathbb{S}^2(k^2), \quad \mathbb{R} \times_{ce^t} \mathbb{H}^2(-k^2), \quad \mathbb{R} \times_{ce^t} \mathbb{R}^2 = \mathbb{H}^3(-1)$$

are Kenmotsu 3-manifolds with η-parallel Ricci operator. The scalar curvature of these warped products are

$$-6 + \frac{2k^2}{c^2}e^{-2t}, \quad -6 - \frac{2k^2}{c^2}e^{-2t}, \quad -6,$$

respectively. It should be remarked that the scalar curvature of a Kenmotsu 3-manifold with η-parallel Ricci operator is η-parallel. Combining this fact with Proposition 2.2, we obtain the following:

Corollary 6.7. *If a Kenmotsu 3-manifold M has η-parallel Ricci operator, then its Riemannian curvature is η-parallel.*

Kenmotsu 3-manifolds with η-parallel Riemannian curvature will be discussed again in the final section.

On the other hand, strong η-parallelism of the Ricci operator implies the local symmetry.

Theorem 6.8 ([15])**.** *A Kenmotsu 3-manifold M has strongly η-parallel Ricci operator if and only if it is of constant curvature -1.*

Now let us investigate almost Kenmotsu 3-manifolds with dominantly η-parallel Ricci operator. First we study those 3-manifolds under certain additional condition on the operator h:

Proposition 6.9. *An almost Kenmotsu 3-manifold M has dominantly η-parallel Ricci operator with $\xi(\mathrm{tr}\, h^2) = 0$ if and only if it is locally symmetric.*

Proof. Let us work on the open set $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$. Then from Theorem 6.8, the open set $\mathcal{U}_0$ is locally isomorphic to $\mathbb{H}^3(-1)$ and hence $\mathcal{U}_0$ is locally symmetric.

Now let us investigate the open set $\mathcal{U}_1$ and take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 3.1. Then one can see that

- M has η-parallel Ricci operator if and only if M satisfies the system of equations (4), (5), (7), (10), (11) and (13).
- M has strongly η-parallel Ricci operator if and only if S is η-parallel and in addition satisfies (16), (17) and (19).
- M has dominantly η-parallel Ricci operator if and only if S is strongly η-parallel and in addition satisfies (6), (8), (12), (14), (18) and (20).

From the additional assumption $\xi(\mathrm{tr}\, h^2) = 0$, we have $\rho_{12} = 2\lambda$ and hence $\xi(\rho_{12}) = 0$. The operator h satisfies $\nabla_\xi h = -2\alpha h\varphi$. The strong η-parallelism implies $g((\nabla_\xi S)e_1, e_2) = 0$. From this we have $\alpha^2\lambda = 0$. Since λ does not vanish on $\mathcal{U}_1$, $\alpha = 0$ holds on $\mathcal{U}_1$. This implies that $\nabla_\xi h = 0$. Moreover, from $g((\nabla_\xi S)e_2, e_2) = 0$, we have $\xi(r) = 0$ on $\mathcal{U}_1$.

Here we recall the divergence formula for S:

$$\mathrm{div}\, S = \frac{1}{2}\,\mathrm{grad}\, r. \tag{22}$$

The left hand side is computed as

$$\mathrm{div}\, S = \mathrm{tr}(\nabla S) = (\nabla_{e_1} S)e_1 + (\nabla_{e_2} S)e_2 + (\nabla_{e_3} S)e_3.$$

If S is strongly η-parallel, then $(\nabla_{e_1} S)e_1$, $(\nabla_{e_2} S)e_2$ have only e_3-components. Since we assumed that S is dominantly η-parallel, we have

$$g((\nabla_{e_1} S)e_1, e_3) = g((\nabla_{e_1} S)e_3, e_1) = 0,\ \ g((\nabla_{e_2} S)e_2, e_3) = g((\nabla_{e_2} S)e_3, e_2) = 0.$$

Hence $\mathrm{div}\, S$ is parallel to ξ and has the expression $\mathrm{div}\, S = (\nabla_\xi S)\xi = \eta((\nabla_\xi S)\xi)\xi$.

On the other hand, the right hand side of the divergence formula is

$$\frac{1}{2}\mathrm{grad}\, r = \frac{1}{2}\left(e_1(r)e_1 + e_2(r)e_2 + e_3(r)e_3\right).$$

Hence we have $e_1(r) = e_2(r) = 0$ and the scalar curvature r is a constant on $\mathcal{U}_1$.

From the assumption $\xi(\mathrm{tr}(h^2)) = 0$, the equation (21) holds on $\mathcal{U}_1$. This together with the dominantly η-parallelism of S, we have $\nabla_\xi S = 0$ on $\mathcal{U}_1$. On the open set $\mathcal{U}_1$

$$\alpha = 0, \quad dr = 0, \quad \xi(\lambda) = 0$$

hold. In addition, from η-parallelism, (4) and (7) hold. From these two equations one can verify that (9) holds on $\mathcal{U}_1$. Analogously, from (10) and (13), one can deduce (15). Henceforth we proved that S is parallel on $\mathcal{U}_1$. Thus M is locally symmetric.

Conversely, when M is locally symmetric, it has always dominantly η-parallel Ricci operator and satisfies $\nabla_\xi h = 0$. □

Thus we have the following partial classification.

Corollary 6.10. *For an almost Kenmotsu 3-manifold M satisfying $\nabla_\xi h = 0$, the dominantly η-parallelism of the Ricci operator S is equivalent to the local symmetry of M.*

Thus an almost Kenmotsu 3-manifold M with dominantly η-parallel Ricci operator and satisfying $\nabla_\xi h = 0$ is locally isomorphic to one of the following spaces:

(i) *the hyperbolic 3-space $\mathbb{H}^3(-1)$ equipped with a homogeneous Kenmotsu structure,*

(ii) *the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$ equipped with a homogeneous strictly Kenmotsu structure, or*

(iii) *the Riemannian product $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ equipped with a homogeneous strictly Kenmotsu structure for some $\gamma \neq 0$.*

Remark 6.11. The system of transversal Killing Ricci operator on the open set $\mathcal{U}_1$ of an almost Kenmotsu 3-manifold M is the system (4)–(7), (10), (11), (13), (14), and $g((\nabla_{e_1} S)e_2, e_3) + g((\nabla_{e_2} S)e_1, e_3) = 0$. Thus clearly, transversal Killing property of S is strictly stronger than the η-parallelism of S. On the other hand, the strong η-parallelism and the transversal Killing property have *no inclusion relations*. The dominant η-parallelism of S is stronger than the transversal Killing property of S.

6.5. Almost Kenmotsu 3-manifolds with η-parallel Ricci operator

In this subsection, we classify almost Kenmotsu 3-manifolds with η-parallel Ricci operator. However, unfortunately, to classify all almost Kenmotsu 3-manifolds with η-parallel Ricci operator is a true challenging issue. We demand some adequate conditions. For this direction, Proposition 6.2 motivates us to study almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0$ which has η-parallel Ricci operator.

Theorem 6.12. *Let M be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0$. Assume that both the Ricci operator and the scalar curvature r are η-parallel. Then it is locally isomorphic to one of the following spaces*:

(i) *the warped products $\mathbb{R} \times_{ce^t} \mathbb{S}^2(k^2)$, $\mathbb{R} \times_{ce^t} \mathbb{H}^2(-k^2)$ or the hyperbolic 3-space $\mathbb{H}^3(-1)$.*

(ii) *the type II Lie group $G(\lambda, 0)$ for some constant $\lambda \neq \pm 1$, 0.*

(iii) *a strictly almost Kenmotsu 3-manifold of constant scalar curvature satisfying $\alpha = 0$, $\mathrm{tr}(h^2) = 2$ whose Ricci operator is strongly η-parallel and satisfies $\nabla_\xi S = \pounds_\xi S = 0$.*

The third class includes the type II Lie groups $G(\pm 1, 0)$ *and type IV Lie group* $G[0, \gamma] = \mathbb{H}^2(-4 - \gamma^2) \times \mathbb{R}$ *for some constant* $\gamma \neq 0$.

Proof. Let M be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0$ and has η-parallel Ricci operator S. Let us work on the open set $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$. Then from Theorem 6.6, the open set $\mathcal{U}_0$ is locally isomorphic to the warped product $I \times_{ce^t} \overline{M}$, where I is an open interval with coordinate t, $\overline{M}$ is a Riemannian 2-manifold of constant curvature and c is a positive constant. Note that the scalar curvature of the warped product is η-parallel.

Now let us investigate the open set $\mathcal{U}_1$ and take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 3.1. From the additional assumption $\nabla_\xi h = 0$ on $\mathcal{U}_1$, we have that $\xi(\lambda) = 0$ and $\alpha = 0$.

Then from η-parallel condition (4), (5), (7), (10), (11) and (13), we deduce

$$\lambda e_1(\lambda) + e_2(\lambda) = 2\lambda(b - c\lambda), \quad e_1(\lambda) + \lambda e_2(\lambda) = 2\lambda(c - b\lambda), \tag{23}$$

and

$$\frac{1}{4} e_1(r) = -\lambda e_1(\lambda) + e_2(\lambda), \quad \frac{1}{4} e_2(r) = e_1(\lambda) - \lambda e_2(\lambda). \tag{24}$$

Now, we assume that the scalar curvature r is η-parallel, then

$$e_1(r) = e_2(r) = 0.$$

From (24) and the above equations, we have

$$e_1(\lambda) = \lambda\, e_2(\lambda), \quad e_2(\lambda) = \lambda\, e_1(\lambda). \tag{25}$$

From (25) we deduce that

$$(1 - \lambda^2) e_1(\lambda) = 0, \quad (1 - \lambda^2) e_2(\lambda) = 0. \tag{26}$$

On the other hand, from (23) and (25), on $\mathcal{U}_1$, we obtain

$$e_1(\lambda) = b - c\lambda, \quad e_2(\lambda) = c - b\lambda. \tag{27}$$

By applying (27) to (25), we have

$$b(1 + \lambda^2) = 2c\lambda, \quad c(1 + \lambda^2) = 2b\lambda.$$

Hence we get

$$b^2(1 + \lambda^2)^2 + c^2(1 + \lambda^2)^2 = 4\lambda^2(b^2 + c^2).$$

This is rewritten as

$$(b^2 + c^2)(1 - \lambda^2)^2 = 0. \tag{28}$$

The equations (26) and (28) suggest us to introduce open sets

$$\mathcal{U}_8 = \{p \in \mathcal{U}_1 \mid \lambda^2 - 1 = 0 \text{ in a neighborhood of } p\},$$
$$\mathcal{U}_9 = \{p \in \mathcal{U}_1 \mid \lambda^2 - 1 \neq 0 \text{ in a neighborhood of } p\}$$

in $\mathcal{U}_1$. One can see that $\mathcal{U}_8 \cup \mathcal{U}_9$ is open and dense in $\mathcal{U}_1$.

On the open set $\mathcal{U}_8$, we have $\lambda = 1$ or $\lambda = -1$. It suffices to investigate the case $\lambda = 1$. The case $\lambda = -1$ is investigated in much the same way to the case $\lambda = 1$. From the Jacobi identity,

$$\xi(b) - c + b = 0, \quad -\xi(c) + b - c = 0. \tag{29}$$

From (27) and (29), we have

$$\xi(b) = \xi(c) = 0, \quad c = b \neq 0. \tag{30}$$

Hence the commutation relations of $\{e_1, e_2, e_3\}$ are

$$[e_1, e_2] = c(e_1 - e_2), \quad [e_2, e_3] = -e_1 + e_2, \quad [e_3, e_1] = -e_1 + e_2.$$

The scalar curvature is given by

$$r = -2\{e_1(c) + e_2(b) + b^2 + c^2 + 4\} = -2\{e_1(b) + e_2(b) + 2b^2 + 4\}.$$

By using the commutation relations we get

$$\begin{aligned}\xi(r) &= -2\{e_3e_1(b) + e_3e_2(b) + 4be_3(b)\}\\ &= -2\{e_1e_3(b) - e_1(b) + e_2(b) + e_2e_3(b) + e_1(b) - e_2(b) + 4be_3(b)\}\\ &= 0,\end{aligned}$$

thus the scalar curvature r is a constant.

The components of S are given by

$$\begin{aligned}&\rho_{11} = \rho_{22} = \frac{r}{2} + 2 = -e_1(b) - e_2(b) - 2b^2 - 2,\\ &\rho_{12} = 2, \quad \rho_{13} = \rho_{23} = -2b, \quad \rho_{33} = -4.\end{aligned}$$

Moreover one can check that the Ricci operator S satisfies (9), (15), (16)-(21). Thus $\nabla_\xi S = 0$. From the table of ∇S, except (6), (8), (12) and (14), other equations of the system of local symmetry are satisfied. Direct computation shows that

$$\begin{aligned}g((\nabla_{e_1}S)e_1, e_3) &= g((\nabla_{e_1}S)e_3, e_1) = -e_1(b) + e_2(b),\\ g((\nabla_{e_1}S)e_2, e_3) &= g((\nabla_{e_1}S)e_3, e_2) = -3e_1(b) - e_2(b),\\ g((\nabla_{e_2}S)e_1, e_3) &= g((\nabla_{e_2}S)e_3, e_1) = -e_1(b) - 3e_2(b),\\ g((\nabla_{e_2}S)e_2, e_3) &= g((\nabla_{e_2}S)e_3, e_2) = e_1(b) - e_2(b).\end{aligned}$$

Thus if both $\sigma(e_1) = \rho_{13}$ and $\sigma(e_2) = \rho_{23}$ are constant on $\mathcal{U}_8$, then $\mathcal{U}_8$ is locally symmetric. In particular, if $b = 0$ on $\mathcal{U}_8$, then $\mathcal{U}_8$ is H-almost Kenmotsu. One can see that $\mathcal{U}_8$ is locally isomorphic to $\mathbb{H}^2(-4) \times \mathbb{R} = G(1, 0)$. Note that if S is transversally Killing, then $g((\nabla_{e_1}S)e_1, e_3) = g((\nabla_{e_2}S)e_2, e_3) = 0$ and $g((\nabla_{e_1}S)e_2, e_3) + g((\nabla_{e_2}S)e_1, e_3) = 0$. Hence, on $\mathcal{U}_8$, b is a constant and $\mathcal{U}_8$ is locally symmetric.

Let us compute the Lie derivative $\pounds_\xi S$. For any vector field X, we have

$$(\pounds_\xi S)X = [\xi, SX] - S[\xi, X] = (\nabla_\xi S)X - \nabla_{SX}\xi + S(\nabla_X \xi) = -\nabla_{SX}\xi + S(\nabla_X \xi)$$

on $\mathcal{U}_8$. By using this we get

$$(\pounds_\xi S)e_1 = (\pounds_\xi S)e_2 = (\pounds_\xi S)e_3 = 0.$$

Now we rotate the orthonormal frame field and get a new one $\{E_1, E_2, E_3\}$ as

$$E_1 = \frac{1}{\sqrt{2}}(e_1 - e_2), \quad E_2 = \frac{1}{\sqrt{2}}(e_1 + e_2), \quad E_3 = e_3.$$

Then we get

$$[E_1, E_2] = \sqrt{2}bE_1, \quad [E_2, E_3] = 0, \quad [E_3, E_1] = -2E_1. \tag{31}$$

In case b is a non-zero constant on $\mathcal{U}_8$, (31) implies that $\mathcal{U}_8$ is locally isomorphic to the Lie group $G[0,\gamma] = \mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ of type IV with $\gamma = \sqrt{2}b \neq 0$.

Next, on the open set $\mathcal{U}_9$, from (26), $e_1(\lambda) = e_2(\lambda) = 0$. Hence λ is a constant. Moreover form (28), we have $b = c = 0$. Thus λ is a nonzero-constant on $\mathcal{U}_9$ such that $\lambda^2 \neq 1$. Hence commutation relations

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -\lambda\, e_1 + e_2, \quad [e_3, e_1] = -e_1 + \lambda\, e_2$$

hold. Thus $\mathcal{U}_9$ is locally isomorphic to the type II Lie group $G(\lambda, 0)$ for some constant $\lambda \neq \pm 1$, 0. □

Note that as pointed out by Perrone [29, Remark 4.2], the classification of almost Kenmotsu 3-manifolds satisfying $\pounds_\xi S = 0$ is still an open question.

For an almost Kenmotsu 3-manifold M, M has strongly η-parallel Ricci operator if and only if S is η-parallel and in addition satisfies (16), (17) and (19). In particular, for an almost Kenmotsu 3-manifold M satisfying $\nabla_\xi h = 0$, (16), (17) and (19) are equivalent to $\xi(r) = 0$. Hence we have:

Corollary 6.13. *Let M be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0$. Assume that the scalar curvature r is a constant. Then M has η-parallel Ricci operator if and only if it has strongly η-parallel Ricci operator.*

The Ricci operator S is transversally Killing if and only if it is η-parallel and, in addition, satisfies (6), (14), and $g((\nabla_{e_1} S)e_2 + (\nabla_{e_2} S)e_1, e_3) = 0$, *i.e.*,

$$e_1(\rho_{23}) + e_2(\rho_{13}) + \lambda(\rho_{11} + \rho_{22} - 2\rho_{33}) - 2\rho_{12} - b\rho_{13} - c\rho_{23} = 0.$$

Theorem 6.12 implies the following corollary.

Corollary 6.14. *Let M be a strictly almost Kenmotsu 3-manifold with η-parallel scalar curvature which satisfies $\nabla_\xi h = 0$. Then the Ricci operator S is transversally Killing if and only if M is locally isomorphic to one of the following spaces*:

(i) *the type II Lie group $G(\lambda, 0)$ for some constant $\lambda \neq 0$.*
(ii) *the type IV Lie group $G[0,\gamma] = \mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ for some constant $\gamma \neq 0$.*

6.6. H-almost Kenmotsu 3-manifolds with η-parallel Ricci operator

Another adequate assumption is the H-almost Kenmotsu property. Note that not all of locally symmetric almost Kenmotsu 3-manifolds are H-almost Kenmotsu. Indeed, the product space $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$ is locally symmetric, but not H-almost Kenmotsu.

In this subsection, we demand both "H-almost Kenmotsu" and "$\nabla_\xi h = -2\alpha h\varphi$" for some constant α (including the case $\alpha = 0$, *i.e.*, $\nabla_\xi h = 0$) for almost Kenmotsu 3-manifolds.

Theorem 6.15. *Let M be an H-almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α. Then M is a Kenmotsu 3-manifold or it is locally isomorphic to one of the type II Lie group $G(\lambda,\alpha)$ for some λ and α.*

Proof. Let us work on the open set $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$. On $\mathcal{U}_0$, we have $h = 0$ and hence both H-almost Kenmotsu condition and $\nabla_\xi h = 0$ are trivially satisfied.

Now let us investigate the open set $\mathcal{U}_1$ and take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 3.1. From the additional assumption $\nabla_\xi h = -2\alpha h\varphi$ for some constant α, $\xi(\lambda) = 0$ holds on $\mathcal{U}_1$. Thus we have

$$\rho_{11} = \frac{r}{2} + \lambda^2 - 2\alpha\lambda + 1, \quad \rho_{22} = \frac{r}{2} + \lambda^2 + 2\alpha\lambda + 1,$$
$$\rho_{12} = 2\lambda, \quad \rho_{13} = \rho_{23} = 0, \quad \rho_{33} = -2(1+\lambda^2)$$

on $\mathcal{U}_1$. Since M is H-almost Kenmotsu, the condition $\rho_{13} = \rho_{23} = 0$ implies

$$e_1(\lambda) = -2\lambda c, \quad e_2(\lambda) = -2\lambda b. \tag{32}$$

Then from the divergence formula (22), we have

$$g(\operatorname{div} S, e_1) = \frac{1}{2} e_1(r).$$

Let us compute the left hand side:

$$\begin{aligned} g(\operatorname{div} S, e_1) &= g((\nabla_{e_1} S)e_1, e_1) + g((\nabla_{e_2} S)e_2, e_1) + g((\nabla_{e_3} S)e_3, e_1) \\ &= (e_1(\rho_{11}) + 2b\rho_{12} + 2\rho_{13}) + (e_2(\rho_{12}) + c(\rho_{11} - \rho_{22}) + \rho_{13} - \lambda\rho_{23}) \\ &\quad + (e_3(\rho_{13}) - \alpha\rho_{23}) \\ &= \frac{1}{2} e_1(r) + 2\lambda e_1(\lambda) - 2\alpha e_1(\lambda) + 4b\lambda + 2e_2(\lambda) - 4\alpha\lambda c \\ &= \frac{1}{2} e_1(r) - 4c\lambda^2. \end{aligned}$$

Analogously, we consider

$$g(\operatorname{div} S, e_2) = \frac{1}{2} e_2(r).$$

Let us compute the left hand side:

$$\begin{aligned} g(\operatorname{div} S, e_2) &= g((\nabla_{e_1} S)e_1, e_2) + g((\nabla_{e_2} S)e_2, e_2) + g((\nabla_{e_3} S)e_3, e_2) \\ &= (e_1(\rho_{12}) - b(\rho_{11} - \rho_{22}) - \lambda\rho_{13} + \rho_{23}) + (e_2(\rho_{22}) + 2c\rho_{12} + 2\rho_{23}) \end{aligned}$$

$$+ (e_3(\rho_{23}) + \alpha\rho_{13})$$
$$= \frac{1}{2}e_2(r) + 2e_1(\lambda) + 2\lambda e_2(\lambda) + 4c\lambda + 2\alpha e_2(\lambda) + 4\alpha\lambda b$$
$$= \frac{1}{2}e_2(r) - 4b\lambda^2.$$

Hence we have $b = c = 0$ on $\mathcal{U}_1$. Moreover, since M is an H-almost Kenmotsu 3-manifold, from (32) we have $e_1(\lambda) = e_2(\lambda) = 0$. Since we know that $\xi(\lambda) = 0$, hence λ is a constant on $\mathcal{U}_1$. Hence $\{e_1, e_2, e_3\}$ satisfies the commutation relations

$$[e_1, e_2] = 0, \quad [e_2, e_3] = (\alpha - \lambda)\, e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\alpha + \lambda)\, e_2.$$

Thus $\mathcal{U}_1$ is locally isomorphic to one of the type II Lie group $G(\lambda, \alpha)$ for some λ. The family $\{G(\lambda, \alpha)\}$ of Lie groups includes locally symmetric spaces $G(\pm 1, 0) = \mathbb{H}^2(-4) \times \mathbb{R}$. □

From Proposition 6.1 and Theorem 6.15, we obtain:

Corollary 6.16. *Let M be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α. If the operator h is η-parallel, then M is a Kenmotsu 3-manifold or it is locally isomorphic to the type II Lie group $G(\lambda, \alpha)$ for some λ and α.*

Proof. It suffices to work on the open set $\mathcal{U}_1$. From Proposition 6.1, the η-parallelism of h implies

$$e_1(\lambda) = e_2(\lambda) = 0, \quad b = c = 0$$

and $\mathcal{U}_1$ is H-almost Kenmotsu. Thus the result follows from Theorem 6.15. □

Now let us consider almost Kenmotsu 3-manifolds with *η-parallel Ricci operator* satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α. From Theorem 6.6 and Theorem 6.15 we have the following partial classification:

Theorem 6.17. *Let M be an H-almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some α. Then M has η-parallel Ricci operator if and only if it is locally isomorphic to one of the following spaces:*

(i) *the warped product $I \times_{ce^t} \overline{M}$. Here I is an open interval with coordinate t, $\overline{M}$ is a Riemannian 2-manifold of constant curvature, c is a positive constant, or*

(ii) *the type II Lie group $G(\lambda, \alpha)$ for some λ and α.*

From Proposition 4.1 and Theorem 6.15, we have:

Corollary 6.18. *Let M be a strictly H-almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α. Then*

(i) *the Ricci operator S is strongly η-parallel when and only when $\lambda = 0$ or $\alpha = 0$. In the former case $G(0, \alpha)$ is isometric to the hyperbolic 3-space $\mathbb{H}^3(-1)$ equipped with a left invariant Kenmotsu structure. Among the* *latter case $G(\pm 1, 0)$ is isometric to $\mathbb{H}^2(-4) \times \mathbb{R}$ equipped with a left invariant strictly H-almost Kenmotsu structure.*

(ii) *the Ricci operator S is dominantly η-parallel if and only if $G(\lambda, \alpha)$ is locally symmetric.*

It should be remarked that the condition $\nabla_\xi h = -2\alpha h\varphi$ can not be removed for this classification stated in Theorem 6.17. In fact the example exhibited in Example 5.3 due to Pastore and Saltarelli is a strictly H-almost Kenmotsu 3-manifold with η-parallel Ricci operator but does not satisfy $\nabla_\xi h = -2\alpha h\varphi$. Next, in [32, Theorem 3.1], Wang claimed that a strictly almost Kenmotsu 3-manifold M satisfying $\nabla_\xi h = 0$ has η-parallel Ricci operator if and only if it is locally isomorphic to $\mathbb{H}^2(-4) \times \mathbb{R}$ or a certain non-unimodular Lie group equipped with a left invariant strictly almost Kenmotsu structure. Namely, in [32, Theorem 3.1], Wang does not assume that M is H-almost Kenmotsu. However [32, Theorem 3.1] is correct under the additional assumption "H-almost Kenmotsu" (see [10, Theorem C] and [29, Remark 4.1]). Our Theorem 6.17 corrects as well as improves [32, Theorem 3.1]. Indeed we give a detailed expression of the Lie group stated in [32, Theorem 3.1].

7. The η-parallelism of the Riemannian curvature

In this section we discuss the η-parallelism of the Riemannian curvature R on almost Kenmotsu 3-manifolds. First we prove the following:

Proposition 7.1. *An almost Kenmotsu 3-manifold M has η-parallel Riemannian curvature if and only if*

$$dH(X) + 2\rho(X, \xi) + 2\rho(h\varphi X, \xi) = 0 \tag{33}$$

for all vector field X orthogonal to ξ.

Moreover M has strongly η-parallel Riemannian curvature if R is η-parallel and $dH(\xi) = 0$ holds.

In particular if M is H-almost Kenmotsu, then R is η-parallel if and only if the holomorphic sectional curvature H is η-parallel. In addition, an H-almost Kenmotsu 3-manifold M has strongly η-parallel Riemannian curvature if and only if H is constant.

Proof. With respect to the local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 3.1, the covariant derivative ∇R is described as

$$\begin{aligned}
(\nabla_{e_1}R)(e_1, e_2)e_1 = &-\{e_1(H) + 2\sigma(e_1) - 2\lambda\sigma(e_2)\}e_2 \\
&-\{e_1(\sigma(e_2)) + \lambda(H - K_{13}) - (\xi(\lambda) + 2\lambda) - b\sigma(e_1)\}\xi, \\
(\nabla_{e_1}R)(e_1, e_2)e_2 = &\{e_1(H) + 2\sigma(e_1) - 2\lambda\sigma(e_2)\}e_1 \\
&+\{e_1(\sigma(e_1)) - (H - K_{23}) + \lambda(\xi(\lambda) + 2\lambda) + b\sigma(e_2)\}\xi, \\
(\nabla_{e_2}R)(e_1, e_2)e_1 = &-\{e_2(H) + 2\sigma(e_2) - 2\lambda\sigma(e_1)\}e_2 \\
&-\{e_2(\sigma(e_2)) - (H - K_{13}) + \lambda(\xi(\lambda) + 2\lambda) + c\sigma(e_1)\}\xi,
\end{aligned}$$

$$\begin{aligned}(\nabla_{e_2}R)(e_1,e_2)e_2 = &\{e_2(H)+2\sigma(e_2)-2\lambda\sigma(e_1)\}e_1\\ &+\{e_2(\sigma(e_1))+\lambda(H-K_{23})-(\xi(\lambda)+2\lambda)-c\sigma(e_2)\}\xi.\end{aligned}$$

From this table, the system of η-parallelism of R is

$$e_1(H)+2\sigma(e_1)-2\lambda\sigma(e_2)=0, \quad e_2(H)+2\sigma(e_2)-2\lambda\sigma(e_1)=0. \tag{34}$$

This system is equivalent to (33). Next, from the formulas:

$$\begin{aligned}(\nabla_{e_3}R)(e_1,e_2)e_1 &= -\xi(H)e_2-\{\xi(\sigma(e_2))+\alpha\sigma(e_1)\}\xi,\\ (\nabla_{e_3}R)(e_1,e_2)e_2 &= \xi(H)e_1+\{\xi(\sigma(e_1))-\alpha\sigma(e_2)\}\xi.\end{aligned}$$

We conclude that R is strongly η-parallel if and only if R is η-parallel and $dH(\xi)=0$. □

Example 7.2 (Homogeneous almost Kenmotsu 3-manifolds)**.** The H-almost Kenmotsu Lie group $G(\lambda,\alpha)$ of type II has constant holomorphic sectional curvature $H=-(1-\lambda^2)$ and hence it has η-parallel Riemannian curvature. As we saw in Proposition 4.1, every $G(\lambda,\alpha)$ has η-parallel Ricci operator as well as η-parallel Riemannian curvature (see Remark 4.2).

On the other hand, the almost Kenmotsu Lie group $G[\alpha,\gamma]$ of type IV has η-parallel Ricci operator when and only when $\alpha=0$, *i.e.*, it is locally symmetric (and hence R is dominantly η-parallel).

Example 7.3 (H-almost Kenmotsu 3-manifolds)**.** The H-almost Kenmotsu 3-manifold M exhibited in Example 5.3 has η-parallel Ricci operator. On the other hand, the holomorphic sectional curvature

$$H=-1+\frac{1}{4z^2}$$

is η-parallel but $dH(\xi)\neq 0$. Note that the scalar curvature r is η-parallel. Hence R is η-parallel but not strongly η-parallel.

In Proposition 2.2, we proved that on an almost contact metric 3-manifold M with η-parallel Riemannian curvature, the Ricci operator S is η-parallel if and only if S satisfies the equation (2). Here we compute the equation (2) on an almost Kenmotsu 3-manifold M.

$$\begin{aligned}(\nabla_{e_1}R)(e_2,e_3)e_1 = &\{e_1(\sigma(e_1))-(H-K_{23})+\lambda(\xi(\lambda)+2\lambda)+b\sigma(e_2)\}e_2\\ &-\{e_1(\xi(\lambda)+2\lambda)-b(K_{13}-K_{23})-\lambda\sigma(e_1)+\sigma(e_2)\}\xi,\\ (\nabla_{e_1}R)(e_2,e_3)e_2 = &-\{e_1(\sigma(e_1))-(H-K_{23})+\lambda(\xi(\lambda)+2\lambda)+b\sigma(e_2)\}e_2\\ &-\{e_1(K_{23})-2b(\xi(\lambda)+2\lambda)-2\sigma(e_1)\}\xi,\\ (\nabla_{e_1}R)(e_3,e_1)e_1 = &\{e_1(\sigma(e_2))+\lambda(H-K_{13})-(\xi(\lambda)+2\lambda)-b\sigma(e_1)\}e_2\\ &+\{e_1(K_{13})+2b(\xi(\lambda)+2\lambda)+2\lambda\sigma(e_2)\}\xi,\\ (\nabla_{e_1}R)(e_3,e_1)e_2 = &-\{e_1(\sigma(e_2))+\lambda(H-K_{13})-(\xi(\lambda)+2\lambda)-b\sigma(e_1)\}e_1\\ &+\{e_1(\xi(\lambda)+2\lambda)-b(K_{13}-K_{23})-\lambda\sigma(e_1)+\sigma(e_2)\}\xi,\\ (\nabla_{e_2}R)(e_2,e_3)e_1 = &\{e_2(\sigma(e_1))+\lambda(H-K_{23})-(\xi(\lambda)+2\lambda)-c\sigma(e_2)\}e_2\end{aligned}$$

$$
\begin{aligned}
& \quad - \{e_2(\xi(\lambda)+2\lambda)+c(K_{13}-K_{23})-\lambda\sigma(e_2)+\sigma(e_1)\}\xi,\\
(\nabla_{e_2}R)(e_2,e_3)e_2 = & \; - \{e_2(\sigma(e_1))+\lambda(H-K_{23})-(\xi(\lambda)+2\lambda)-c\sigma(e_2)\}e_1\\
& \quad - \{e_2(K_{23})+2c(\xi(\lambda)+2\lambda)+2\lambda\sigma(e_1)\}\xi,\\
(\nabla_{e_2}R)(e_3,e_1)e_1 = & \; \{e_2(\sigma(e_2))-(H-K_{13})+\lambda(\xi(\lambda)+2\lambda)+c\sigma(e_1)\}e_2\\
& \quad + \{e_2(K_{13})-2c(\xi(\lambda)+2\lambda)-2\sigma(e_2)\}\xi,\\
(\nabla_{e_2}R)(e_3,e_1)e_2 = & \; - \{e_2(\sigma(e_2))+\lambda(\xi(\lambda)+2\lambda)-(H-K_{13})+c\sigma(e_1)\}e_1\\
& \quad + \{e_2(\xi(\lambda)+2\lambda)+c(K_{13}-K_{23})-\lambda\sigma(e_2)+\sigma(e_1)\}\xi.
\end{aligned}
$$

From this table we obtain the following proposition.

Proposition 7.4. *Let M be an almost Kenmotsu 3-manifold with η-parallel Riemannian curvature. Take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 3.1. Then the Ricci operator S is η-parallel if and only if the following system of equations is satisfied.*

$$e_1(\xi(\lambda)+2\lambda)-b(K_{13}-K_{23})-\lambda\sigma(e_1)+\sigma(e_2)=0, \tag{35}$$

$$e_1(K_{13})+2b(\xi(\lambda)+2\lambda)+2\lambda\sigma(e_2)=0, \tag{36}$$

$$e_1(K_{23})-2b(\xi(\lambda)+2\lambda)-2\sigma(e_1)=0, \tag{37}$$

$$e_2(\xi(\lambda)+2\lambda)+c(K_{13}-K_{23})-\lambda\sigma(e_2)+\sigma(e_1)=0, \tag{38}$$

$$e_2(K_{13})-2c(\xi(\lambda)+2\lambda)-2\sigma(e_2)=0, \tag{39}$$

$$e_2(K_{23})+2c(\xi(\lambda)+2\lambda)+2\lambda\sigma(e_1)=0. \tag{40}$$

One can verify that the above system (35)–(40) together with (34) is equivalent to the system (4), (5), (7), (10), (11) and (13) of η-parallelism for S.

In case M is H-almost Kenmotsu, the system (35)–(40) is reduced to the following one.

Corollary 7.5. *Let M be an H-almost Kenmotsu 3-manifold with η-parallel Riemannian curvature. Take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 3.1. Then the Ricci operator S is η-parallel if and only if the following system of equations is satisfied.*

$$e_1(\xi(\lambda)+2\lambda)-b(K_{13}-K_{23})=0, \tag{41}$$

$$e_1(K_{13})+2b(\xi(\lambda)+2\lambda)=0, \tag{42}$$

$$e_1(K_{23})-2b(\xi(\lambda)+2\lambda)=0, \tag{43}$$

$$e_2(\xi(\lambda)+2\lambda)+c(K_{13}-K_{23})=0, \tag{44}$$

$$e_2(K_{13})-2c(\xi(\lambda)+2\lambda)=0, \tag{45}$$

$$e_2(K_{23})+2c(\xi(\lambda)+2\lambda)=0. \tag{46}$$

Note that from (42), (43), (44) and (45) imply that ρ_{33} is η-parallel.

Proposition 2.2 says that the η-parallelism of S and that of R are not equivalent, in general. Let us look for a class of almost Kenmotsu 3-manifolds on which the η-parallelism of S is equivalent to that of R.

From Theorem 6.17, an H-almost Kenmotsu 3-manifold M satisfying $\nabla_\xi h = 0$ has η-parallel Ricci operator if and only if M is locally isomorphic to either

(i) the warped products $\mathbb{R} \times_{ce^t} \overline{M}$, where $\overline{M}$ is a Riemannian 2-manifold of constant curvature or

(ii) the type II Lie group $G(\lambda, 0)$ for some constant λ.

Note that M is a Kenmotsu 3-manifold of constant curvature -1 in the first item with flat $\overline{M}$ and the type II Lie group $G(0, 0)$. One can see that on the all examples in this list the equivalence of η-parallelism and that of R holds. In other words, these spaces have η-parallel Ricci operator as well as η-parallel Riemannian curvature.

However up to now the classification of almost Kenmotsu 3-manifolds on which η-parallelisms of S and R are equivalent is not yet obtained. As a partial classification, we determine almost Kenmotsu 3-manifolds on which two η-parallelisms are equivalent under an additional condition $\nabla_\xi h = 0$. More precisely we show that warped products $\mathbb{R} \times_{ce^t} \overline{M}$ and type II Lie groups locally exhaust the class of almost Kenmotsu 3-manifolds satisfying $\nabla_\xi h = 0$ on which η-parallelisms of S and R are equivalent.

Corollary 7.6. *Let M be an H-almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0$. Assume that both the Ricci operator S and the Riemannian curvature R are η-parallel. Then it is locally isomorphic to one of the following spaces*:

(i) *the warped products $\mathbb{R} \times_{ce^t} \mathbb{S}^2(k^2)$, $\mathbb{R} \times_{ce^t} \mathbb{H}^2(-k^2)$ or the hyperbolic 3-space $\mathbb{H}^3(-1)$.*

(ii) *the type II Lie group $G(\lambda, 0)$ for some constant $\lambda \neq 0$.*

Proof. Since we assumed that M is an H-almost Kenmotsu 3-manifold with η-parallel Ricci operator S under the assumption $\nabla_\xi h = 0$, M is locally isomorphic to either the warped products with constant curvature fiber or a type II Lie group $G(\lambda, 0)$ by Theorem 6.17. One can check that these warped products and $G(\lambda, 0)$ has η-parallel Riemannian curvature. □

Finally, let us pick up Kenmotsu 3-manifolds. A Kenmotsu 3-manifold M has η-parallel Riemannian curvature if and only if H is η-parallel. Since $r = 2(H - 2)$, the η-parallelism of R is equivalent to that of r. Here we prove the following:

Theorem 7.7. *Let M be a Kenmotsu 3-manifold. Then M has η-parallel Riemannian curvature if and only if it is locally isomorphic to the warped product $I \times_{ce^t} \overline{M}$, where I is an open interval with coordinate t, $\overline{M}$ is a Riemannian 2-manifold of constant curvature and c is a positive constant. In particular, the η-parallelism of R is equivalent to that of S on Kenmotsu 3-manifolds.*

Proof. Take a point $p \in M$, there exists a neighborhood U_p of p such that U is a warped product $U_p = (-\varepsilon, \varepsilon) \times_{ce^t} \overline{U}$, where ε is a positive constant, t is a coordinate of the interval $(-\varepsilon, \varepsilon)$ and $\overline{U} = (\overline{U}, \bar{g}, J)$ is a Riemannian 2-manifold equipped with a Kähler structure. The characteristic vector field is expressed locally as $\xi = \partial/\partial t$. The holomorphic sectional curvature H is locally expressed as

$$H = \frac{1}{c^2 e^{2t}} \left(\overline{K} - c^2 e^{2t}\right),$$

where $\overline{K}$ is the Gaussian curvature of $\overline{U}$. This formula implies that the η-parallelism of H is equivalent to the constancy of $\overline{K}$. From Theorem 6.6, we conclude that the η-parallelism of R is equivalent to that of S. □

Here we should give a remark concerning on Theorem 7.7 and Theorem 6.8. In [12, Theorem 4.1], De claimed that if a Kenmotsu 3-manifold M satisfies (3) for all vector fields X, Y, Z and W orthogonal to ξ if and only if the scalar curvature is constant. In our terminology, he claimed that if a Kenmotsu 3-manifold M has η-parallel Riemannian curvature, then M is of constant scalar curvature. However the conclusion of [12, Theorem 4.1] should be corrected as "M has η-parallel scalar curvature". One can see that the conclusion of [12, Theorem 4.1] is true under the assumption R is strongly η-parallel. Hence we obtain the following result.

Proposition 7.8. *A Kenmotsu* 3*-manifold satisfies the condition*

$$\varphi^2\{(\nabla_W R)(X,Y)Z\} = 0$$

for all vector fields X*,* Y*,* Z *orthogonal to* ξ *and any vector field* W *on* M *if and only if* M *is of constant curvature* -1*.*

This proposition corrects [14, Corollary 4] and [15, Corollary 2.2].

It should be remarked that every $G(\lambda, \alpha)$ of type II has η-parallel Ricci operator as well as η-parallel Riemannian curvature. The additional condition $\nabla_\xi h = 0$ forces α to be 0 and λ to be constant for almost Kenmotsu 3-manifolds discussed in Corollary 7.6. On the other hand, the H-almost Kenmotsu 3-manifold discussed in Example 5.3 and Example 7.3 has η-parallel S, η-parallel R and satisfies $\alpha = 0$ but not $\nabla_\xi h = 0$. Since the condition $\nabla_\xi h = 0$ is equivalent to $\alpha = \xi(\lambda) = 0$, to relax the assumption $\nabla_\xi h = 0$ to $\alpha = 0$ in Corollary 7.6 seems to be an interesting problem.

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Jun-ichi Inoguchi
Department of Mathematics
Hokkaido University
Sapporo, 060-0810, Japan
Email address: inoguchi@math.sci.hokudai.ac.jp

Ji-Eun Lee
Department of Mathematics
Chonnam National University
Gwangju 61186, Korea
Email address: jieunlee12@jnu.ac.kr