ON NONNIL-EXACT SEQUENCES AND NONNIL-COMMUTATIVE DIAGRAMS

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Abstract. In this paper, we investigate the nonnil-exact sequences and nonnil-commutative diagrams and show that they behave in a way similar to the classical ones in Abelian categories.

1. Introduction

Throughout this paper, it is assumed that all rings are commutative and associative with non-zero identity and all modules are unitary. If $R$ is a ring, then $\text{Nil}(R)$ denotes the set of nilpotent elements of $R$, and $\text{Z}(R)$ denotes the set of zero-divisors of $R$. A ring with $\text{Nil}(R)$ being divided prime is called a $\phi$-ring. In this paper, if the nilradical $\text{Nil}(R)$ of a ring $R$ is prime, then $R$ is called a PN-ring. If $\text{Z}(R) = \text{Nil}(R)$, then $R$ is called a ZN-ring. We recommend [1–20] for the study of the ring-theoretic characterizations on $\phi$-rings, and [21–28] for the study of the module-theoretic characterizations on $\phi$-rings.

In order to extend the homological methods to commutative rings with the nilradical as a prime ideal, the authors in [26] introduced nonnil-exact sequences and nonnil-commutative diagrams. Since commutative diagrams play an important role in homological theory, we investigate nonnil-commutative diagrams in order to lay the foundation for future work on homological theories over PN-rings. We show that nonnil-commutative diagrams behave in a way similar to the classical ones. For example, Five Lemma, Snake Lemma and $3 \times 3$ Lemma show the same pattern in PN-rings. To this end, we introduce some necessary concepts and symbols.

Let $R$ be a PN-ring and $M$ an $R$-module. We set

$\text{NN}(R) = \{I \mid I$ is a nonnil ideal of $R\}$,
and 
\[ \text{Ntor}(M) = \{ x \in M \mid Ix = 0 \text{ for some } I \in NN(R) \} \].

If \( \text{Ntor}(M) = M \), \( M \) is called a nonnil-torsion \( R \)-module, and if \( \text{Ntor}(M) = 0 \), \( M \) is called a nonnil-torsion-free \( R \)-module. Let \( f : A \to B \) be a homomorphism of \( R \)-modules. Set 
\[ \text{N Ker}(f) = \{ a \in A \mid sf(a) = 0 \text{ for some } s \in R \setminus \text{Nil}(R) \} \],
\[ \text{N Im}(f) = \{ b \in B \mid sb = sf(a) \text{ for some } a \in A \text{ and } s \in R \setminus \text{Nil}(R) \} \].

Since \( \text{Nil}(R) \) is a prime ideal, \( \text{N Ker}(f) \) is a submodule of \( A \) and \( \text{N Im}(f) \) is a submodule of \( B \). We set \( \text{NCoker}(f) = B / \text{N Im}(f) \).

Recall from [26] that the submodule \( \text{N Ker}(f) \) of \( A \) is called the nonnil-kernel of \( f \) and the submodule \( \text{N Im}(f) \) of \( B \) is called the nonnil-image of \( f \). The homomorphism \( f : A \to B \) is called a nonnil-monomorphism if \( \text{N Ker}(f) = \text{Ntor}(A) \) and it is called a nonnil-epimorphism if \( \text{N Im}(f) = B \). A sequence of \( R \)-modules and homomorphisms \( A \xrightarrow{f} B \xrightarrow{g} C \) is called a nonnil-complex (resp., a nonnil-exact sequence) if \( \text{N Im}(g) \subseteq \text{N Ker}(f) \) (resp., \( \text{N Im}(f) = \text{N Ker}(g) \)).

Every \( R \)-module \( M \) has a free nonnil-resolution, that is, there exists a nonnil-exact sequence 
\[ \cdots \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0, \]
where each \( F_i \) is free.

Let \( f : A \to B \), \( g : B \to D \), \( h : A \to C \) and \( k : C \to D \) be homomorphisms of \( R \)-modules. Then the following diagram
\[ \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C & \xrightarrow{k} & D
\end{array} \]
is said to be nonnil-commutative if \( \text{N Im}(gf - kh) = \text{Ntor}(D) \), that is, there exists \( s_a \in R \setminus \text{Nil}(R) \) such that \( s_agf(a) = s_akh(a) \) for any \( a \in A \), where the choice of \( s_a \) depends on \( a \).

An \( R \)-module homomorphism \( f : A \to B \) is called a nonnil-isomorphism if there exists a homomorphism \( g : B \to A \) such that \( \text{N Im}(1_A - gf) = \text{Ntor}(A) \) and \( \text{N Im}(1_B - fg) = \text{Ntor}(B) \), that is, the following diagram
\[ \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{1_A} & & \downarrow{1_B} \\
A & \xrightarrow{g} & B
\end{array} \]
is nonnil-commutative. If there exists a nonnil-isomorphism \( f : A \to B \), we say that \( A, B \) are nonnil-isomorphic, denoted by \( A \overset{N}{\simeq} B \). An \( R \)-module homomorphism \( f : A \to B \) is called a weakly nonnil-isomorphism if \( \text{N Ker}(f) = \text{Ntor}(A) \)
and $\text{Nlm}(f) = B$, in this case, we say that $A$ is weakly nonnil-isomorphic to $B$ \(\text{W}_N\) by $f$, denoted by $f : A \xrightarrow{\sim} B$.

In this paper, $R$ always denotes a PN-ring.

2. Main results

**Theorem 2.1** (Five Lemma in PN-rings). Consider the following non-nil-commutative diagram with non-nil-exact rows:

\[
\begin{array}{cccccc}
D & \xrightarrow{h} & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{k} & E \\
\downarrow{\delta} & & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\mu} \\
D' & \xrightarrow{h'} & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{k'} & E'
\end{array}
\]

(a) If $\alpha$ and $\gamma$ are non-nil-monomorphisms and $\delta$ is a non-nil-epimorphism, then $\beta$ is a non-nil-monomorphism.

(b) If $\alpha$ and $\gamma$ are non-nil-epimorphisms and $\mu$ is a non-nil-monomorphism, then $\beta$ is a non-nil-epimorphism.

(c) If $\delta$ is a non-nil-epimorphism, $\mu$ is a non-nil-monomorphism and $\alpha, \gamma$ are weakly non-nil-isomorphisms, then $\beta$ is a weakly non-nil-isomorphism.

(d) If $\delta, \alpha, \gamma, \mu$ are weakly non-nil-isomorphisms, then $\beta$ is a weakly non-nil-isomorphism.

**Proof.** (a) Let $b \in \text{Nker}(\beta)$. Then there is some $s \in R \setminus \text{Nil}(R)$ such that $s\beta(b) = 0$. Since the diagram is non-nil-commutative, we have that $s_1s\gamma g(b) = s_1sg'\beta(b) = 0$ for some $s_1 \in R \setminus \text{Nil}(R)$. Thus

\[g(b) \in \text{Nker}(\gamma).\]

Since $\text{Nker}(\gamma) = \text{Ntor}(C)$, there is some $s_2 \in R \setminus \text{Nil}(R)$ such that $s_2g(b) = 0$. Thus

\[b \in \text{Nker}(g).\]

Since $\text{Nker}(g) = \text{Nlm}(f)$, there are some $a \in A$ and $s_3 \in R \setminus \text{Nil}(R)$ such that $s_3b = s_3f(a)$. Hence $ss_3s_4f'\alpha(a) = ss_3s_4\beta f(a) = ss_3s_4\beta(b) = 0$ for some $s_4 \in R \setminus \text{Nil}(R)$. Thus

\[\alpha(a) \in \text{Nker}(f').\]

Since $\text{Nker}(f') = \text{Nlm}(h')$, there are some $d' \in D'$ and $s_5 \in R \setminus \text{Nil}(R)$ such that $s_5a = s_5h'(d')$. Since $\delta$ is a non-nil-epimorphism, there are some $d \in D$ and $s_6 \in R \setminus \text{Nil}(R)$ such that $s_6d' = s_6\delta(d)$. So we have that $s_5s_6s_7\alpha(a) = s_5s_6s_7h'(d') = s_5s_6s_7\delta(d) = s_5s_6s_7\alpha h(d)$ for some $s_7 \in R \setminus \text{Nil}(R)$. Hence

\[a - h(d) \in \text{Nker}(\alpha).\]

Thus there exists some $s_8 \in R \setminus \text{Nil}(R)$ such that $s_8\alpha = s_8h(d)$. Since $s_9s_8s_3b = s_9s_8s_3f(a) = s_9s_8s_3fh(d) = 0$ for some $s_9 \in R \setminus \text{Nil}(R)$,

\[b \in \text{Ntor}(B).\]
Therefore,
\[ \text{NKer}(\beta) = \text{Ntor}(B), \]
which implies that \( \beta \) is a nonnil-monomorphism.

(b) Let \( b' \in B' \). Since \( \gamma \) is a nonnil-epimorphism, there are some \( c \in C \) and \( s \in R \setminus \text{Nil}(R) \) such that
\[ s\gamma(c) = sg'(b'). \]
Non-nil-commutativity of the right square gives \( s_1s_2s_\mu k(c) = s_1s_2sk'\gamma(c) = s_1s_2sk'g'(b') = 0 \) for some \( s_1, s_2 \in R \setminus \text{Nil}(R) \). Since \( \mu \) is a non-nil-monomorphism, \( s_3k(c) = 0 \) for some \( s_3 \in R \setminus \text{Nil}(R) \). Because of the non-nil-exactness of the top row, there are some \( b \in B \) and \( s_4 \in R \setminus \text{Nil}(R) \) such that
\[ s_4g(b) = s_4c. \]
Hence \( ss_4s_5g(b') = ss_4s_5\gamma(c) = ss_4s_5\gamma g(b) = ss_4s_5g'(b) \) for some \( s_5 \in R \setminus \text{Nil}(R) \). Hence
\[ b' - \beta(b) \in \text{NKer}(g'). \]
Since \( \text{NKer}(g') = \text{NIm}(f') \), \( s_6(b' - \beta(b)) = s_6f'(a') \) for some \( a' \in A' \), \( s_6 \in R \setminus \text{Nil}(R) \) by the non-nil-exactness of the bottom row. Since \( \alpha \) is a non-nil-epimorphism, there are some \( a \in A \) and \( s_7 \in R \setminus \text{Nil}(R) \) with \( s_7\alpha(a) = s_7a' \).
Hence \( s_6s_7s_8(b' - \beta(b)) = s_6s_7s_8f'(a') = s_6s_7s_8f'\alpha(a) = s_6s_7s_8\beta f(a) \) for some \( s_8 \in R \setminus \text{Nil}(R) \). Thus \( s_6s_7s_8b' = s_6s_7s_8\beta(b + f(a)) \). Therefore
\[ b' \in \text{NIm}(\beta), \]
which implies that \( \beta \) is a non-nil-epimorphism.

It is easy to see that (c) follows from (a) and (b), while (d) follows from (c).

\[ \square \]

**Theorem 2.2** (Snake Lemma in PN-rings). Consider the following non-nil-commutative diagram with non-nil-exact rows:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{} & 0 \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\
0 & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{} & \\
\end{array}
\]

(a) There is a non-nil-exact sequence
\[ \text{NKer}(\alpha) \to \text{NKer}(\beta) \to \text{NKer}(\gamma) \to \text{NCoker}(\alpha) \to \text{NCoker}(\beta) \to \text{NCoker}(\gamma). \]

(b) If \( f \) is a non-nil-monomorphism, then
\[ 0 \to \text{NKer}(\alpha) \to \text{NKer}(\beta) \to \text{NKer}(\gamma) \to \text{NCoker}(\alpha) \to \text{NCoker}(\beta) \to \text{NCoker}(\gamma) \]
is non-nil-exact.

(c) If \( g' \) is a non-nil-epimorphism, then
\[ \text{NKer}(\alpha) \to \text{NKer}(\beta) \to \text{NKer}(\gamma) \to \text{NCoker}(\alpha) \to \text{NCoker}(\beta) \to \text{NCoker}(\gamma) \to 0 \]
is non-nil-exact.
Proof. (a) The proof is completed by the following three steps.

(1) Let $a \in \text{N Ker}(\alpha)$. Then there exists $s \in R\setminus \text{Nil}(R)$ such that $s\alpha(a) = 0$. Because of nonnil-commutativity, there exists some $s_1 \in R\setminus \text{Nil}(R)$ such that $s_1\beta(f(a) = s_1f'\alpha(a) = 0$. Hence $f(a) \in \text{N Ker}(\beta)$. Define $f_1 : \text{N Ker}(\alpha) \rightarrow \text{N Ker}(\beta)$ by

$$f_1(a) = f(a).$$

Then $f_1$ is well-defined. Similarly, we get a homomorphism $g_1 : \text{N Ker}(\beta) \rightarrow \text{N Ker}(\gamma)$ by $g_1(b) = g(b)$ for $b \in B$.

For $a' \in A'$, define

$$f_2(a' + \text{N Im}(\alpha)) = f'((a') + \text{N Im}(\beta)).$$

It is easy to check that $f_2 : A'/\text{N Im}(\alpha) \rightarrow B'/\text{N Im}(\beta)$ is well-defined. In fact, if $a' \in \text{N Im}(\alpha)$, then $s(a') = sa(a)$ for some $a \in A$ and $s \in R\setminus \text{Nil}(R)$. So there exists $s_1 \in R\setminus \text{Nil}(R)$ such that $s_1f'(a') = s_1f'\alpha(a') = s_1\beta(f(a))$. Thus $f'(a') \in \text{N Im}(\beta)$. Similarly, we get a homomorphism $g_2 : B'/\text{N Im}(\beta) \rightarrow C'/\text{N Im}(\gamma)$ by $g_2(b' + \text{N Im}(\beta)) = g'(b') + \text{N Im}(\gamma)$ for $b' \in B'$.

Let $c \in \text{N Ker}(\gamma)$. Since $g$ is a nonnil-epimorphism, there exist $b \in B$ and $s, s_1 \in R\setminus \text{Nil}(R)$ such that $s_1\gamma(c) = 0$ and $sg(b) = sc$. Thus $s_1s_2\beta(g(b) = s_1s_2\gamma(g(b) = s_1s_2\gamma(c) = 0$ for some $s_2 \in R\setminus \text{Nil}(R)$. Hence $\beta(b) \in \text{N Ker}(g') = \text{N Im}(f')$. In this sense, there exist some $a' \in A'$ and $s_3 \in R\setminus \text{Nil}(R)$ such that $s_3f'(a') = s_3\beta(b)$. Define

$$\delta(c) = a' + \text{N Im}(\alpha).$$

If $sg(b) = 0$, then there are some $a \in A$ and $s_4 \in R\setminus \text{Nil}(R)$ such that $s_4f(a) = s_4b$ by the fact that $\text{N Im}(f) = \text{N Ker}(g)$. Since $s_4s_5\beta(b) = s_4s_5\beta(f(a) = s_4s_5f'\alpha(a)$ for some $s_5 \in R\setminus \text{Nil}(R)$, we have $s_4a' = s_4\alpha(a)$ for some $s_6 \in R\setminus \text{Nil}(R)$ and $a' \in \text{N Im}(\alpha)$. Therefore, $\delta$ is well-defined.

(2) Next, we show that the following sequence is a nonnil-complex of $R$-modules and homomorphisms:

$$\text{N Ker}(\alpha) \xrightarrow{f_1} \text{N Ker}(\beta) \xrightarrow{g_1} \text{N Ker}(\gamma) \xrightarrow{\delta} \text{N Coker}(\alpha) \xrightarrow{f_2} \text{N Coker}(\beta) \xrightarrow{g_2} \text{N Coker}(\gamma).$$

For any $a \in \text{N Ker}(\alpha)$, we have that $s_1g_1f_1(a) = s_1gf(a) = 0$ for some $s_1 \in R\setminus \text{Nil}(R)$. Similarly, $s_2g_2f_2(a' + \text{N Im}(\alpha)) = s_2g'f'(a') + \text{N Im}(\gamma) = 0$ for some $s_2 \in R\setminus \text{Nil}(R)$.

Let $b \in \text{N Ker}(\beta)$. Then there exists some $s \in R\setminus \text{Nil}(R)$ such that $s\beta(b) = 0$. Hence $\beta(b) \in \text{N Ker}(g') = \text{N Im}(f')$. Thus there are some $a' \in A'$ and $s_3 \in R\setminus \text{Nil}(R)$ such that $s_3f'(a') = s_3\beta(b) = 0$. We have that $a' \in \text{N Ker}(f') = \text{N Tor}(A')$. So

$$\delta g_1(b) = a' + \text{N Im}(\alpha) = 0.$$

Let $c \in \text{N Ker}(\gamma)$. Then there exists some $s \in R\setminus \text{Nil}(R)$ such that $s\gamma(c) = 0$ and $sc = sg(b)$. Because there exist some $s_1, s_2 \in R\setminus \text{Nil}(R)$ such that $s_1s_2\beta(b) = s_1s_2\gamma(b) = s_1s_2\gamma(c) = 0$, $\beta(b) \in \text{N Ker}(g') = \text{N Im}(f')$. Hence we
have that \( s_3 \beta (b) = s_3 f'(a') \), where \( s_3 \in R \setminus \text{Nil}(R) \). Thus
\[
f_2 \delta(c) = f_2(a' + \text{NIm}(\alpha)) = f'(a') + \text{NIm}(\beta) = 0.
\]

(3) Notice we have that
\[
\text{N Ker}(g_1) = \{ b \in \text{N Ker}(\beta) \mid s g_1(b) = sg(b) = 0, s \in R \setminus \text{Nil}(R) \}
\]
and
\[
\text{NIm}(f_1) = \{ b \in \text{N Ker}(\beta) \mid sb = sf_1(a) = sf(a), a \in \text{N Ker}(\alpha), s \in R \setminus \text{Nil}(R) \}.
\]

In verifying that \( \text{N Ker}(g_1) = \text{NIm}(f_1) \), we show only that \( \text{N Ker}(g_1) \subseteq \text{NIm}(f_1) \).

Let \( b \in \text{N Ker}(g_1) \). Then \( b \in \text{N Ker}(\beta) \) and \( b \in \text{N Ker}(g) = \text{NIm}(f) \). Thus there exist some \( s \in R \setminus \text{Nil}(R) \) and \( a \in \text{N Ker}(\alpha) \) such that \( sb = sf(a) \). Hence \( ss_1f'\alpha(a) = ss_1\beta f(a) = ss_1\beta(b) \) for some \( s_1 \in R \setminus \text{Nil}(R) \). So \( ss_1s_2f'\alpha(a) = ss_1s_2\beta(b) = 0 \) for some \( s_2 \in R \setminus \text{Nil}(R) \). Thus \( \alpha(a) \in \text{N Ker}(f') = \text{N Tor}(A') \).

Since there exists some \( s_3 \in R \setminus \text{Nil}(R) \) such that \( s_3 \alpha(a) = 0 \). In this sense, \( b \in \text{NIm}(f_1) \). Therefore \( \text{NIm}(f_1) = \text{N Ker}(g_1) \).

Notice that
\[
\text{NIm}(g_1) = \{ c \in \text{N Ker}(\gamma) \mid sc = sg_1(b) = sg(b), b \in \text{N Ker}(\beta), s \in R \setminus \text{Nil}(R) \}
\]
and
\[
\text{N Ker}(\delta) = \{ c \in \text{N Ker}(\gamma) \mid sc = sg(b), s_1 \beta(b) = s_1f'(a'), \alpha' \in \text{NIm}(\alpha), s_1 \in R \setminus \text{Nil}(R) \}.
\]

Let \( c \in \text{N Ker}(\delta) \). Then \( c \in \text{N Ker}(\gamma) \) and there exist some \( a' \in \text{N Im}(\alpha) \) and \( s_1 \in R \setminus \text{Nil}(R) \) such that \( sc = sg(b_1), s_1 \beta(b_1) = s_1f'(a') \). So there exists some \( s_2 \in R \setminus \text{Nil}(R) \) such that \( s_2a' = s_1a \). Thus we have that \( s_1s_2\beta(b_1) = s_1s_2f'(a') = s_1s_2f'\alpha(a) = s_1s_2\beta(f(a)) \). Hence \( b_1 - f(a) \in \text{N Ker}(\beta) \).

Set\[
b = b_1 - f(a).
\]
We have that \( b \in \text{N Ker}(\beta) \) and \( sc = sg_1(b) = sg(b) \). Thus \( c \in \text{N Im}(g_1) \), which implies that \( \text{NIm}(g_1) = \text{N Ker}(\delta) \).

Notice that
\[
\text{N Ker}(f_2) = \{ a' + \text{NIm}(\alpha) \mid s f_2(a' + \text{NIm}(\alpha)) = 0, a' \in A', s \in R \setminus \text{Nil}(R) \}
\]
\[
= \{ a' + \text{NIm}(\alpha) \mid s f'(a') \in \text{NIm}(\beta), a' \in A', s \in R \setminus \text{Nil}(R) \}
\]
\[
= \{ a' + \text{NIm}(\alpha) \mid s_1s f'(a') = s_1 \beta(b), a' \in A', b \in B, s, s_1 \in R \setminus \text{Nil}(R) \}.
\]
If \( s_1s f'(a') = s_1 \beta(b) \), then \( s' s_1s g' \beta(b) = s' s_1s g' f'(a') = 0 \) for some \( s' \in R \setminus \text{Nil}(R) \). So we have that \( \beta(b) \in \text{N Ker}(g') = \text{NIm}(f') \). Thus \( s \beta(b) = s f'(a') \) for some \( s \in R \setminus \text{Nil}(R) \) and \( a' \in A \). Therefore, we have
\[
\text{N Ker}(f_2) = \{ a' + \text{NIm}(\alpha) \mid s \beta(b) = s f'(a'), b \in B, a' \in A', s \in R \setminus \text{Nil}(R) \}.
\]
Set $c = g(b)$. We have $s\gamma(c) = sg'f(a') = 0$. Hence $c \in \text{NKer}(\gamma)$. Therefore $\text{NKer}(f_2) = \text{NIm}(\delta)$.

Similarly,

$$\text{NKer}(g_2) = \{b' + \text{NIm}(\beta) \mid s\gamma(c) = sg'(b'), c \in C, b' \in B', s \in R\setminus\text{Nil}(R)\}.$$ 

Notice that

$$\text{NIm}(f_2) = \{b' + \text{NIm}(\beta) \mid sb' = s f' (a') + s_1 \beta(b), a' \in A', b \in B, b' \in B', s, s_1 \in R\setminus\text{Nil}(R)\}.$$ 

If $\overline{b} \in \text{NKer}(g_2)$, then there exist $c \in C$, $b' \in B'$ and $s \in R\setminus\text{Nil}(R)$ such that $s\gamma(c) = sg'(b')$. Because $c \in C = \text{NIm}(g)$, we have that $s_1c = s_1g(b)$ for some $b \in B$, $s_1 \in R\setminus\text{Nil}(R)$. Thus

$$s_1sg'(b') = s_1s\gamma(c) = s_1s\gamma(b) = s_1sg'(b).$$

So

$$b' - \beta(b) \in \text{NKer}(g') = \text{NIm}(f').$$

Hence $sb' - s\beta(b) = s f' (a')$ for some $a' \in A'$, $s \in R\setminus\text{Nil}(R)$. We have $\overline{b} \in \text{NIm}(f_2)$, which implies that $\text{NIm}(f_2) = \text{NKer}(g_2)$.

(b) Consider that $\text{NKer}(f_1) = \text{NKer}(\alpha) \cap \text{NKer}(f) = \text{Ntor}(A)$.

(c) Consider that $\text{NIm}(g_2) \supseteq (\text{NIm}(g') + \text{NIm}(\gamma))/\text{NIm}(\gamma) = C'/\text{NIm}(\gamma)$.


\begin{corollary}
Consider the following nonnil-commutative diagram with nonnil-exact rows:

\begin{center}
\begin{tikzcd}
0 \arrow{r} & A \arrow{d}{\alpha} \arrow{r}{f} & B \arrow{d}{\beta} \arrow{r}{g} & C \arrow{d}{\gamma} \arrow{r} & 0 \\
0 \arrow{r} & A' \arrow{r}{f'} & B' \arrow{r}{g'} & C' \arrow{r} & 0.
\end{tikzcd}
\end{center}

(a) If $\alpha$ is a nonnil-epimorphism, then the sequence

$$0 \to \text{NKer}(\alpha) \to \text{NKer}(\beta) \to \text{NKer}(\gamma) \to 0$$

is nonnil-exact.

(b) If $\gamma$ is a nonnil-monomorphism, then the sequence

$$0 \to \text{NCoker}(\alpha) \to \text{NCoker}(\beta) \to \text{NCoker}(\gamma) \to 0$$

is nonnil-exact.

\end{corollary}

\begin{proof}
(a) It holds by Theorem 2.2.

(b) Suppose that $\gamma$ is a nonnil-monomorphism. For $a' + \text{NIm}(\alpha) \in \text{NKer}(f_2)$, $a' \in A'$, there exists $s_1 \in R\setminus\text{Nil}(R)$ such that $s_1f'(a') = s_1\beta(b)$, where $b \in B$. Since the diagram is nonnil-commutative, we have some $s_2 \in R\setminus\text{Nil}(R)$ such that $s_2\gamma(b) = s_2g'(b) = s_2g f'(a') = 0$. Thus

$$g(b) \in \text{NKer}(\gamma) = \text{Ntor}(C).$$

\end{proof}
Hence $s_3 g(b) = 0$ for some $s_3 \in R \setminus \text{Nil}(R)$. So $b \in \text{N Ker}(g) = \text{N Im}(f)$, we have some $s_4 \in R \setminus \text{Nil}(R)$, $a \in A$ such that $s_4 b = s_4 f(a)$. It is clear that $s_1 s_4 f'(a') = s_1 s_4 \beta(b) = s_1 s_4 \beta f(a) = s_1 s_4 f' \alpha(a)$. So we have that $$a' - \alpha(a) \in \text{N Ker}(f') = \text{N tor}(A').$$ Hence there exists some $s_5 \in R \setminus \text{Nil}(R)$ such that $s_5 a' = s_5 \alpha(a)$ and $a' \in \text{N Im}(\alpha)$. Therefore $a' + \text{N Im}(\alpha) = 0$, which implies that $0 \rightarrow \text{N Coker}(\alpha) \rightarrow \text{N Coker}(\beta) \rightarrow \text{N Coker}(\gamma) \rightarrow 0$ is nonnil-exact. \hfill \qed

**Lemma 2.4.** Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a nonnil-exact sequence of $R$-modules. If $A, C$ are nonnil-torsion, then $B$ is also nonnil-torsion.

**Proof.** For any $b \in B$, since $g(b) \in C$ is nonnil-torsion, $sg(b) = 0$ for some $s \in R \setminus \text{Nil}(R)$. Hence $b \in \text{N Ker}(g) = \text{N Im}(f)$. So there are some $a \in A$ and $s_1 \in R \setminus \text{Nil}(R)$ such that $$s_1 b = s_1 f(a).$$ Since $a \in A$ is nonnil-torsion, there is some $s_2 \in R \setminus \text{Nil}(R)$ such that $s_2 a = 0$. Thus $$s_1 s_2 b = s_1 s_2 f(a) = 0.$$ Therefore $b$ is nonnil-torsion, which implies that $B$ is a nonnil-torsion $R$-module. \hfill \boxed

**Corollary 2.5.** Consider the following nonnil-commutative diagram with nonnil-exact rows:

\[
\begin{array}{cccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{\gamma} & 0 \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} \\
0 & \xrightarrow{f'} & A' & \xrightarrow{g'} & B' & \xrightarrow{\gamma'} & C' \\
\end{array}
\]

(a) If $\alpha$ and $\gamma$ are nonnil-monomorphisms, then $\beta$ is a nonnil-monomorphism.

(b) If $\alpha$ and $\gamma$ are nonnil-epimorphisms, then $\beta$ is a nonnil-epimorphism.

(c) If $\alpha$ and $\gamma$ are weakly nonnil-isomorphisms, then $\beta$ is a weakly nonnil-isomorphism.

**Proof.** (a) If $\alpha$ and $\gamma$ are nonnil-monomorphisms, then $\text{N Ker}(\alpha)$ and $\text{N Ker}(\gamma)$ are nonnil-torsion. So $\text{N Ker}(\beta)$ is also nonnil-torsion by Theorem 2.2 and Lemma 2.4. Thus $\beta$ is a nonnil-monomorphism.

(b) If $\alpha$ and $\gamma$ are nonnil-epimorphisms, then $\text{N Coker}(\alpha) = \text{N Coker}(\gamma) = 0$. Hence $A' = \text{N Im}(\alpha)$ and $C' = \text{N Im}(\gamma)$. For any $b' \in B'$, we have that $$g_2(b') = g'(b') = 0.$$ Thus $g'(b') \in \text{N Im}(\gamma)$. Hence $sg'(b') = s\gamma(c)$ for some $c \in C$, $s \in R \setminus \text{Nil}(R)$. Suppose that $s_1 c = s_1 g(b)$, $b \in B$, $s_1 \in R \setminus \text{Nil}(R)$. We have $$s s_1 s_2 g'(b') = s s_1 s_2 \gamma g(b) = s s_1 s_2 \gamma \beta(b)$$ for some $s_3 \in R \setminus \text{Nil}(R)$. Hence $$b' - \beta(b) \in \text{N Ker}(g') = \text{N Im}(f').$$
In this sense, there are some \( a' \in A' \) and \( s_3 \in R \setminus \text{Nil}(R) \) such that
\[
s_3 b' - s_3 \beta(b) = s_3 f'(a').
\]
Because \( a' \in A' = \text{Nil}(\alpha) \), we have that \( s_4 a' = s_4 \alpha(a) \) for some \( a \in A \), \( s_4 \in R \setminus \text{Nil}(R) \). Hence
\[
s_3 s_4 b' = s_3 s_4 \beta(b) + s_3 s_4 \beta f(a).
\]
This is to say that \( b' \in \text{Nil}(\beta) \). Therefore \( \text{NCoker}(\beta) = B'/\text{Nil}(\beta) = 0 \), which implies that \( \beta \) is a nonnil-epimorphism.
(c) follows from (a) and (b).

**Theorem 2.6** (\( 3 \times 3 \) Lemma in PN-rings). Consider the following nonnil-commutative diagram:

\[
\begin{array}{c}
0 & 0 & 0 \\
0 & A_1 & f_1 & B_1 & g_1 & C_1 & 0 \\
0 & A & f & B & g & C & 0 \\
0 & A_2 & f_2 & B_2 & g_2 & C_2 & 0 \\
0 & 0 & 0 & 0
\end{array}
\]

in which all columns (resp., rows) are nonnil-exact.

(a) If the first and the second rows (resp., columns) are nonnil-exact, then the third row (resp., column) is nonnil-exact.

(b) If the second and the third rows (resp., columns) are nonnil-exact, then the first row (resp., column) is nonnil-exact.

**Proof.** (a) Suppose that the first and the second rows (resp., columns) are nonnil-exact. For any \( a_2 \in \text{N Ker}(f_2) \), there is some \( s_1 \in R \setminus \text{Nil}(R) \) such that \( s_1 f_2(a_2) = 0 \). Since \( a_2 \in A_2 = \text{Nil}(\alpha) \), there are some \( a \in A \) and \( s_2 \in R \setminus \text{Nil}(R) \) such that \( s_2 a_2 = s_2 \alpha(a) \). Thus we have some \( s_3 \in R \setminus \text{Nil}(R) \) such that \( s_1 s_2 s_3 \beta f(a) = s_1 s_2 s_3 f_2 \alpha(a) = s_1 s_2 s_3 f_2(a_2) = 0 \) by the nonnil-commutativity. Hence
\[
f(a) \in \text{N Ker}(\beta).
\]

Since \( \text{N Ker}(\beta) = \text{N Im}(\beta) \), there are some \( b_1 \in B_1 \) and \( s_4 \in R \setminus \text{Nil}(R) \) such that \( s_4 f(a) = s_4 \beta(b_1) \). Because the diagram is nonnil-commutative, we have some \( s_5 \in R \setminus \text{Nil}(R) \) such that \( s_5 g_1(b_1) = s_5 g \beta(b_1) \). Hence
\[
s_4 s_5 s_6 g_1(b_1) = s_4 s_5 s_6 g \beta(b_1) = s_4 s_5 s_6 g f(a) = 0 \text{ for some } s_5 \in R \setminus \text{Nil}(R).
\]
Thus
\[
g_1(b_1) \in \text{N Ker}(\gamma_1).
\]
Since $\text{N Ker}(\gamma) = \text{N Tor}(C_1)$, there is some $s_7 \in R \setminus \text{Nil}(R)$ such that $s_7g_1(b_1) = 0$. Hence

$$b_1 \in \text{N Ker}(g_1).$$

Since $\text{N Ker}(g_1) = \text{N Im}(f_1)$, there are some $a_1 \in A_1$ and $s_8 \in R \setminus \text{Nil}(R)$ such that $s_8b_1 = s_8f_1(a_1)$. Hence $s_4s_8s_9f(a) = s_4s_8s_9\beta f_1(a_1) = s_4s_8s_9f\alpha_1(a_1)$ for some $s_9 \in R \setminus \text{Nil}(R)$. Thus we have

$$a - \alpha_1(a_1) \in \text{N Ker}(f).$$

Since $\text{N Ker}(f) = \text{N Tor}(A)$, $s_{10}(a - \alpha_1(a_1)) = 0$ for some $s_{10} \in R \setminus \text{Nil}(R)$. Thus we have that $s_{10}a = s_{10}\alpha_1(a_1)$. Hence $s_{11}s_2s_{10}a_2 = s_{11}s_2s_{10}\alpha_1(a_1) = 0$ for some $s_{11} \in R \setminus \text{Nil}(R)$. Therefore $a_2 \in \text{N Tor}(A_2)$, which implies that

$$\text{N Ker}(f_2) = \text{N Tor}(A_2).$$

For any $b_2 \in \text{N Ker}(g_2)$, there is some $s_4 \in R \setminus \text{Nil}(R)$ such that $s_4g_2(b_2) = 0$. Since $b_2 \in B_2 = \text{N Im}(\beta)$, there are some $b \in B$ and $s_2 \in R \setminus \text{Nil}(R)$ such that $s_2b_2 = s_2\beta(b)$. By the non-nil-commutativity, there is some $s_3 \in R \setminus \text{Nil}(R)$ such that $s_3s_2s_3\gamma g(b) = s_3s_2s_3\beta(b) = s_3s_2s_3g_2(b_2) = 0$. Hence

$$g(b) \in \text{N Ker}(\gamma).$$

Since $\text{N Ker}(\gamma) = \text{N Im}(\gamma)$, there are some $c_1 \in C_1$ and $s_4 \in R \setminus \text{Nil}(R)$ such that $s_4g(b) = s_4\gamma_1(c_1)$. Since $c_1 \in C_1 = \text{N Im}(g_1)$, there are some $b_1 \in B_1$ and $s_6 \in R \setminus \text{Nil}(R)$ such that $s_6c_1 = s_6g_1(b_1)$. Thus $s_4s_5s_6g(b) = s_4s_5s_6\gamma_1g_1(b_1) = s_4s_5s_6g\beta_1(b_1)$ for some $s_6 \in R \setminus \text{Nil}(R)$. So

$$b - \beta_1(b_1) \in \text{N Ker}(g).$$

Since $\text{N Ker}(g) = \text{N Im}(f)$, there are some $a \in A$ and $s_7 \in R \setminus \text{Nil}(R)$ such that $s_7f(a) = s_7\beta_1(b_1)$. Hence $s_7\beta(b - \beta_1(b_1)) = s_7f(a)$. So $s_7s_8\beta(a) = s_7s_8f_2\alpha(a)$ for some $s_8 \in R \setminus \text{Nil}(R)$. Thus we have that $s_2s_7b_2 = s_2s_7\beta(b) = s_2s_7\alpha(a)$. Therefore $b_2 \in \text{N Im}(f_2)$, which implies that

$$\text{N Ker}(g_2) \subseteq \text{N Im}(f_2).$$

Conversely, for any $a_2 \in A_2 = \text{N Im}(\alpha)$, there are some $a \in A$ and $s \in R \setminus \text{Nil}(R)$ such that $sa_2 = s\alpha(a)$. Hence

$$s's_2f_2(a_2) = s's_2f_2(a) = s's_2s_2f\alpha(a) = s's_2s_7gf(a) = 0$$

for some $s' \in R \setminus \text{Nil}(R)$. Thus $f_2(a_2) \in \text{N Ker}(g_2)$, which implies that $\text{Im}(f_2) \subseteq \text{N Ker}(g_2)$. We have that $\text{N Im}(f_2) = \text{Im}(f_2) + \text{N Tor}(B_2) \subseteq \text{N Ker}(g_2)$. Therefore

$$\text{N Ker}(g_2) = \text{N Im}(f_2).$$

For any $c_2 \in C_2 = \text{N Im}(\gamma)$, there are some $c \in C$ and $s \in R \setminus \text{Nil}(R)$ such that $sc_2 = s\gamma(c)$. Since $c \in C = \text{N Im}(g)$, we have that $s_1c = s_1g(b)$ for some $b \in B$, $s_1 \in R \setminus \text{Nil}(R)$. Thus

$$ss_1c_2 = ss_1\gamma g(b) = ss_1g_2\beta(b),$$

that is, $c_2 \in \text{N Im}(g_2)$. So

$$\text{N Im}(g_2) = C_2.$$
Therefore, the third row is also nonnil-exact.
(b) Similar to (a), we can show (b) by diagram-chases. □

Using known nonnil-commutative diagrams, we can transfer related properties between modules. However, a given diagram is often not nonnil-commutative, and so we have to adapt the nonnil-commutative diagram to complete the job. An $R$-module $M$ is said to be NN-torsion-free if it is nonnil-isomorphic to some nonnil-torsion-free module.

**Theorem 2.7.** Consider the following nonnil-commutative diagram of $R$-modules and homomorphisms with nonnil-exact rows:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{\gamma} & 0 \\
\alpha & & \beta & & \gamma & & \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'
\end{array}
$$

where the square on the left is nonnil-commutative and $C'$ is NN-torsion-free. Then there is $\gamma : C \to C'$ such that the square on the right is nonnil-commutative.

**Proof.** Suppose that $C' \cong D$, $D$ is a nonnil-torsion-free module and the following diagram

$$
\begin{array}{ccc}
C' & \xrightarrow{\mu} & D \\
\downarrow{\nu} & & \downarrow{1_D} \\
C' & \xrightarrow{\mu} & D
\end{array}
$$

is nonnil-commutative. Let $c \in C$. Since $g$ is a nonnil-epimorphism, there are $b \in B$ and $s \in R \setminus \text{Nil}(R)$ such that $sg(b) = sc$. Define

$$
\gamma(c) = v\mu g'\beta(b).
$$

If $sg(b) = sc = 0$, then there exist $a \in A$ and $s_1 \in R \setminus \text{Nil}(R)$ such that $s_1f(a) = s_1b$. Thus

$$
s_2s_1\mu g'\beta(b) = s_2s_1\mu g'\beta f(a) = s_2s_1\mu g'f'\alpha(a) = 0
$$

for some $s_2 \in R \setminus \text{Nil}(R)$. Since $D$ is a nonnil-torsion-free module, $\mu g'\beta(b) = 0$. Hence $\gamma(c) = v\mu g'\beta(b) = 0$. Then $\gamma$ is a well-defined homomorphism. It is easy to see that the square on the right is nonnil-commutative. □

If the sequence $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ is only a nonnil-complex, then the result in Theorem 2.7 also holds.
Theorem 2.8. Consider the following nonnil-commutative diagram of $R$-modules and homomorphisms with nonnil-exact rows:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} \\
0 & \xrightarrow{f'} & A' & \xrightarrow{g'} & C'
\end{array}
\]

where the square on the right is nonnil-commutative and $A'$ is $NN$-torsion-free. Then there is $\alpha : A \to A'$ such that the square on the left is nonnil-commutative.

Proof. Suppose that $A' \cong L$, $L$ is a nonnil-torsion-free module and the following diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{\mu} & L \\
\downarrow{1_{A'}} & & \downarrow{1_L} \\
A' & \xrightarrow{\nu} & L
\end{array}
\]

is nonnil-commutative. Let $a \in A$. Then $s_1 g' \beta f(a) = s_1 g f(a) = 0$ for some $s_1 \in R \setminus \text{Nil}(R)$. Thus there are $a' \in A'$ and $s_2 \in R \setminus \text{Nil}(R)$ such that $s_2 f'(a') = s_2 \beta f(a)$. Define

$$\alpha(a) = \nu \mu(a').$$

Since $L$ is a nonnil-torsion-free module, we have $\alpha(a) = \nu \mu(a') = 0$ if $a = 0$. Therefore, $\alpha$ is a well-defined homomorphism. It is easy to see that the square on the left is nonnil-commutative. \hfill $\square$

If the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is only a nonnil-complex, then the result in Theorem 2.8 also holds.

Theorem 2.9. The following diagram with a nonnil-exact row:

\[
\begin{array}{ccc}
0 & \xrightarrow{} & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{} & 0 \\
& & \downarrow{h} & & \downarrow{\beta} \\
& & E
\end{array}
\]

where $C$ is $NN$-torsion-free, can be completed to the following nonnil-commutative diagram with nonnil-exact rows:

\[
\begin{array}{ccc}
0 & \xrightarrow{} & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{} & 0 \\
& & \downarrow{h} & & \downarrow{\beta} \\
0 & \xrightarrow{} & E & \xrightarrow{\alpha} & M & \xrightarrow{\gamma} & C & \xrightarrow{} & 0
\end{array}
\]
Proof. Suppose that \( C \cong D \), \( D \) is a nonnil-torsion-free module and the following diagram

\[
\begin{array}{c}
C \\ \\
\downarrow^\alpha \\
C \\
\end{array}
\begin{array}{c}
\uparrow \cong \\
\downarrow^\mu \\
D \\
\end{array}
\begin{array}{c}
D \\ \\
\downarrow^\gamma \\
D \\
\end{array}
\begin{array}{c}
\downarrow^\nu \\
\downarrow^1 \\
1 \\
\end{array}
\]

is nonnil-commutative. Define \( N = \{(b, -e) \mid sb = sf(a), se = sh(a), a \in A, s \in R \setminus \text{Nil}(R)\} \) and \( M = (B \oplus E)/N \). For \( b \in B \) and \( e \in E \), define

\[ \alpha(e) = (0, e) + N, \quad \beta(b) = (b, 0) + N. \]

Then \( \alpha h = \beta f \). Therefore, the square on the left is commutative and thus it is nonnil-commutative.

Define \( \gamma : M \to C \) by \( \gamma((b, e) + N) = \nu \mu g(b) \). If \( (b, e) \in N \), then there are \( a \in A \) and \( s_1 \in R \setminus \text{Nil}(R) \) such that \( s_1 f(a) = s_1 b \) and \( s_1 h(a) = -s_1 e \). So \( s_1 s_2 g(b) = s_1 s_2 g f(a) = 0 \) for some \( s_2 \in R \setminus \text{Nil}(R) \). Since \( D \) is a nonnil-torsion-free module, we have \( \mu g(b) = 0 \). Thus

\[ \gamma((b, e) + N) = \nu \mu g(b) = 0. \]

Therefore, \( \gamma \) is a well-defined map.

For any \( b \in B \), we have \( g(b) \in C \) and there exists an element \( s_3 \in R \setminus \text{Nil}(R) \) such that \( s_3 \nu g(b) = s_3 g(b) \). So

\[ s_3 \gamma \beta(b) = s_3 \nu g(b) = s_3 g(b). \]

Therefore, the square on the right is nonnil-commutative.

Since \( g \) is a nonnil-epimorphism, \( \gamma \) is also a nonnil-epimorphism.

If \( \alpha(e) = 0 \), then \( (0, e) \in N \). So there exist \( a \in A \) and \( s_4 \in R \setminus \text{Nil}(R) \) such that \( s_4 f(a) = 0 \) and \( s_4 h(a) = -s_4 e \). Since \( f \) is a nonnil-monomorphism, \( s_4 a = 0 \) for some \( s_5 \in R \setminus \text{Nil}(R) \). Thus \( -s_5 s_4 e = s_5 s_4 h(a) = 0 \) and hence \( e \in \text{Ntor}(E) \). Therefore, \( \alpha \) is a nonnil-monomorphism.

It is easy to see that \( \gamma \alpha = 0 \). Thus \( \text{NIm}(\alpha) \subseteq \text{NKer}(\gamma) \).

If \( s_6 \gamma((b, e) + N) = s_6 \nu g(b) = s_6 g(b) = 0 \) for some \( s_6 \in R \setminus \text{Nil}(R) \), then there are \( a \in A \) and \( s_7 \in R \setminus \text{Nil}(R) \) such that \( s_7 f(a) = s_7 b \). So

\[
\begin{align*}
(\gamma \alpha)(e + h(a)) + N &= s_7 f(a)(e + h(a)) + N \\
&= s_7 f(a) e + s_7 f(a) h(a) + N \\
&= s_7 \alpha e + s_7 \alpha h(a).
\end{align*}
\]

Therefore, \( \text{NKer}(\gamma) = \text{NIm}(\alpha) \), that is, the bottom row is nonnil-exact. \( \square \)
Theorem 2.10. The following diagram with a non-nil-exact row:

\[
\begin{array}{ccc}
E & \xrightarrow{n} & 0 \\
\downarrow & & \\
0 & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{n} & 0
\end{array}
\]

can be completed to the following non-nil-commutative diagram with non-nil-exact rows:

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{n} & 0 \\
\downarrow & & & & & & \\
0 & \xrightarrow{\alpha} & L & \xrightarrow{\gamma} & E & \xrightarrow{n} & 0
\end{array}
\]

Proof. Set \( L = \{ (b, e) \in B \oplus E \mid sg(b) = sh(e), s \in R \setminus \text{Nil}(R) \} \) and define \( \beta(b, e) = b, \gamma(b, e) = e, b \in B, e \in E. \)

Then trivially, for \((b, e) \in B \oplus E\), there exists \( s \in R \setminus \text{Nil}(R) \) such that \( sg\beta(b, e) = sh\gamma(b, e) \). Define \( \alpha : A \to L \) by \( \alpha(a) = (f(a), 0) \). Then \( \beta\alpha = f \).

Since \( f \) is a non-nil-monomorphism, \( \alpha \) is a non-nil-monomorphism.

For \( e \in E \), since \( g \) is a non-nil-epimorphism, we choose \( b \in B \) and \( s_1 \in R \setminus \text{Nil}(R) \) such that \( s_1g(b) = s_1h(e) \).

Therefore, \( \gamma \) is an epimorphism.

For \( a \in A \), we have \( \gamma\alpha(a) = \gamma(f(a), 0) = \gamma(0) = 0 \). And for \((b, e) \in \text{Nker}(\gamma)\), we have some \( s_2 \in R \setminus \text{Nil}(R) \) such that \( s_2\gamma(b, e) = s_2e = 0 \).

Thus, there exists some \( s_3 \in R \setminus \text{Nil}(R) \) such that \( s_2s_3g(b) = s_2s_3h(e) = 0 \).

Therefore, the top row is non-nil-exact. \( \square \)

Theorem 2.11. Consider the following diagram \( \Gamma \) of modules and homomorphisms

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow{\beta} & & \downarrow{f} \\
C & \xrightarrow{g} & D
\end{array}
\]

and the sequence \( \Delta \) of modules and homomorphisms

\[
0 \to A \xrightarrow{\sigma} B \oplus C \xrightarrow{\rho} D \to 0,
\]

where \( \sigma(a) = (\alpha(a), -\beta(a)) \) and \( \rho(b, c) = f(b) + g(c) \) for \( a \in A, b \in B, c \in C. \)
(a) The homomorphism $\rho$ is a nonnil-epimorphism if and only if
$$D = N\text{Im}(f) + N\text{Im}(g).$$

(b) The homomorphism $\sigma$ is a nonnil-monomorphism if and only if
$$\text{NKer}(\alpha) \cap \text{NKer}(\beta) = \text{Ntor}(A).$$

(c) The sequence $\Delta$ is a nonnil-complex if and only if the diagram $\Gamma$ is nonnil-commutative.

(d) $\text{NKer}(\rho) \subseteq N\text{Im}(\sigma)$ if and only if there exist $a \in A$ and $s \in R\backslash\text{Nil}(R)$ such that $sb = s\alpha(a)$ and $sc = s\beta(a)$ if $s'f(b) = s'g(c)$ for some $s' \in R\backslash\text{Nil}(R)$.

Proof. (a) If $\rho$ is a nonnil-epimorphism, then there exist $b \in B$, $c \in C$ and $s_1 \in R\backslash\text{Nil}(R)$ such that $s_1d = s_1\rho(b, c) = s_1f(b) + s_1g(c)$ for any element $d \in D$. Thus
$$d = f(b) + g(c) + t \in N\text{Im}(f) + N\text{Im}(g),$$
where $t \in \text{Ntor}(D)$. Therefore $D = N\text{Im}(f) + N\text{Im}(g)$.

Conversely, suppose $D = N\text{Im}(f) + N\text{Im}(g)$. For any $d \in D$, there exist $d_1 \in N\text{Im}(f)$ and $d_2 \in N\text{Im}(g)$ such that $d = d_1 + d_2$. Hence there exist $b, c \in B, c \in C$ and $s_2, s_3 \in R\backslash\text{Nil}(R)$ such that $s_2d_1 = s_2f(b)$ and $s_3d_2 = s_3g(c)$. Thus we have
$$s_2s_3d = s_2s_3(f(b) + g(c)) = s_2s_3\rho(b, c).$$

So $d \in \text{NIm}(\rho)$. Therefore, the homomorphism $\rho$ is a nonnil-epimorphism.

(b) It is clear that $\text{Ntor}(A) \subseteq \text{NKer}(\alpha) \cap \text{NKer}(\beta)$. If $a \in \text{NKer}(\alpha) \cap \text{NKer}(\beta)$, then there exist $s_4, s_5 \in R\backslash\text{Nil}(R)$ such that $s_4\alpha(a) = 0$, $s_5\beta(a) = 0$. So
$$s_4s_5\sigma(a) = s_4s_5(\alpha(a), -\beta(a)) = 0.$$
Since $\sigma$ is a nonnil-monomorphism, we have $a \in \text{Ntor}(A)$. Therefore, $\text{NKer}(\alpha) \cap \text{NKer}(\beta) = \text{Ntor}(A)$.

Conversely, if $s_6\sigma(a) = s_6(\alpha(a), -\beta(a)) = 0$ for some $s_6 \in R\backslash\text{Nil}(R)$, then $a \in \text{NKer}(\alpha) \cap \text{NKer}(\beta) = \text{Ntor}(A)$. Therefore, $\sigma$ is a nonnil-monomorphism.

(c) The sequence $\Delta$ is a nonnil-complex if and only if $N\text{Im}(\sigma) \subseteq \text{NKer}(\rho)$, if and only if there exists $s_7 \in R\backslash\text{Nil}(R)$ such that
$$s_7\rho\sigma(a) = s_7(f\alpha(a) - g\beta(a)) = 0,$$
if and only if the diagram $\Gamma$ is nonnil-commutative.

(d) Suppose that $\text{NKer}(\rho) \subseteq N\text{Im}(\sigma)$. If $s'f(b) = s'g(c)$ for some $s' \in R\backslash\text{Nil}(R)$, then $s'\rho(b, c) = 0$. Hence $(b, c) \in \text{NKer}(\rho) \subseteq N\text{Im}(\sigma)$. Thus there exist an element $a \in A$ and $s \in R\backslash\text{Nil}(R)$ such that $s(b, c) = s(\alpha(a), \beta(a))$.

Conversely, if $(b, c) \in \text{NKer}(\rho)$, that is, $s'f(b) = s'g(-c)$ for some $s' \in R\backslash\text{Nil}(R)$, then there exist $a \in A$ and $s \in R\backslash\text{Nil}(R)$ such that $sb = s\alpha(a)$ and $-sc = s\beta(a)$. Thus $s(b, c) = s(\alpha(a))$. Therefore $(b, c) \in N\text{Im}(\sigma)$, which implies that $\text{NKer}(\rho) \subseteq N\text{Im}(\sigma)$. □
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