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THE AUTOMORPHISM GROUPS OF ARTIN GROUPS OF EDGE-SEPARATED CLTTF GRAPHS

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Abstract. This work is a continuation of Crisp's work on automorphism groups of CLTTF Artin groups, where the defining graph of a CLTTF Artin group is connected, large-type, and triangle-free. More precisely, we provide an explicit presentation of the automorphism group of an edge-separated CLTTF Artin group whose defining graph has no separating vertices.

1. Introduction

1.1. CLTTF Artin groups

Let Γ be a simple graph such that every edge e carries an integer label $m(e) \geq 2$. An *Artin group* A_Γ with a defining graph Γ is generated by vertices of Γ and related by

$$\underbrace{sts\cdots}_{m(e)} = \underbrace{tst\cdots}_{m(e)}$$

for each edge e joining s and t. A set of generators is called that of *Artin generators* if a defining graph can be recovered by using them as vertices. For example, the 4-strand braid group is an Artin group defined by the triangle with edge labels 2, 3, 3. If all edge labels are 2, A_Γ is called a *right-angled Artin group*. An Artin group is *rigid* if it has a unique defining graph, or equivalently, if a set of Artin generators is sent to any other set of Artin generators by an automorphism of the Artin group. Right-angled Artin groups [5] and Artin groups of finite type [1] are known to be rigid. In general, Artin groups need not be rigid.

From now on, we fix a finite set V of vertices and assume that a graph Γ is edge-labeled whose set of edges is denoted by $E(\Gamma)$. Suppose that a graph Γ has two subgraphs Γ_1 and Γ_2 with intersection Γ_0 such that A_{Γ_0} is an Artin subgroup of finite type. In [2], the author proposes a typical way of obtaining a new defining graph from Γ under this circumstance. Recall that there is

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a unique element λ in A_{Γ_0}, which is the longest element in the associated Coxeter group, such that the conjugation by λ permutes elements of Γ_2. A new set S' obtained from V by replacing elements of $V(\Gamma_2)$ by their conjugates by λ generates A_Γ and then S' determines a new defining graph Δ that is called an *edge-twist* of Γ with respect to the triple $(\Gamma_1, \Gamma_0, \Gamma_2)$. In fact Δ is obtained from Γ by replacing each edges joining a vertex v in Γ_0 and a vertex w in Γ_2 by a new edge joining v and $\lambda w \lambda^{-1}$. We may identify $V(\Delta)$ with V since only edges are altered. There is an obvious isomorphism : $A_\Gamma \to A_\Delta$ called a *twist isomorphism*, that sends each $v \in V(\Gamma_2)$ to $\lambda v \lambda^{-1}$ and fixes other generators. It is a conjecture that two defining graphs of an Artin group are *twist-equivalent*, that is, related via a series of twists.

There have been extensive researches on automorphism groups of free abelian groups, free groups, and more generally, right-angled Artin groups. In particular isometric actions on appropriate spaces by outer automorphisms are studied to understand geometric structures of groups of outer automorphisms. There are also many complete results on automorphism groups of some Artin groups of finite type. Nielsen automorphisms or Whitehead automorphisms on free groups can be adapted to form a set of generators of automorphism groups when they are appropriate. They are usually classified as one of the following types: permutations of generators, inversions, transvections, and partial conjugations. For right-angled Artin groups, peak reduction arguments can be employed to obtain a complete set of relations among generators [3, 4].

On the other hand, there are very few results on automorphism groups of non-rigid Artin groups. J. Crisp gave the first noticeable result in [2]. He considered CLTTF Artin groups defined by graphs that are Connected, has edge labels ≥ 3 (Large Type), and is Triangle Free. CLTTF Artin groups form a somewhat manageable family of non-rigid Artin groups in studying their automorphism groups. In fact there are no transvections and twists occur only along edges with odd labels. Furthermore two defining graphs of a CLTTF group are twist-equivalent [2]. He showed that the isomorphism groupoid of a CLTTF Artin group is generated by graph automorphisms, inversions, partial conjugations, and twist isomorphisms. However given a CLTTF Artin group, it is not feasible to obtain a presentation of its automorphism group by using the groupoid since its automorphism can be given by any circuit including loops in the graph of groupoid.

1.2. Results

In this paper, we provide concrete and explicit description and group presentations of the (outer) automorphism group, whose generators are vertices, edge-twists, and certain graph isomorphisms. To this end, we first define a directed rooted tree $(\mathcal{C}\mathsf{h}_\Gamma, *_\Gamma)$ for each edge-separating CLTTF graph Γ, whose vertices are *chunks* and separating edges of Γ and whose edges are given by inclusions between separating edges and chunks. Roughly speaking, chunks are maximal indecomposable subgraphs. We call $(\mathcal{C}\mathsf{h}_\Gamma, *_\Gamma)$ called the *chunk tree*. See Definitions 2.2, 2.3 and Theorem 2.4.

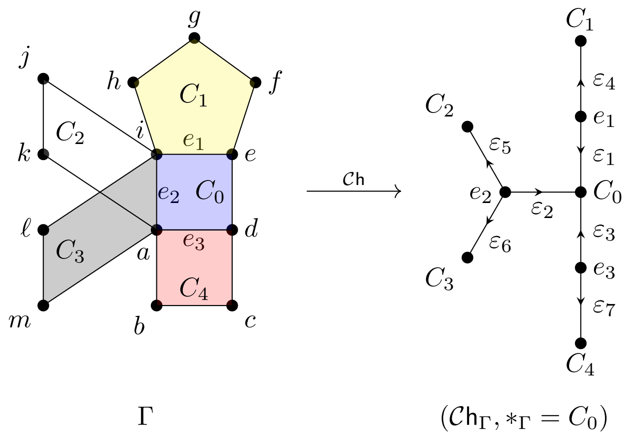

FIGURE 1. An edge-separated CLTTF graph Γ and its chunk tree $\mathcal{C}\mathsf{h}_\Gamma$

Then we may permute vertices or twist edges attached to each separating edge in order to obtain new CLTTF graphs. See Definition 2.9 and the following picture for what twisting edges means. We define a category $\mathcal{G}$ of CLTTF graphs whose morphisms correspond to these operations so that the automorphism group $\mathrm{Aut}_\mathcal{G}(\Gamma)$ of $\Gamma \in \mathcal{G}$ can be used to compute the (outer) automorphism group of the CLTTF Artin group A_Γ.

We say that a CLTTF graph Γ is *discretely rigid* if any non-trivial composition of edge-twists on Γ yields a CLTTF graph Δ, which is non-isomorphic to Γ.

Corollary 1.1 (Corollary 2.37)**.** *Let Γ be a discretely rigid CLTTF graph. Then the automorphism group $\mathrm{Aut}_\mathcal{G}(\Gamma)$ is the semidirect product of the free abelian group generated by edge-twists and the automorphism group of Γ:*

$$\mathrm{Aut}_\mathcal{G}(\Gamma) \cong \mathbb{Z}^{\#(\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]})} \rtimes \mathrm{Aut}(\Gamma).$$

We also define a category $\mathcal{A}$ of edge-separated CLTTF Artin groups with morphisms given by graph isomorphisms and partial conjugations, which are essentially part of generators that Crisp gave. Then the canonical assignment $\Gamma \mapsto A_\Gamma$ will define a functor $\mathcal{F} : \mathcal{G} \to \mathcal{A}$, which is indeed an equivalence of categories.

Theorem 1.2 (Theorem 3.9)**.** *The induced functor $\mathcal{F} : \mathcal{G} \to \mathcal{A}$ is an equivalence of categories.*

In particular, for each Γ, there is a group isomorphism

$$\mathrm{Aut}_{\mathcal{G}}(\Gamma) \cong \mathrm{Aut}_{\mathcal{A}}(\Gamma).$$

Theorem 1.3 (§ 4.1.2, Theorem 4.5)**.** *There is an isomorphism*

$$\mathrm{Out}(A_\Gamma) \cong (\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma) \rtimes \mathbb{Z}_2,$$

where

$$Z_\Gamma \coloneqq \mathrm{Inn}(A_\Gamma) \cap \mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \cong \begin{cases} 1 & *_\Gamma \text{ is a chunk;} \\ \mathbb{Z} & *_\Gamma \text{ is a separating edge.} \end{cases}$$

In particular, when $*_\Gamma$ is a chunk or Γ is discretely rigid, we have the following consequence.

Corollary 1.4 (Corollaries 4.6 and 4.7)**.** *If $*_\Gamma$ is a chunk, then*

$$\mathrm{Aut}(A_\Gamma) \cong \mathrm{Inn}(A_\Gamma) \rtimes \mathrm{Out}(A_\Gamma), \qquad \mathrm{Out}(A_\Gamma) \cong \mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \rtimes \mathbb{Z}_2.$$

Moreover, if Γ is discretely rigid and $*_\Gamma$ is a chunk, then

$$\begin{aligned} \mathrm{Aut}(A_\Gamma) &\cong \mathrm{Inn}(A_\Gamma) \rtimes \left(\left(\mathbb{Z}^{\#(\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]})} \rtimes \mathrm{Aut}(\Gamma) \right) \rtimes \mathbb{Z}_2 \right), \\ \mathrm{Out}(A_\Gamma) &\cong \left(\mathbb{Z}^{\#(\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]})} \rtimes \mathrm{Aut}(\Gamma) \right) \rtimes \mathbb{Z}_2. \end{aligned}$$

In general, explicit group presentations for both automorphism and outer automorphism groups of arbitrary edge-separated CLTTF Artin groups are given as follows:

Theorem 1.5 (Theorem 4.11)**.** *Let $\Gamma = (V, E, m)$ be a CLTTF graph. Then the automorphism group $\mathrm{Aut}(A_\Gamma)$ and outer automorphism group admit the following finite group presentations*:

$$\begin{aligned} \mathrm{Aut}(A_\Gamma) &\cong \left\langle V, S, \iota \;\middle|\; R_0, R_1, R_2, R_3, R_4, \widetilde{R}_\Phi \right\rangle, \\ \mathrm{Out}(A_\Gamma) &\cong \left\langle V, S, \iota \;\middle|\; R_0, R_1, R_2, R_3, R_4, \widetilde{R}_\Phi, V \right\rangle \\ &\cong \langle S, \iota \mid R_1, R_2, R_3, R_4, R_\Phi \rangle. \end{aligned}$$

Here, the sets $S, R_0, \ldots, R_4, \widetilde{R}_\Phi$ and R_Φ are briefly described in Table 1.

1.3. Outline

The rest of the paper is organized as follows. In Section 2, we review basics on CLTTF graphs including chunk trees, graph isomorphisms, edge-twists and their pull-backs and push-forwards. We also define the subgroup $\mathrm{Twist}(\Gamma)$ of the permutation group $\mathfrak{S}_V$ consisting of graph isomorphisms whose source and target are edge-twist equivalent. We further define the category $\mathcal{G}$ of CLTTF graphs whose morphisms are isomorphisms freely generated by graph isomorphisms and edge-twists.

In Section 3, we review CLTTF Artin groups and their isomorphisms, and define the category $\mathcal{A}$ of CLTTF Artin groups whose morphisms are generated

TABLE 1. The sets $S, R_0, R_1, R_2, R_3, R_4, \widetilde{R}_\Phi$ and R_Φ

S	even edge-twists and twist isomorphisms
R_0	relations coming from (the action on) $\mathrm{Inn}(A_\Gamma)$
R_1	even edge-twists commute with each other
R_2	twist isomorphisms conjugate even edge-twists to their push-forwards
R_3	twisted intersection products of products of twist isomorphisms
R_4	actions of the global inversion ι
$\widetilde{R}_\Phi$	the special automorphism in $\mathrm{Aut}(A_\Gamma)$
R_Φ	the special automorphism in $\mathrm{Out}(A_\Gamma)$

by graph isomorphisms and partial conjugations. We prove the equivalence between categories $\mathcal{G}$ and $\mathcal{A}$, and the relationship between our and Crisp's categories of CLTTF Artin groups is briefly explained.

In Section 4, we introduce the *twisted intersection product* between graph isomorphisms in $\mathrm{Twist}(\Gamma)$ and finally we provide the group presentations for both $\mathrm{Aut}(A_\Gamma)$ and $\mathrm{Out}(A_\Gamma)$.

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2. CLTTF Graphs

2.1. CLTTF graphs and chunk trees

Let $\Gamma = (V, E, m)$ be a simple graph with an edge-label $m : E \to \mathbb{Z}_{\geq 2}$. We call Γ CLTTF if it satisfies the following:

- it is Connected,
- it is of Large Type, i.e., $m(e) \geq 3$ for every edge $e \in E$,
- it is Triangle-Free, i.e., there are no full subgraphs of three vertices which look like a triangle.

A *decomposition* of Γ along a subgraph Γ_0 is a triple $(\Gamma_1, \Gamma_0, \Gamma_2)$ such that $\Gamma_i = (V_i, E_i)$'s are full subgraphs different from Γ_0 whose union and intersection are Γ and Γ_0, respectively,

$$\Gamma = (V, E) = (V_1 \cup V_2, E_1 \cup E_2) = \Gamma_1 \cup \Gamma_2,$$
$$\Gamma_0 = (V_0, E_0) = (V_1 \cap V_2, E_1 \cap E_2) = \Gamma_1 \cap \Gamma_2.$$

We call a vertex v or an edge e *separating* if there exists a decomposition with $\Gamma_0 = v$ or e and we say that a CLTTF graph Γ is *edge-separated* if there are no separating vertices. See Figure 1 for an example. Note that we usually suppress labels m on edges if $m = 3$.

Throughout this paper, we assume the following.

Assumption 2.1. We assume that every CLTTF graph is edge-separated, and for a finite fixed set V with $\#(V) \geq 3$, we define the set $\mathcal{G}$ of all CLTTF graphs with the set V of vertices.

Definition 2.2 (Chunk)**.** Let C be a connected full subgraph of $\Gamma \in \mathcal{G}$. We say that C is *indecomposable* if, for every decomposition (Γ_1, e, Γ_2) of Γ over a separating edge e, either $C \subset \Gamma_1$ or $C \subset \Gamma_2$.

By a *chunk* of Γ we mean a maximal indecomposable subgraph of Γ.

Notice that a chunk C of a CLTTF graph Γ is again a CLTTF graph with at least 3 vertices without separating vertices and edges. Moreover, any two chunks of Γ intersect, if at all, along a single separating edge. Hence we can construct a new graph from Γ consisting of chunks and separating edges as follows:

Definition 2.3 (Chunk graph)**.** The *chunk graph* $\mathcal{C}\mathsf{h}_\Gamma = (\mathcal{V}_\Gamma, \mathcal{E}_\Gamma, m_\Gamma)$ for $\Gamma = (V, E, m) \in \mathcal{G}$ is a directed edge-labeled graph constructed as follows:

- The set $\mathcal{V}_\Gamma$ of vertices consists of chunks and separating edges

  $$\mathcal{V}_\Gamma = \{C \subset \Gamma \mid C \text{ is a chunk}\} \cup \{e \subset \Gamma \mid e \text{ is a separating edge}\}.$$

- The set $\mathcal{E}_\Gamma$ of directed edges consists of pairs of a separating edge e and a chunk C for $e \subset C$

$$\mathcal{E}_\Gamma = \{(e, C) \mid e \subset C, e \text{ is a separating edge}, C \text{ is a chunk}\}, \quad m_\Gamma(e, C) = m(e).$$

As seen in Figure 1, one can easily observe the following: the chunk graph $\mathcal{C}\mathsf{h}_\Gamma$ is

- simple and connected,
- bipartite with respect to being a separating edge and being a chunk, and
- a tree whose leaves are chunks.

The first two observations are obviously true for any edge-separated CLTTF graphs by the construction of the chunk graph. Indeed, $\mathcal{C}\mathsf{h}_\Gamma$ has no multiple edges since any two chunks have at most one common edge, no edges connecting two chunks or two separating edges, and no loops.

The connectivity of Γ implies the connectivity of $\mathcal{C}\mathsf{h}_\Gamma$. Moreover, since each separating edge of Γ should be contained in at least two chunks of Γ, all univalent vertices of $\mathcal{C}\mathsf{h}_\Gamma$ are chunks.

Theorem 2.4. *The chunk graph $\mathcal{C}\mathsf{h}_\Gamma$ is a tree whose leaves are chunks of Γ, and we will call $\mathcal{C}\mathsf{h}_\Gamma$ the* chunk tree *for Γ.*

Proof. Suppose that $\mathcal{C}\mathsf{h}_\Gamma$ is not a tree. Since $\mathcal{C}\mathsf{h}_\Gamma$ is simple and bipartite, any embedded cycle in $\mathcal{C}\mathsf{h}_\Gamma$ has four or more vertices, of which more than one vertex correspond to separating edges. Therefore $\mathcal{C}\mathsf{h}_\Gamma$ can not be disconnected removing one separating edge of the cycle, which is a contradiction. $\square$

One of the direct consequence of the theorem is that $\mathcal{C}\mathsf{h}_\Gamma$ has a unique vertex $*_\Gamma$ such that any vertex in $\mathcal{C}\mathsf{h}_\Gamma$ is far from $*_\Gamma$ at most $\mathrm{Diam}(\mathcal{C}\mathsf{h}_\Gamma)/2$, where $\mathrm{Diam}(\mathcal{C}\mathsf{h}_\Gamma)$ is the diameter of $\mathcal{C}\mathsf{h}_\Gamma$ with respect to the edge-length. Hence the vertex $*_\Gamma$ plays the role of the *center* of $\mathcal{C}\mathsf{h}_\Gamma$.

Definition 2.5 (Center of the chunk tree)**.** We call the vertex $*_\Gamma$ the *center* of the chunk tree $\mathcal{C}\mathsf{h}_\Gamma$.

Let $\varepsilon = (e, C)$ be an edge of the chunk tree $\mathcal{C}\mathsf{h}_\Gamma$. By cutting ε in $\mathcal{C}\mathsf{h}_\Gamma$, we have two disjoint subgraphs $\mathcal{C}\mathsf{h}_{\Gamma,1}(\varepsilon)$ and $\mathcal{C}\mathsf{h}_{\Gamma,2}(\varepsilon)$ containing e and C, respectively. Then it induces a decomposition $(\Gamma_1(\varepsilon), e, \Gamma_2(\varepsilon))$ such that each $\Gamma_i(\epsilon)$ is the union of all chunks corresponding to vertices in $\mathcal{C}\mathsf{h}_{\Gamma,i}(\varepsilon)$. See Figure 2 for example.

Remark 2.6. Each edge in $\mathcal{C}\mathsf{h}_\Gamma$ induces a decomposition of Γ, but not every decomposition of Γ comes from an edge in $\mathcal{C}\mathsf{h}_\Gamma$.

2.2. Modifications of CLTTF graphs

We introduce two ways of modifications which are graph isomorphisms and edge-twists.

2.2.1. *Graph isomorphisms.* Let $\mathfrak{S}_V$ be the group of permutations on V and let $(\alpha : V \to V) \in \mathfrak{S}_V$. Then for each graph $\Gamma = (V, E_\Gamma, m_\Gamma)$, there exists a unique graph $\Delta = (V, E_\Delta, m_\Delta)$ such that the permutation α induces a graph isomorphism $\Gamma \to \Delta$, denoted by α again. Here we mean by a *graph isomorphism* α from Γ to Δ a permutation on V which preserves edges with labels, i.e., for every pair $\{s, t\} \subset V$, $s \neq t$,

$$\{s, t\} \in E_\Gamma \iff \{\alpha(s), \alpha(t)\} \in E_\Delta$$

and for each $e \in E_\Gamma$, $m_\Gamma(e) = m_\Delta(\alpha(e))$. We also denote by $\Gamma \cong \Delta$ if Γ and Δ are isomorphic as CLTTF graphs. For each $\Gamma \in \mathcal{G}$, let us denote $[\Gamma]$ the graph isomorphism class of Γ in $\mathcal{G}$:

$$[\Gamma] \coloneqq \{\Delta \in \mathcal{G} \mid \Delta \cong \Gamma\}.$$

Remark 2.7. The set $[\mathcal{G}] = \{[\Gamma] \mid \Gamma \in \mathcal{G}\}$ of isomorphism classes is the same as the set of CLTTF graphs up to isomorphism, or the set of unlabelled CLTTF graphs.

Since each graph isomorphism preserves the connectivity of subgraphs and labels, it maps chunks and separating edges to themselves, respectively. In other words, it preserves the chunk tree.

Theorem 2.8. *Let $\alpha \in \mathfrak{S}_V$. For each Γ and $\Delta = \alpha(\Gamma)$, there is an induced isomorphism between rooted trees*

$$\mathcal{C}\mathsf{h}(\alpha) : (\mathcal{C}\mathsf{h}_\Gamma, *_\Gamma) \to (\mathcal{C}\mathsf{h}_\Delta, *_\Delta).$$

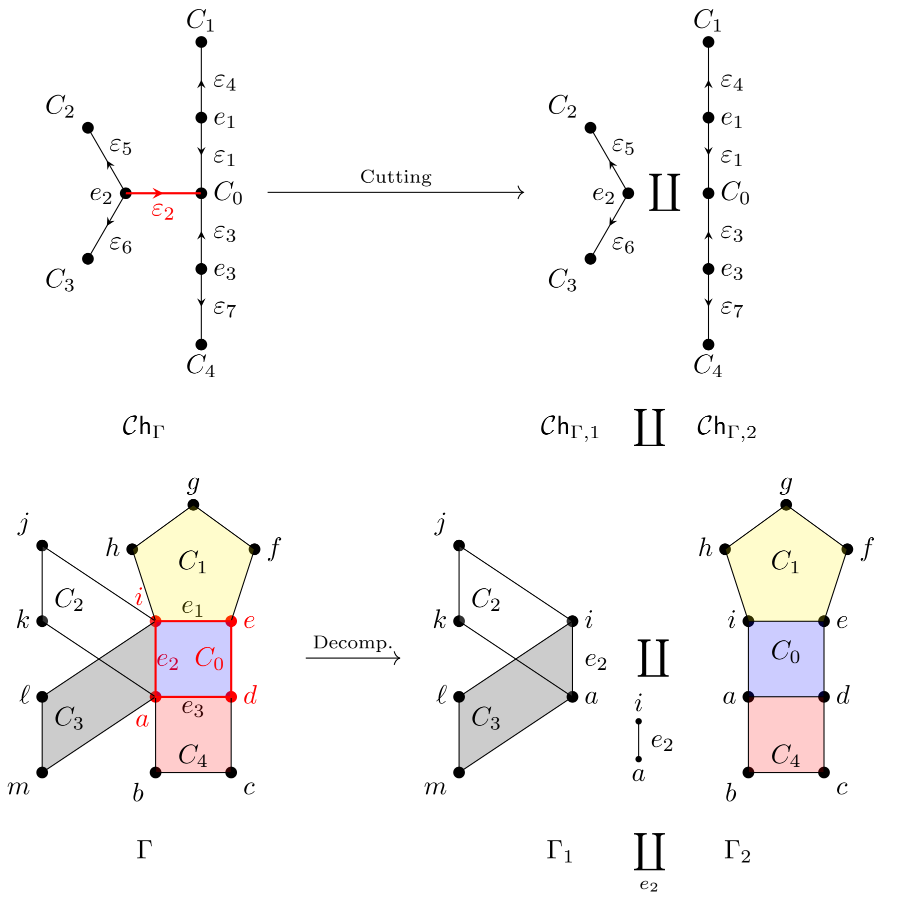


FIGURE 2. An edge in $\mathcal{C}\mathsf{h}_\Gamma$ and a decomposition

2.2.2. *Edge-twists.* Another way to obtain a new CLTTF graph is an *edge-twist.*

Definition 2.9 (Edge-twists)**.** Let $(\Gamma_1, e = \{s,t\}, \Gamma_2)$ be a decomposition of $\Gamma = (V, E_\Gamma, m_\Gamma)$. The *edge-twist* of Γ with respect to the decomposition (Γ_1, e, Γ_2) is the graph $\Delta = (V, E_\Delta, m_\Delta)$ with the label $m_\Delta : E_\Delta \to \mathbb{Z}_{\geq 2}$ obtained as follows:

- if $m_\Gamma(e)$ is even, then $(E_\Delta, m_\Delta) \coloneqq (E_\Gamma, m_\Gamma)$,
- if $m_\Gamma(e)$ is odd, then

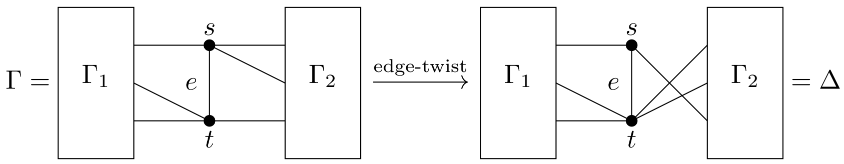
$$\begin{aligned} E_\Delta \coloneqq \{e\} &\cup \{f \mid f \in E_\Gamma, f \not\subset \Gamma_2 \text{ or } f \cap e = \varnothing\} \\ &\cup \{\{v,s\},\{w,t\} \mid v,w \in \Gamma_2, \{v,t\},\{w,s\} \in E_\Gamma\}, \end{aligned}$$

$$m_\Delta(f) \coloneqq \begin{cases} m_\Gamma(e) & f = e; \\ m_\Gamma(f) & f \not\subset \Gamma_2 \text{ or } f \cap e = \varnothing; \\ m_\Gamma(\{v,s\}) & f = \{v,t\}, v \in \Gamma_2; \\ m_\Gamma(\{w,t\}) & f = \{w,s\}, w \in \Gamma_2. \end{cases}$$

We say that Γ and Δ are *edge-twist equivalent* if Δ is obtained from Γ by a sequence of edge-twists, denoted by $\Gamma \sim \Delta$ and let $[\![\Gamma]\!]$ be an edge-twist equivalence class of an edge-separated CLTTF graph Γ:

$$[\![\Gamma]\!] \coloneqq \{\Delta \mid \Gamma \sim \Delta\}.$$

Roughly speaking, the edge-twist along $(\Gamma_1, e = \{s,t\}, \Gamma_2)$ with $m(e)$ odd will interchange the connectivities with s and t only for vertices in Γ_2. An intuitive example is depicted as follows:



Lemma 2.10. *Let* (Γ_1, e, Γ_2) *be a decomposition of* Γ *and* Δ *be the edge-twist* Δ *of* Γ *with respect to* (Γ_1, e, Γ_2)*. If* $m(e)$ *is odd, then* $\Delta \neq \Gamma$*.*

Proof. This is obvious since Γ is triangle-free. $\square$

As mentioned above, each edge $\varepsilon = (e, C) \in \mathcal{E}_\Gamma$ in the chunk tree gives us a decomposition $(\Gamma_1(\varepsilon), e, \Gamma_2(\varepsilon))$, whose corresponding edge-twist will be denoted by $\bar{\varepsilon}$. If we obtain a new graph Δ, then we write

$$\Delta = \bar{\varepsilon}(\Gamma), \qquad \text{or} \qquad \bar{\varepsilon} : \Gamma \to \Delta.$$

Notice that the decomposition (Γ_1, e, Γ_2) of Γ can be regarded as a decomposition of Δ as well. Therefore, chunks and separating edges in $\Gamma_i \subset \Gamma$ are again chunks and separating edges in $\Gamma_i \subset \Delta$. Namely, we have an induced isomorphism between chunk trees.

Theorem 2.11. *Let* (Γ_1, e, Γ_2) *be a decomposition of* Γ*. The edge-twist* $\Gamma \to \Delta$ *with respect to the decomposition* (Γ_1, e, Γ_2) *induces an isomorphism* $(\mathcal{C}\mathsf{h}_\Gamma, *_\Gamma) \to (\mathcal{C}\mathsf{h}_\Delta, *_\Delta)$ *between rooted trees.*

In particular, for each $\varepsilon \in \mathcal{E}_\Gamma$*, the induced isomorphism will be denoted by* $\mathcal{C}\mathsf{h}(\varepsilon)$*:*

$$\mathcal{C}\mathsf{h}(\varepsilon) : (\mathcal{C}\mathsf{h}_\Gamma, *_\Gamma) \to (\mathcal{C}\mathsf{h}_\Delta, *_\Delta).$$

Proof. This follows obviously from the above discussion and we omit the proof. $\square$

The direct consequence of this theorem is as follows:

Corollary 2.12. *Let Γ be a CLTTF graph. Then the pair $(\mathcal{C}\mathsf{h}_{[\![\Gamma]\!]}, *_{[\![\Gamma]\!]})$ of the chunk tree $\mathcal{C}\mathsf{h}_{[\![\Gamma]\!]}$ and the central vertex $*_{[\![\Gamma]\!]}$ for the class $[\![\Gamma]\!]$ is well-defined.*

In other words, there is a canonical identification $(\mathcal{C}\mathsf{h}_\Gamma, *_\Gamma) \cong (\mathcal{C}\mathsf{h}_\Delta, *_\Delta)$ if $\Gamma \sim \Delta$.

Furthermore, each edge-twist $\bar{\varepsilon} : \Gamma \to \Delta$ induces a label-preserving bijection $\bar{\varepsilon} : (E_\Gamma, m_\Gamma) \to (E_\Delta, m_\Delta)$. That is, for each edge $e \in E_\Gamma$, we have an edge $\bar{\varepsilon}(e)$ in Δ so that $m_\Gamma(e) = m_\Delta(\bar{\varepsilon}(e))$. In particular, for each separating edge $e \subset E_\Gamma$, then its label remains the same in its edge-twist equivalence class. Moreover, for each edge $\varepsilon = (e, C) \in \mathcal{E}_{[\![\Gamma]\!]}$, the label $m_{[\![\Gamma]\!]}(\varepsilon)$ is well-defined as $m_\Gamma(e)$.

Due to Remark 2.6, edge-twists with respect to arbitrary decompositions of Γ form a strictly larger class than those with respect to decompositions coming from edges in the chunk tree $\mathcal{C}\mathsf{h}_\Gamma$. However, one can easily check that every edge-twist is actually a composition of the latter edge-twists. See Figure 3 for example. Note that in Figure 3, the chunk trees are identified in the obvious way.

More precisely, let $e \subset \Gamma$ be an odd-labeled separating edge and let $C_1, \ldots, C_n \subset \Gamma$ be all chunks containing e. In the chunk tree $\mathcal{C}\mathsf{h}_\Gamma$, the vertex e is of n-valent and edges $\varepsilon_i = (e, C_i)$ are adjacent to e. Suppose that a decomposition (Γ_1, e, Γ_2) is given such that for some $\ell < n$. That is,

$$C_1, \ldots, C_\ell \subset \Gamma_1 \quad \text{and} \quad C_{\ell+1}, \ldots, C_n \subset \Gamma_2.$$

Then the edge-twist with respect to (Γ_1, e, Γ_2) is nothing but the composition

$$\bar{\varepsilon}_{\ell+1}\bar{\varepsilon}_\ell \cdots \bar{\varepsilon}_n : \Gamma \to \Delta.[1]$$

In this sense, it is enough to consider edge-twists along edges in the chunk tree. Then one can easily check that for every $i, j \in \{1, \ldots, n\}$,

$$\bar{\varepsilon}_i(\bar{\varepsilon}_j(\Gamma)) = \bar{\varepsilon}_j(\bar{\varepsilon}_i(\Gamma)) \quad \text{and} \quad \bar{\varepsilon}_i(\bar{\varepsilon}_i(\Gamma)) = \Gamma. \tag{2.1}$$

On the other hand, edge-twists along all edges in $\mathcal{C}\mathsf{h}_\Gamma$ are sometimes too many. If we take edge-twists on Γ with respect to all ε_i's, then the result graph

$$\Delta = (\bar{\varepsilon}_1\bar{\varepsilon}_2 \cdots \bar{\varepsilon}_n)(\Gamma)$$

is obtained from Γ by interchanging the roles of two vertices of e. That is, there is a graph isomorphism $\alpha \in \mathfrak{S}_V$ with $\alpha(\Gamma) = \Delta$ defined by

$$\alpha(v) = \begin{cases} v & v \neq s, t; \\ t & v = s; \\ s & v = t. \end{cases}$$

Therefore, up to graph isomorphisms, one may reduce one of edge-twists $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_n$.

[1] Here we omit the notation $\circ$ for compositions.

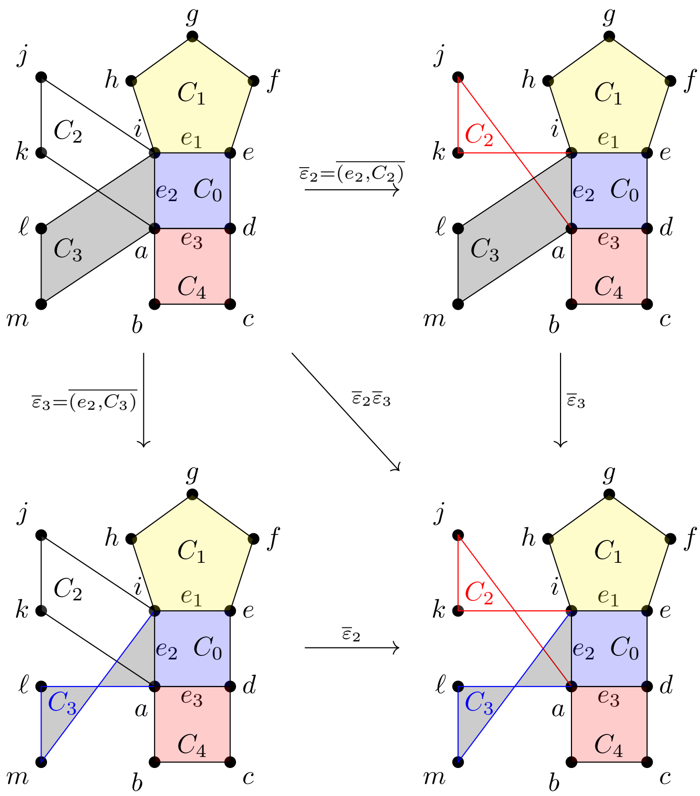


FIGURE 3. A composition of edge-twists

We will provide a systematical way of doing this as follows: recall that $\mathcal{C}\mathsf{h}_\Gamma$ has the central vertex $*_\Gamma$. Then we have another orientation on edges of $\mathcal{C}\mathsf{h}_\Gamma$ given by the *away-from-center* convention. We say that an edge $\varepsilon = (e, C) \in \mathcal{E}_\Gamma$ is *outward* or *inward* if C is farther or closer than e from $*_\Gamma$, respectively. We denote the subset of outward and inward edges in $\mathcal{C}\mathsf{h}_\Gamma$ by $\mathcal{E}_\Gamma^{\mathsf{out}}$ and $\mathcal{E}_\Gamma^{\mathsf{in}}$, respectively.

For example, the chunk tree $\mathcal{C}\mathsf{h}_\Gamma$ in Figure 1 has three inward edges $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and four outward edges $\varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7$.

$$\mathcal{E}_\Gamma^{\mathsf{in}} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, \qquad \mathcal{E}_\Gamma^{\mathsf{out}} = \{\varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}.$$

One simple but important observation is as follows: in the chunk tree $\mathcal{C}\mathsf{h}_\Gamma$ and a separating edge e, there are at most one inward edge adjacent to e. Indeed, the case without inward edge is when $*_\Gamma = e$ is the central vertex of $\mathcal{C}\mathsf{h}_\Gamma$.

In summary, it suffices to consider edge-twists with respect to outward edges in $\mathcal{C}\mathsf{h}_\Gamma$ since every edge-twist is a combination of those and graph isomorphisms.

Remark 2.13. Even though we consider outward edges only in $\mathcal{C}\mathsf{h}_\Gamma$, there are still possibilities that compositions of edge-twists involving different separating edges become graph isomorphisms as well. We will consider these cases later in Section 2.3.

Let $\varepsilon \in \mathcal{E}_\Gamma$ and (Γ_1, e, Γ_2) be the decomposition corresponding to ε. For a sake of convenience, we define for each $i = 1, 2$,

$$\Gamma_i(\varepsilon) \coloneqq \Gamma_i \subset \Gamma \quad \text{and} \quad V_i(\varepsilon) \coloneqq V(\Gamma_i) \subset V.$$

Lemma 2.14. *Edge-twists along outward edges in the chunk tree are commutative and involutive.*

Proof. By definition of edge-twists, it is obvious that for $\varepsilon \in E^{\mathsf{out}}(\mathcal{C}\mathsf{h}_{[\![\Gamma]\!]})$ and any $\Delta \sim \Gamma$,

$$\overline{\varepsilon}^2(\Delta) = (\overline{\varepsilon}\overline{\varepsilon})(\Delta) = \Delta \tag{2.2}$$

and so edge-twists along outward edges are involutive.

Let $\varepsilon = (e, C), \varepsilon' = (e', C') \in \mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]}$. If $e = e'$ or $e \cap e' = \emptyset$, then corresponding edge-twists will commute. Otherwise, there are three cases (i) $C \not\subset \Gamma_2(\varepsilon')$ and $C' \not\subset \Gamma_2(\varepsilon)$, (ii) $C \not\subset \Gamma_2(\varepsilon')$ and $C' \subset \Gamma_2(\varepsilon)$, and (iii) $C \subset \Gamma_2(\varepsilon')$ and $C' \not\subset \Gamma_2(\varepsilon)$, according to whether $C \subset \Gamma_2(\varepsilon')$ or $C' \subset \Gamma_2(\varepsilon)$ for decomposition $(\Gamma_1(\varepsilon), e, \Gamma_2(\varepsilon))$ and $(\Gamma_1(\varepsilon'), e', \Gamma_2(\varepsilon'))$ induced from ε and ϵ', respectively.

Notice that when $C \subset \Gamma_2(\varepsilon')$ and $C' \subset \Gamma_2(\varepsilon)$, then $C = C'$ and at least one of ε and ε' should be *inward*.

For each case, edge-twists are commutative as shown in Figure 4 and we omit the detail. □

Remark 2.15. We remark that for any $\varepsilon, \varepsilon' \in \mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]}$, two sets $V \setminus V_1(\varepsilon)$ and $V \setminus V_1(\varepsilon')$ are either disjoint or nested.

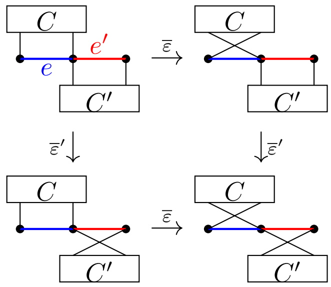
Remark 2.16. We can exclude the case (i) corresponding to the schematic picture depicted in Figure 4a since it always contains a separating vertices $v = e \cap e'$, but the commutativity still holds for this situation as well.

For each $i \in \{0, 1\}$, let us define the subset $\mathcal{E}^{\mathsf{out},i}_{[\![\Gamma]\!]}$ of $\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]}$ as

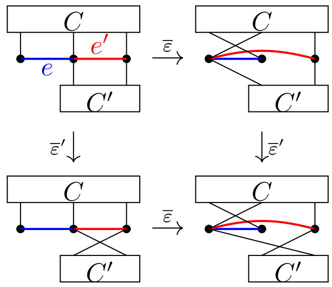
$$\mathcal{E}^{\mathsf{out},i}_{[\![\Gamma]\!]} = \{\varepsilon \in \mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]} \mid m_{[\![\Gamma]\!]}(\varepsilon) \equiv i \mod 2\},$$

and for each function $\overline{\eta} : \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]} \to \mathbb{Z}_2$, we define the composition $\overline{\mathcal{E}}(\overline{\eta})$ of edge-twists as

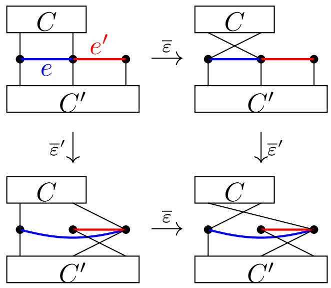
$$\overline{\mathcal{E}}(\overline{\eta}) = \prod_{\varepsilon \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]}} \overline{\varepsilon}^{\overline{\eta}(\varepsilon)}.$$



(a) $C \not\subset \Gamma_2(\varepsilon')$ and $C' \not\subset \Gamma_2(\varepsilon)$



(b) $C \not\subset \Gamma_2(\varepsilon')$ and $C \subset \Gamma_2(\epsilon)$



(c) $C \subset \Gamma_2(\varepsilon')$ and $C' \not\subset \Gamma_2(\varepsilon)$

FIGURE 4. Commutativity of edge-twists

Proposition 2.17. *For each $\Delta \in [\![\Gamma]\!]$, there exists a unique $\overline{\eta} : \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]} \to \mathbb{Z}_2$ with $\overline{\mathcal{E}}(\overline{\eta}) : \Gamma \to \Delta$.*

Proof. By definition of edge-twist equivalence, the existence is obvious. Let $\overline{\eta}$ and $\overline{\eta}'$ be two such functions. That is,

$$\overline{\mathcal{E}} = \overline{\mathcal{E}}(\overline{\eta}) : \Gamma \to \Delta \quad \text{and} \quad \overline{\mathcal{E}}' = \overline{\mathcal{E}}(\overline{\eta}') : \Gamma \to \Delta.$$

We define $(\overline{\eta} + \overline{\eta}') : \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]} \to \mathbb{Z}_2$ as the point-wise addition over $\mathbb{Z}_2$

$$(\overline{\eta} + \overline{\eta}')(\varepsilon) \coloneqq \overline{\eta}(\varepsilon) + \overline{\eta}'(\varepsilon) = \begin{cases} 0 & \overline{\eta}(\varepsilon) = \overline{\eta}'(\varepsilon); \\ 1 & \overline{\eta}(\varepsilon) \neq \overline{\eta}'(\varepsilon), \end{cases}$$

and then by Lemma 2.14, we have $\overline{\mathcal{E}}'^{-1}\overline{\mathcal{E}} = \overline{\mathcal{E}}(\overline{\eta} + \overline{\eta}') : \Gamma \to \Gamma$.

Now let $\varepsilon = (e = \{s,t\}, C = (V_C, E_C)) \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]}$ be one of the farthest odd-labeled edge from the center $*_{[\![\Gamma]\!]}$ with $(\overline{\eta} + \overline{\eta}')(\varepsilon) = 1$. Pick an edge $\{v,s\} \in E_C$ with $v \neq s,t$. Then by definition of edge-twist, $\{v,t\} \in E_C$. This is a contradiction since the set $\{v,s,t\}$ of vertices forms a triangle but Γ is triangle-free. Hence there are no such ε and so $(\overline{\eta} + \overline{\eta}') \equiv 0$, or equivalently, $\overline{\eta}$ coincides with $\overline{\eta}'$. $\square$

2.2.3. *Pull-backs and push-forwards.* Now let us describe how graph isomorphisms and edge-twists interact with each other.

(a) Let $\alpha : \Gamma \to \Delta$ be a graph isomorphism and $\varepsilon = (e, C) \in \mathcal{E}_\Gamma$ with $\Gamma' = \overline{\varepsilon}(\Gamma)$. We define the *push-forwards* $\alpha_*(\overline{\varepsilon}) : \Delta \to \Delta'$ of $\overline{\varepsilon}$ via α as the edge-twist $\alpha_*(\overline{\varepsilon}) \coloneqq \overline{\alpha_*(\varepsilon)}$, where

$$\alpha_*(\varepsilon) \coloneqq \mathcal{C}\mathsf{h}(\alpha)(\varepsilon) = (\alpha(e), \alpha(C)) \in \mathcal{E}_\Delta.$$

Then it is obvious that $\alpha \in \mathfrak{S}_V$ induces a graph isomorphism $\alpha : \Gamma' \to \Delta'$, which fits into the diagram in Figure 5a.

(b) Let $\overline{\varepsilon}' = \overline{(e', C')} : \Delta \to \Delta'$ be an edge-twist. Then the *pull-back* $\alpha^*(\overline{\varepsilon}')$ of ε' via α is defined as the edge-twist $\alpha^*(\overline{\varepsilon}) \coloneqq \overline{\alpha^*(\varepsilon)}$, where

$$\alpha^*(\varepsilon') = \alpha_*^{-1}(\varepsilon') = (\alpha^{-1}(e'), \alpha^{-1}(C)) \in \mathcal{E}_\Gamma.$$

Then as before, the permutation $\alpha' \in \mathfrak{S}_V$ induces a graph isomorphism $\alpha : \Gamma' \to \Delta'$. See Figure 5b.

(c) For a graph isomorphism $\alpha' : \Gamma' \to \Delta'$ and an edge-twist $\overline{\varepsilon} : \Gamma \to \Gamma'$, we define $\Delta = \alpha'(\Gamma)$. Then the push-forward $\alpha_*(\overline{\varepsilon}) : \Delta \to \Delta'$ is the edge-twist, which fits into the diagram in Figure 5c.

(d) For an edge-twist $\overline{\varepsilon}' : \Delta \to \Delta'$, by using the inverse α'^{-1} as before, there exist a graph $\Gamma = \alpha^{-1}(\Delta')$ and an edge-twist $\alpha^*(\overline{\varepsilon}') : \Gamma \to \Gamma'$. See Figure 5d.

Example 2.18. Recall the graph Γ in Figure 1. Let $(\alpha_0 : \Gamma \to \Gamma) \in \mathrm{Aut}(\Gamma)$ be a graph automorphism which switches vertices j and k with ℓ and m,

$$\alpha_0(j) = \ell, \qquad \alpha_0(\ell) = j, \qquad \alpha_0(k) = m, \qquad \alpha_0(m) = k,$$

and let $\overline{\varepsilon} = \overline{(e_2, C_2)} : \Gamma \to \Gamma'$ be an edge-twist. Then we have a CLTTF graph $\Delta' = \alpha_0(\Gamma')$ and an edge-twist $(\alpha_0)_*(\overline{\varepsilon}) = \overline{(e_2, C_3)} : \Delta \to \Delta'$ as depicted in Figure 6.

Remark 2.19. Notice that in the previous example, the graph isomorphism $\alpha_0 : \Gamma' \to \Delta'$ is not an automorphism anymore, i.e., $E_{\Gamma'} \neq E_{\Delta'}$.

2.2.4. *Discrete rigidities.* We consider the following rigidities of CLTTF graphs. Recall that two edge-twist equivalent graphs have the same set of vertices.

Definition 2.20 (Rigidity of CLTTF graphs)**.** A CLTTF graph Γ is said to be

(1) *rigid* if

$$\Gamma \sim \Delta \Longrightarrow \Gamma \cong \Delta, \quad \text{ or equivalently,} \quad [\![\Gamma]\!] \cap [\Gamma] = [\![\Gamma]\!],$$

(2) *discretely rigid* if

$$\Gamma \sim \Delta \text{ and } \Gamma \cong \Delta \Longrightarrow \Gamma = \Delta, \quad \text{ or equivalently,} \quad [\![\Gamma]\!] \cap [\Gamma] = \{\Gamma\}.$$

Remark 2.21. One can see these rigidity as follows: rigid if and only if $[\![\Gamma]\!]$ up to graph isomorphism is a singleton, and discretely rigid if $[\![\Gamma]\!]$ up to graph isomorphism is the same as $[\![\Gamma]\!]$ itself.

$$\begin{tikzcd} \Gamma \arrow[r, "\alpha"] \arrow[d, "\overline{\varepsilon}"'] & \Delta \\ \Gamma' & \end{tikzcd} \Longrightarrow \begin{tikzcd} \Gamma \arrow[r, "\alpha"] \arrow[d, "\overline{\varepsilon}"'] & \Delta \arrow[d, "\alpha_*(\overline{\varepsilon})"] \\ \Gamma' \arrow[r, "\alpha"] & \alpha(\Gamma') \end{tikzcd}$$

(a)

$$\begin{tikzcd} \Gamma \arrow[r, "\alpha"] & \Delta \arrow[d, "\overline{\varepsilon}'"] \\ & \Delta' \end{tikzcd} \Longrightarrow \begin{tikzcd} \Gamma \arrow[r, "\alpha"] \arrow[d, "\alpha^*(\overline{\varepsilon}')"'] & \Delta \arrow[d, "\overline{\varepsilon}'"] \\ \alpha^{-1}(\Delta') \arrow[r, "\alpha"] & \Delta' \end{tikzcd}$$

(b)

$$\begin{tikzcd} \Gamma \arrow[d, "\varepsilon"'] & \\ \Gamma' \arrow[r, "\alpha'"] & \Delta' \end{tikzcd} \Longrightarrow \begin{tikzcd} \Gamma \arrow[r, "\alpha'"] \arrow[d, "\overline{\varepsilon}"'] & \alpha'(\Gamma) \arrow[d, "\alpha'_*(\overline{\varepsilon})"] \\ \Gamma' \arrow[r, "\alpha'"] & \Delta' \end{tikzcd}$$

(c)

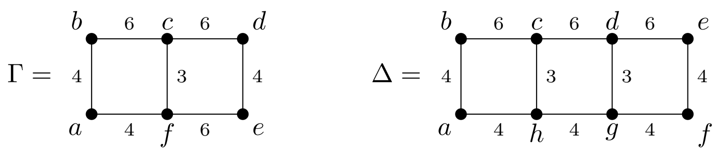
$$\begin{tikzcd} & \Delta \arrow[d, "\overline{\varepsilon}'"] \\ \Gamma' \arrow[r, "\alpha'"] & \Delta' \end{tikzcd} \Longrightarrow \begin{tikzcd} \alpha'^{-1}(\Delta) \arrow[r, "\alpha'"] \arrow[d, "\alpha^*(\overline{\varepsilon}')"'] & \Delta \arrow[d, "\overline{\varepsilon}'"] \\ \Gamma' \arrow[r, "\alpha'"] & \Delta' \end{tikzcd}$$

(d)

Figure 5. Push-forwards and pull-backs

There are examples of rigid but not discretely rigid CLTTF graphs, and *vice versa.*

Example 2.22. The CLTTF graph Γ below is rigid but not discretely rigid, while Δ is discretely rigid but not rigid:



Lemma 2.23. *Let $\Gamma = (V, E, m)$ be a rigid and discretely rigid CLTTF graph. Then for each separating edge e, the label $m(e)$ is even.*

Proof. Since Γ is rigid and discretely rigid, $\Gamma \sim \Delta$ implies $\Gamma = \Delta$. However, if there is an odd-labelled edge $e \in E$, then an edge-twist involving e yields an edge-twist equivalent graph Δ different from Γ as mentioned earlier. This contradiction completes the proof. $\square$

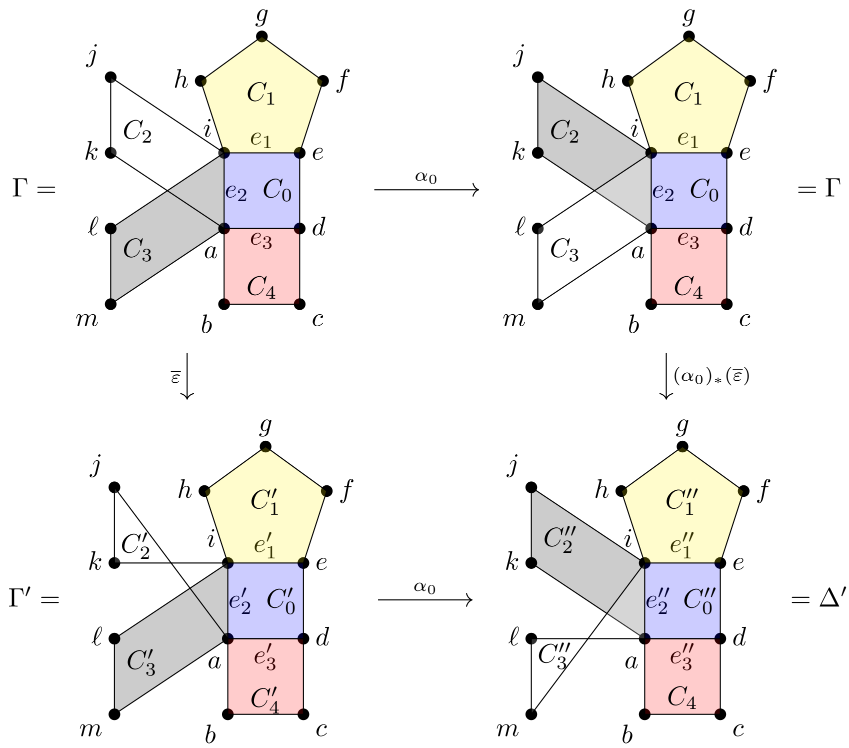


FIGURE 6. An example of push-forward

2.3. The group Twist(Γ)

As mentioned earlier in Remark 2.13, we will consider graph isomorphisms which can be expressed as compositions of edge-twists as well in this section. Namely, those are graph isomorphisms between edge-twist equivalent graphs. Let Δ be a graph which is edge-twist equivalent to and isomorphic to Γ. Namely,

$$\Delta \sim \Gamma \quad \text{and} \quad \Delta \cong \Gamma \quad \text{or equivalently,} \quad \Delta \in [\![\Gamma]\!] \cap [\Gamma].$$

Hence there are two ways of obtaining Δ from Γ so that for some $\alpha \in \mathfrak{S}_V$ and $\overline{\eta} : \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]} \to \mathbb{Z}_2$,

$$\Delta = \overline{\mathcal{E}}(\overline{\eta})(\Gamma) = \alpha(\Gamma).$$

We define the set Twist(Γ) consisting of such graph isomorphisms

$$\mathrm{Twist}(\Gamma) = \{\alpha \in \mathfrak{S}_V \mid \Gamma \sim \alpha(\Gamma)\} \subset \mathfrak{S}_V.$$

Remark 2.24. By Proposition 2.17, the composition $\overline{\mathcal{E}}$ is uniquely determined only by Δ. Hence we will not lose any information even though we throw out $\overline{\mathcal{E}}$.

Lemma 2.25. *The set* $\mathrm{Twist}(\Gamma)$ *is closed under the composition.*

Proof. For $i = 1, 2$, let $\alpha_i \in \mathfrak{S}_V$ with $\alpha_i(\Gamma) = \Delta_i$ be elements in $\mathrm{Twist}(\Gamma)$. Then we need to show that $\Gamma \sim \Delta = \alpha(\Gamma)$, where $\alpha = \alpha_2\alpha_1$ is the composition.

By definition of $\mathrm{Twist}(\Gamma)$ and Proposition 2.17, there exist unique compositions $\overline{\mathcal{E}}_{\alpha_1}$ and $\overline{\mathcal{E}}_{\alpha_2}$ of edge-twists such that $\overline{\mathcal{E}}_{\alpha_i} : \Gamma \to \Delta_i$ for $i = 1, 2$. Then as seen in the previous section, we have the push-forward $\overline{\mathcal{E}}'_{\alpha_1} \coloneqq (\alpha_2)_*(\overline{\mathcal{E}}_{\alpha_1}) : \Delta_2 \to \Delta$ for $\Delta = \alpha_2(\Delta_1)$, which fits into the diagram in Figure 7a. Therefore the graph Δ is edge-twist equivalent to Γ via the composition

$$\overline{\mathcal{E}}_\alpha \coloneqq \overline{\mathcal{E}}'_{\alpha_1}\overline{\mathcal{E}}_{\alpha_2} : \Gamma \to \Delta$$

and we are done. $\square$

Theorem 2.26. *The set* $\mathrm{Twist}(\Gamma)$ *has a group structure with respect to the composition.*

Proof. Obviously, the identity isomorphism $\mathrm{Id} : \Gamma \to \Gamma$ is in $\mathrm{Twist}(\Gamma)$ and plays the role of the identity under the composition.

Let $\alpha \in \mathfrak{S}_V$ be in $\mathrm{Twist}(\Gamma)$ and let $\overline{\mathcal{E}}_\alpha : \Gamma \to \alpha(\Gamma)$ be a unique composition of edge-twists by Proposition 2.17. We will consider the inverse $\alpha^{-1} \in \mathfrak{S}_V$. Since $\alpha^{-1}\alpha = \mathrm{Id}$, it suffices to show that $\alpha^{-1}(\Gamma) \sim \Gamma$. Indeed, as seen in the diagram in Figure 7b, the graph isomorphism $\alpha^{-1}(\Gamma)$ is edge-twist equivalent to Γ via $\alpha_*^{-1}(\overline{\mathcal{E}}_\alpha)$ and so $\alpha^{-1} \in \mathrm{Twist}(\Gamma)$.

Finally, since the composition is associative, the set $\mathrm{Twist}(\Gamma)$ has a group structure as claimed. $\square$

$$\begin{tikzcd}
\Gamma \arrow[r, "\alpha_1"] & \Delta_1 \arrow[r, "\alpha_2"] & \Delta \\
 & \Gamma \arrow[u, "\overline{\mathcal{E}}_{\alpha_1}"] \arrow[r, "\alpha_2"] & \Delta_2 \arrow[u, "(\alpha_2)_*(\overline{\mathcal{E}}_{\alpha_1})"'] \\
 & & \Gamma \arrow[u, "\overline{\mathcal{E}}_{\alpha_2}"']
\end{tikzcd}$$

(a) Composition

$$\begin{tikzcd}
\Gamma \arrow[r, "\alpha"] & \Delta \arrow[r, "\alpha^{-1}"] & \Gamma \\
 & \Gamma \arrow[u, "\overline{\mathcal{E}}_{\alpha}"] \arrow[r, "\alpha^{-1}"] & \alpha^{-1}(\Gamma) \arrow[u, "\alpha_*^{-1}(\overline{\mathcal{E}}_{\alpha})"']
\end{tikzcd}$$

(b) Inverse

FIGURE 7. A composition and inverse of two graph isomorphisms in $\mathrm{Twist}(\Gamma)$

Remark 2.27. The group $\mathrm{Twist}(\Gamma)$ is isomorphic to the hom-set of the twist equivalence groupoid $\mathit{Twist}(\mathcal{G})(\Gamma, \Gamma)$ described in [2].

One can check easily that the (graph) automorphism group $\mathrm{Aut}(\Gamma)$ of Γ is a subgroup of $\mathrm{Twist}(\Gamma)$ so that

$$[\mathrm{Twist}(\Gamma) : \mathrm{Aut}(\Gamma)] = \#(\llbracket\Gamma\rrbracket \cap [\Gamma]). \tag{2.3}$$

However, it is not necessarily normal in general.

Corollary 2.28. *For a discretely rigid graph* Γ*, we have* $\mathrm{Twist}(\Gamma) \cong \mathrm{Aut}(\Gamma)$.

Proof. The hypothesis implies that for each $\alpha \in \mathrm{Twist}(\Gamma)$, we have $\alpha(\Gamma) = \Gamma$, i.e., $\alpha \in \mathrm{Aut}(\Gamma)$ and therefore $\mathrm{Twist}(\Gamma) \cong \mathrm{Aut}(\Gamma)$. □

Example 2.29. Recall the graph Γ depicted in Figure 1. One can easily check that

$$\mathrm{Aut}(\Gamma) = \langle \alpha_0 \mid \alpha_0^2 \rangle \cong \mathbb{Z}_2,$$

where α_0 is given in Example 2.18. Then the group $\mathrm{Twist}(\Gamma)$ is generated by graph isomorphisms

$$\{\alpha_0, \alpha_1, \ldots, \alpha_4\} \subset \mathfrak{S}_V$$

as depicted in Figure 8.

One can check that for each $0 \le i \le j \le 4$,

$$\alpha_i\alpha_j = \begin{cases} \mathrm{Id} & i = j; \\ \alpha_3\alpha_0 & i = 0,\ j = 2; \\ \alpha_2\alpha_0 & i = 0,\ j = 3; \\ \alpha_j\alpha_i & \text{otherwise} \end{cases} \tag{2.4}$$

and therefore we have an isomorphism

$$\mathrm{Twist}(\Gamma) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times ((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2),$$

where each factor from the left is generated by $\alpha_1, \alpha_4, \alpha_2, \alpha_3$ and α_0.

2.4. The category $\mathcal{G}$ of CLTTF graphs

From now on, we mean edge-twists by edge-twists along outward edges in the chunk tree unless mentioned otherwise.

Definition 2.30. Let $\widetilde{\mathcal{G}}$ be the category of CLTTF graphs defined as follows:

- The set of objects is $\mathcal{G}$.
- The hom-set is freely generated by graph isomorphisms and edge-twists.[2]

In other words, for any morphism $f \in \hom(\Gamma, \Delta)$ is a composition

$$f : \Gamma = \Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} \Gamma_n = \Delta,$$

where $f_i : \Gamma_{i-1} \to \Gamma_i$ is either a graph isomorphism or an edge-twist.

Notice that the hom-set of the category $\widetilde{\mathcal{G}}$ is *freely* generated. In particular, any $\varepsilon \in \mathcal{E}^{\mathsf{out}}_{\llbracket\Gamma\rrbracket}$ with $m_{\llbracket\Gamma\rrbracket}(\varepsilon)$ even induces an endomorphism $\overline{\varepsilon} : \Gamma \to \Gamma$ but it will

[2] The empty product will be regarded as the identity morphism.

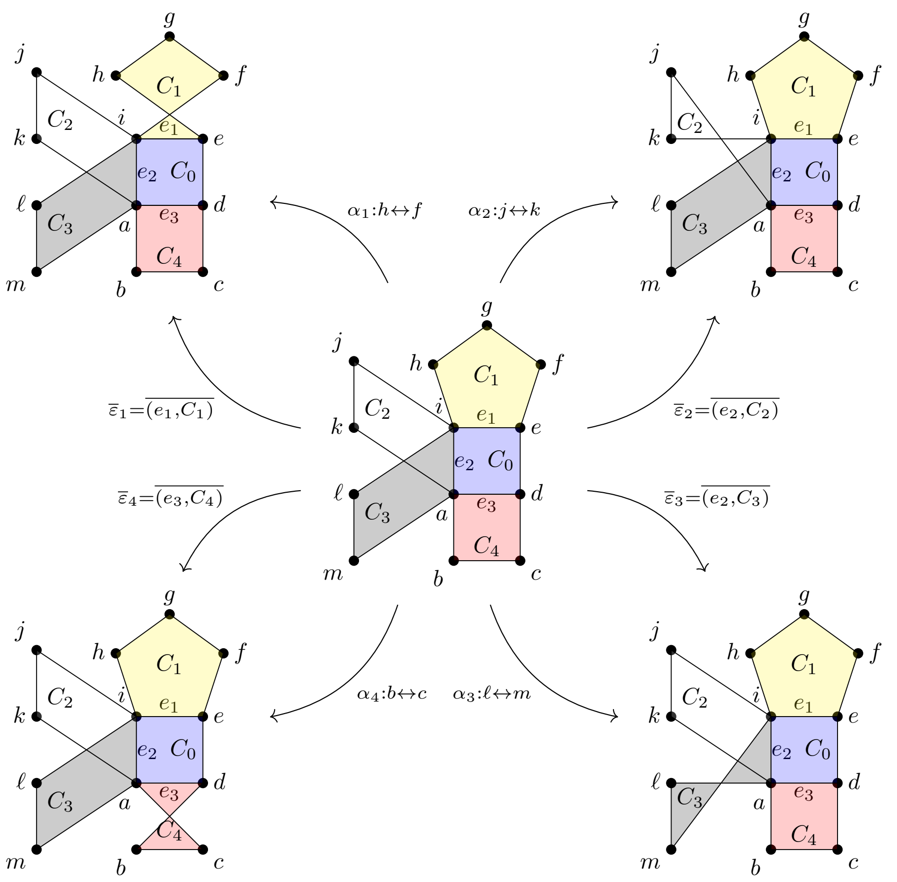


Figure 8. Examples of elements of Twist(Γ)

never be regarded as the identity. Furthermore, if $m_{[\![\Gamma]\!]}(\varepsilon)$ is odd, then there are edge-twists $\bar\varepsilon : \Gamma \to \Delta$ and $\bar\varepsilon : \Delta \to \Gamma$. However, in $\widetilde{\mathcal{G}}$, the composition $\bar\varepsilon^2$ is *not* the identity. Therefore, we will denote each edge-twist in the category $\widetilde{\mathcal{G}}$ by ε instead of $\bar\varepsilon$ in order to avoid the confusion as above.

Now let $\mathcal{E}$ be the set of morphisms generated by edge-twists. Then by localizing $\widetilde{\mathcal{G}}$ with respect to $\mathcal{E}$, we obtain the category

$$\overline{\mathcal{G}} = \widetilde{\mathcal{G}}[\mathcal{E}^{-1}].$$

In other words, in the category $\overline{\mathcal{G}}$, we have the formal inverse $\varepsilon^{-1} \in \mathrm{Hom}_{\overline{\mathcal{G}}}(\Delta, \Gamma)$ of each edge-twist $\varepsilon \in \mathrm{Hom}_{\overline{\mathcal{G}}}(\Gamma, \Delta)$. Hence any morphism $f \in \mathrm{Hom}_{\overline{\mathcal{G}}}(\Gamma, \Delta)$ is a composition

$$f : \Gamma = \Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} \Gamma_n = \Delta,$$

where $f_i : \Gamma_{i-1} \to \Gamma_i$ is either

(1) a graph isomorphism α_i,
(2) an edge-twist ε_i, or
(3) a formal inverse ε_i^{-1} of an edge-twist ε_i.

We also define an equivalence relation on the hom-set of $\overline{\mathcal{G}}$ generated by the following three types of relations:

(1) for two graph isomorphisms $\alpha : \Gamma \to \Gamma'$ and $\beta : \Gamma' \to \Gamma''$ whose composition is $\gamma : \Gamma \to \Gamma''$, we declare the relation in $\hom_{\overline{\mathcal{G}}}(\Gamma, \Gamma'')$ as

$$\beta\alpha \sim \gamma.$$

(2) for each pull-back (or push-forward) diagram of graph isomorphisms and edge-twists

$$\begin{tikzcd} \Gamma \arrow[r, "\alpha"] \arrow[d, "\varepsilon"'] & \Delta \arrow[d, "\varepsilon'"] \\ \Gamma' \arrow[r, "\alpha'"] & \Delta', \end{tikzcd}$$

we declare the relation in $\hom_{\overline{\mathcal{G}}}(\Gamma, \Delta')$ as

$$\varepsilon'\alpha \sim \alpha'\varepsilon. \tag{2.5}$$

(3) for each pull-back (or push-forward) diagram of edge-twists

$$\begin{tikzcd} \Gamma \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2"'] & \Gamma_1 \arrow[d, "\varepsilon_2"] \\ \Gamma_2 \arrow[r, "\varepsilon_1"] & \Gamma', \end{tikzcd}$$

we declare the relation in $\hom_{\overline{\mathcal{G}}}(\Gamma, \Gamma')$ as

$$\varepsilon_2\varepsilon_1 \sim \varepsilon_1\varepsilon_2. \tag{2.6}$$

Definition 2.31. The quotient category $\overline{\mathcal{G}}/\sim$ will be denoted by $\mathcal{G}$ again.

$$\mathcal{G} = \overline{\mathcal{G}}/\sim = \left(\widetilde{\mathcal{G}}[\mathcal{E}^{-1}]\right)/\sim .$$

Let $\Gamma = (V, E, m) \in \mathcal{G}$ be a CLTTF graph. By the relations (2.5) and (2.6), any isomorphism $f \in \mathrm{Hom}_{\mathcal{G}}(\Gamma, \Delta)$ is a composition

$$f = \mathcal{E}\alpha, \quad \mathcal{E} = \mathcal{E}(\eta) = \prod_{\varepsilon \in \mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]}} \varepsilon^{\eta(\varepsilon)}, \tag{2.7}$$

where α is a graph isomorphism and $\eta : E^{\mathsf{out}}(\mathcal{C}\mathsf{h}_{[\![\Gamma]\!]}) \to \mathbb{Z}$ is a function.

Definition 2.32 (Even edge-twists)**.** We say that a composition $\mathcal{E} = \mathcal{E}(\eta)$ of edge-twists in $\mathcal{G}$ is *even* if $\eta(\varepsilon)$ is even for every $\varepsilon \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]}$.

A set $\mathrm{Dehn}_{\mathcal{G}}(\Gamma)$ of morphisms is defined as the set of even compositions of edge-twists.

Lemma 2.33. *Let* $\mathcal{E} : \Gamma \to \Delta$ *be a composition of edge-twists in* $\mathcal{G}$*. Then* $\Gamma = \Delta$ *if and only if* $\mathcal{E} \in \mathrm{Dehn}_{\mathcal{G}}(\Gamma)$*.*

In particular, $\mathrm{Dehn}_{\mathcal{G}}(\Gamma)$ *is a subset of* $\mathrm{Aut}_{\mathcal{G}}(\Gamma)$*.*

Proof. Assume that $\mathcal{E} = \mathcal{E}(\eta)$ for some $\eta : \mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]} \to \mathbb{Z}$. If $\mathcal{E}$ is not even, then there exists $\varepsilon \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]}$ with $\eta(\varepsilon) \equiv 1 \mod 2$. Then the resulting graph never be the same as Γ as seen in Lemma 2.10. Therefore $\mathcal{E}$ should be even.

Conversely, any even $\mathcal{E}$ obviously gives us an automorphism $\mathcal{E} : \Gamma \to \Gamma$. □

Corollary 2.34. *The set* $\mathrm{Dehn}_{\mathcal{G}}(\Gamma)$ *is a normal subgroup of* $\mathrm{Aut}_{\mathcal{G}}(\Gamma)$ *and isomorphic to the free abelian group* $\mathbb{Z}^{\#(\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]})}$*.*

Proof. By the above lemma, even edge-twists form a group, which is free abelian by the relation (2.6) and normal in $\mathrm{Aut}_{\mathcal{G}}(\Gamma)$ since the conjugate of an even edge-twist by a graph automorphism is again even.

Finally, the group of even edge-twists is isomorphic to a free abelian group generated by the set

$$\left\{\varepsilon \mid \varepsilon \in \mathcal{E}^{\mathsf{out},0}_{[\![\Gamma]\!]}\right\} \cup \left\{\varepsilon^2 \mid \varepsilon \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]}\right\},$$

which has one-to-one correspondence with $\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]}$ and we are done. □

Proposition 2.35. *Let* $\alpha, \mathcal{E} \in \mathrm{Hom}_{\mathcal{G}}(\Gamma, \Delta)$ *such that* α *is a graph isomorphism and* $\mathcal{E}$ *is a composition of edge-twists and their inverses. Suppose that* $\alpha = \mathcal{E}$ *in* $\mathrm{Hom}_{\mathcal{G}}(\Gamma, \Delta)$*. Then* $\Gamma = \Delta$ *and both* α *and* $\mathcal{E}$ *are the identities.*

This proposition is evident since the only relations in (2.5) and (2.6) do not cancel a graph isomorphism with a composition of edge-twists. However, we will give a concrete proof later.

Under the aid of Proposition 2.35, we have the following theorem.

Theorem 2.36. *For each isomorphism* $f \in \mathrm{Hom}_{\mathcal{G}}(\Gamma, \Delta)$*, there is a unique pair of a graph isomorphism* α *and a composition* $\mathcal{E}$ *of edge-twists or inverses such that*

$$f = \mathcal{E}\alpha.$$

Proof. Suppose that f has two such expressions $f = \mathcal{E}\alpha = \mathcal{E}'\alpha'$. By pre- and post-compositions of α'^{-1} and $\mathcal{E}^{-1}$, we have $\alpha\alpha'^{-1} = \mathcal{E}^{-1}\mathcal{E}'$, which should be the identity by Proposition 2.35 and so $\alpha = \alpha'$ and $\mathcal{E} = \mathcal{E}'$ as desired. □

As an immediate consequence, we have the following corollary.

Corollary 2.37. *Let Γ be a discretely rigid* CLTTF *graph. Then the automorphism group* $\mathrm{Aut}_{\mathcal{G}}(\Gamma)$ *is the semidirect product of the free abelian group generated by edge-twists and the automorphism group of* Γ:

$$\mathrm{Aut}_{\mathcal{G}}(\Gamma) \cong \mathrm{Dehn}_{\mathcal{G}}(\Gamma) \rtimes \mathrm{Aut}(\Gamma) \cong \mathbb{Z}^{\#(\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]})} \rtimes \mathrm{Aut}(\Gamma).$$

Proof. Let $f = \mathcal{E}\alpha \in \mathrm{Aut}_{\mathcal{G}}(\Gamma)$ with a graph isomorphism $\alpha : \Gamma \to \Delta$ and a composition of edge-twists $\mathcal{E} : \Delta \to \Gamma$ so that $\Gamma \sim \Delta$ and $\Gamma \cong \Delta$. Since Γ is discretely rigid, $\Gamma = \Delta$ and so both α and $\mathcal{E}$ are automorphisms. Therefore $\mathrm{Aut}_{\mathcal{G}}(\Gamma)$ is generated by $\mathrm{Aut}(\Gamma)$ and $\mathrm{Dehn}_{\mathcal{G}}(\Gamma)$ by Lemma 2.33.

Finally, by Corollary 2.34, Proposition 2.35 and Theorem 2.36, we are done. □

Example 2.38 (Special automorphism Φ)**.** Let $*_\Gamma$ be the central vertex of $\mathcal{C}\mathsf{h}_\Gamma$. If $*_\Gamma$ is a chunk, then $\Phi \in \mathrm{Aut}_{\mathcal{G}}(\Gamma)$ will be defined to be the identity.

Suppose that $*_\Gamma = e = \{s, t\}$ is a separating edge. Let

$$\{\varepsilon_1, \dots, \varepsilon_N \mid \varepsilon_i = (e, C_i)\} \subset \mathcal{E}^{\mathsf{out}}_\Gamma$$

be the subset of edges adjacent to $*_\Gamma$ of $\mathcal{C}\mathsf{h}_\Gamma$. We define the composition

$$\mathcal{E}_{*_\Gamma} = \varepsilon_1\varepsilon_2\cdots\varepsilon_N \tag{2.8}$$

of all edge-twists $\varepsilon_1, \dots, \varepsilon_N$.

If the label $m(e)$ is even, then $\Delta = \Gamma$ and so $\mathcal{E}$ is an automorphism and let $\Phi = \mathcal{E}_{*_\Gamma}$.

Otherwise, notice that Δ is obtained by interchanging the roles of vertices s and t, and isomorphic to Γ. The precise graph isomorphism $\alpha_{*_\Gamma} : \Gamma \to \Delta$ is given by

$$\alpha_{*_\Gamma}(v) = \begin{cases} v & v \notin \{s,t\}; \\ t & v = s; \\ s & v = t. \end{cases}$$

Then we define Φ to be the composition $\mathcal{E}_{*_\Gamma}\alpha_{*_\Gamma} : \Gamma \to \Gamma$.

In summary, the special automorphism Φ is defined as

$$\Phi := \begin{cases} \mathrm{Id} & *_\Gamma \text{ is a chunk;} \\ \mathcal{E}_{*_\Gamma} & *_\Gamma \text{ is an even-labeled separating edge;} \\ \mathcal{E}_{*_\Gamma}\alpha_{*_\Gamma} & *_\Gamma \text{ is an odd-labeled separating edge.} \end{cases} \tag{2.9}$$

Remark 2.39. Observe that if $*_\Gamma$ is an odd-labeled separating edge, then Γ can not be discretely rigid. Conversely, for any discretely rigid CLTTF graph Γ, the central vertex $*_\Gamma$ is either a chunk or an even-labeled separating edge.

3. CLTTF Artin groups

3.1. CLTTF Artin groups and their isomorphisms

Let $\Gamma = (V, E, m)$ be a CLTTF graph. An *Artin group* A_Γ with a defining graph Γ is given by the group presentation

$$A_\Gamma = \langle V \mid (s, t; m(e)) = (t, s; m(e)) \text{ for each } e = \{s, t\} \in E\rangle, \tag{3.1}$$

where $(s, t; m)$ is the alternating product of generators s and t of length m. For example,

$$\begin{array}{llll} (s,t;1) = s, & (s,t;2) = st, & (s,t;3) = sts, & \dots \\ (t,s;1) = t, & (t,s;2) = ts, & (t,s;3) = tst, & \dots \end{array}$$

For each $e = \{s, t\} \in E$, let us denote the subgroup $G(e)$ generated by $\{s, t\}$. Then the element $x_e = (s, t; m(e)) \in G(e)$ preserves the set $\{s, t\}$ of generators under the conjugation. That is,

$$x_e^{-1}\{s, t\}x_e = \{s, t\}$$

and we call x_e the *quasi-center* of $G(e)$ or simply the *quasi-center* for e. On the other hand, the conjugation by x_e preserves each generator s and t if and only if $m(e)$ is even. Therefore the element z_e defined as

$$z_e = \begin{cases} x_e^2 & m(e) \text{ is odd;} \\ x_e & m(e) \text{ is even,} \end{cases}$$

generates the center of $G(e)$.

According to the Crisp's result in [2], there are four types of elementary isomorphisms which generate every isomorphism between CLTTF Artin groups as follows:

(1) An isomorphism $\alpha_\# : A_\Gamma \to A_\Delta$ defined as $\alpha_\#(v) = \alpha(v)$ for each $v \in V$, where $(\alpha : \Gamma \to \Delta) \in \mathfrak{S}_V$ is a graph isomorphism. The induced isomorphism $\alpha_\#$ will be called a *graph isomorphism* again.
(2) The *global inversion* $\iota : A_\Gamma \to A_\Gamma$ defined as $\iota(v) = v^{-1}$ for each $v \in V$.
(3) An *inner automorphism* $g_\# : A_\Gamma \to A_\Gamma$ for some $g \in A_\Gamma$ defined as $g_\#(v) = g^{-1}vg$ for each $v \in V$.
(4) A *partial conjugation* $\varepsilon_\# : A_\Gamma \to A_\Delta$ for each decomposition $\varepsilon = (\Gamma_1, e, \Gamma_2)$ defined as

$$\varepsilon_\#(v) = \begin{cases} v & v \in V_1; \\ x_e^{-1}vx_e & v \notin V_1, \end{cases}$$

where $\Gamma_i = (V_i, E_i)$ and the graph Δ is obtained by edge-twists with respect to the decomposition ε.

Remark 3.1. The above classification is slightly different from that described in [2]. Indeed, all graphs are *up to isomorphism* in [2] and so all graph isomorphisms above should be translated into graph automorphisms. Actually, this can be done by fixing a reference graph isomorphism $\Gamma \to \Delta$ for each $\Delta \cong \Gamma$.

Remark 3.2. When Γ has a leaf, then the leaf can be inverted separately, called the *leaf inversion*. However, by Assumption 2.1, there are no leaves in Γ.

Definition 3.3 (Rigidity of CLTTF Artin groups)**.** A CLTTF Artin group A_Γ is said to be *rigid* if it has a unique defining graph Γ up to isomorphism.

Theorem 3.4 ([1])**.** *A CLTTF Artin group A_Γ is rigid if and only if so is Γ.*

3.2. The category $\mathcal{A}$ of CLTTF Artin groups

Let us define the category $\mathcal{A}$ of CLTTF Artin groups, whose objects and morphisms are as follows:

(1) Objects are Artin group A_Γ for all CLTTF graphs Γ given by the group presentation as described in (3.1).
(2) Morphisms are compositions of partial conjugations and graph isomorphisms.

Now let us consider the functor $\widetilde{\mathcal{F}} : \widetilde{\mathcal{G}} \to \mathcal{A}$ as follows: for each CLTTF graph Γ, we assign the Artin group A_Γ

$$\widetilde{\mathcal{F}}(\Gamma) = A_\Gamma$$

given by the group presentation as mentioned at the beginning. For each graph isomorphism α and edge-twist ε in $\mathrm{Hom}_{\widetilde{\mathcal{G}}}(\Gamma, \Delta)$, we assign a graph isomorphism and a partial conjugation

$$\widetilde{\mathcal{F}}(\alpha) = \alpha_\# \quad \text{and} \quad \widetilde{\mathcal{F}}(\varepsilon) = \varepsilon_\#,$$

respectively. Then since the morphisms in $\widetilde{\mathcal{G}}$ are freely generated by graph isomorphisms and edge-twists, the functor $\widetilde{\mathcal{F}}$ is well-defined.

Proposition 3.5. *The functor $\widetilde{\mathcal{F}} : \widetilde{\mathcal{G}} \to \mathcal{A}$ factors through the localization $\overline{\mathcal{G}}$ and the quotient category $\mathcal{G}$. Namely, there exist unique functors up to natural isomorphisms*

$$\overline{\mathcal{F}} : \overline{\mathcal{G}} \to \mathcal{A} \quad \textit{and} \quad \mathcal{F} : \mathcal{G} \to \mathcal{A},$$

which fit into the following commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{G}} \arrow[d] \arrow[dr, bend left, "\widetilde{\mathcal{F}}"] & \\
\overline{\mathcal{G}} = \widetilde{\mathcal{G}}[\mathcal{E}^{-1}] \arrow[d, two heads] \arrow[r, "\exists\overline{\mathcal{F}}"] & \mathcal{A} \\
\mathcal{G} = \overline{\mathcal{G}}/\sim \arrow[ur, bend right, "\exists\mathcal{F}"'] &
\end{tikzcd}$$

Proof. The existence and the uniqueness of the functor

$$\overline{\mathcal{F}} : \overline{\mathcal{G}} = \widetilde{\mathcal{G}}[\mathcal{E}^{-1}] \to \mathcal{A}$$

come from the universal property of the localized category since each edge-twist maps to a partial conjugation which is an isomorphism in $\mathcal{A}$.

Since $\mathcal{G} = \overline{\mathcal{G}}/\sim$ is the quotient category, by the universal property of the quotient category, it suffices to prove that for pull-back diagrams

$$\begin{array}{ccc} \Gamma & \xrightarrow{\alpha} & \Delta \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon'} \\ \Gamma' & \xrightarrow{\alpha} & \Delta', \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Gamma & \xrightarrow{\varepsilon_1} & \Gamma_1 \\ {\scriptstyle\varepsilon_2}\downarrow & & \downarrow{\scriptstyle\varepsilon_2'} \\ \Gamma_2 & \xrightarrow{\varepsilon_1'} & \Gamma', \end{array}$$

the compositions of induced maps are identical in $\mathcal{A}$. Namely,

$$\varepsilon'_{\#}\alpha_{\#} = \alpha_{\#}\varepsilon_{\#} \qquad \text{and} \qquad \varepsilon'_{2\#}\varepsilon_{1\#} = \varepsilon'_{1\#}\varepsilon_{2\#}$$

in $\mathrm{Isom}_{\mathcal{A}}(A_\Gamma, A_\Delta)$.

Let α, α' and $\varepsilon = (e, C)$, $\varepsilon' = (e', C')$ be graph isomorphisms and edge-twists that fit into a pull-back diagram. We denote by $x_e \in A_\Gamma$ and $x_{e'} \in A_\Delta$ the quasi-centers for e and e', respectively. Let $V_i(\varepsilon)$ and $V_i(\varepsilon')$ be the sets of vertices of $\Gamma_i(\varepsilon)$ and $\Gamma_i(\varepsilon')$, respectively. By definition of the pull-back or push-forward, two subsets are canonically identified via α.

Then the maps $\varepsilon'_{\#}\alpha_{\#}$ and $\varepsilon'_{\#}\alpha_{\#}$ are defined as follows: for each $v \in V_\Gamma$,

$$(\varepsilon'_{\#}\alpha_{\#})(v) = \begin{cases} \alpha(v) & \alpha(v) \in V_1(\varepsilon'); \\ x_{e'}^{-1}\alpha(v)x_{e'} & \alpha(v) \notin V_1(\varepsilon'), \end{cases}$$

$$(\alpha_{\#}\varepsilon_{\#})(v) = \begin{cases} \alpha(v) & v \in V_1(\varepsilon); \\ \alpha_{\#}(x_e^{-1}vx_e) & v \notin V_1(\varepsilon). \end{cases}$$

We observe that since the restriction $\alpha| : V_1(\varepsilon) \to V_1(\varepsilon')$ is a bijection, we have

$$\alpha(v) \notin V_1(\varepsilon') \Longleftrightarrow v \notin V_1(\varepsilon)$$

and since $\alpha_{\#}(x_e) = x_{e'}$,

$$\alpha_{\#}(x_e^{-1}vx_e) = \alpha_{\#}(x_e)^{-1}\alpha(v)\alpha_{\#}(x_e) = x_{e'}^{-1}\alpha(v)x_{e'}$$

as desired.

Let $\varepsilon_1 = (e_1, C_1)$, $\varepsilon_1' = (e_1', C_1')$, $\varepsilon_2 = (e_2, C_2)$ and $\varepsilon_2' = (e_2', C_2')$ be edge-twists that fit into a pull-back diagram. Note that $\varepsilon_1 = \varepsilon_1'$ and $\varepsilon_2 = \varepsilon_2'$ in $\mathsf{Ch}_{[\![\Gamma]\!]}$ and so $V_i(\varepsilon_j) = V_i(\varepsilon_j')$ for each $i, j = 1, 2$. Moreover, we have identifications $x_{e_1} = x_{e_1'}$ and $x_{e_2} = x_{e_2'}$ as words of V.

As seen in Remark 2.15, $V \setminus V_1(\varepsilon_1)$ and $V \setminus V_1(\varepsilon_2)$ are either disjoint or nested. If $(V \setminus V_1(\varepsilon_1)) \cap (V \setminus V_1(\varepsilon_2)) = \varnothing$, or equivalently, $V_1(\varepsilon_1) \cup V_1(\varepsilon_2) = V$, then for each $v \in V$,

$$\varepsilon'_{2\#}\varepsilon_{1\#}(v) = \varepsilon'_{1\#}\varepsilon_{2\#}(v) = \begin{cases} v & v \in V_1(\varepsilon_1) \cap V_1(\varepsilon_2); \\ x_{e_1}^{-1} v x_{e_1} & v \in V_1(\varepsilon_2) \setminus V_1(\varepsilon_1); \\ x_{e_2}^{-1} v x_{e_2} & v \in V_1(\varepsilon_1) \setminus V_1(\varepsilon_2). \end{cases}$$

On the other hand, if $V \setminus V_1(\varepsilon_1)$ and $V \setminus V_1(\varepsilon_2)$ are nested, then we may assume that $(V \setminus V_1(\varepsilon_2)) \subset (V \setminus V_1(\varepsilon_1))$, or equivalently, $V_1(\varepsilon_1) \subset V_1(\varepsilon_2)$. Hence, $\varepsilon'_{2\#}$ preserves vertices of e_1 and so x_{e_1} as well. Therefore for each $v \in V$,

$$\begin{aligned} (\varepsilon'_{2\#}\varepsilon_{1\#})(v) &= \begin{cases} v & v \in V_1(\varepsilon_1); \\ \varepsilon'_{2\#}\left(x_{e_1}^{-1} v x_{e_1}\right) & v \notin V_1(\varepsilon_1), \end{cases} \\ &= \begin{cases} v & v \in V_1(\varepsilon_1); \\ x_{e_1}^{-1} v x_{e_1} & v \in V_1(\varepsilon_2) \setminus V_1(\varepsilon_1); \\ x_{e_1}^{-1}\varepsilon'_{2\#}(v) x_{\varepsilon_1} & v \notin V_1(\varepsilon_2), \end{cases} \\ &= \begin{cases} v & v \in V_1(\varepsilon_1); \\ x_{e_1}^{-1} v x_{e_1} & v \in V_1(\varepsilon_2) \setminus V_1(\varepsilon_1); \\ x_{e_1}^{-1} x_{e_2}^{-1} v x_{e_2} x_{e_1} & v \notin V_1(\varepsilon_2). \end{cases} \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} (\varepsilon'_{1\#}\varepsilon_{2\#})(v) &= \begin{cases} v & v \in V_1(\varepsilon_2); \\ \varepsilon'_{1\#}\left(x_{e_2}^{-1} v x_{e_2}\right) & v \notin V_1(\varepsilon_2), \end{cases} \\ &= \begin{cases} v & v \in V_1(\varepsilon_1); \\ \varepsilon'_{1\#}(v) & v \in V_1(\varepsilon_2) \setminus V_1(\varepsilon_1); \\ \varepsilon'_{1\#}\left(x_{e_2}^{-1} v x_{e_2}\right) & v \notin V_1(\varepsilon_2), \end{cases} \\ &= \begin{cases} v & v \in V_1(\varepsilon_1); \\ x_{e_1}^{-1} v x_{e_1} & v \in V_1(\varepsilon_2) \setminus V_1(\varepsilon_1); \\ \left(x_{e_1}^{-1} x_{e_2}^{-1} x_{e_1}\right)\left(x_{e_1}^{-1} v x_{e_1}\right)\left(x_{e_1}^{-1} x_{e_2} x_{e_1}\right) & v \notin V_1(\varepsilon_2). \end{cases} \end{aligned}$$

Therefore,

$$(\varepsilon'_{1\#}\varepsilon_{2\#})(v) = \begin{cases} v & v \in V_1(\varepsilon_1); \\ x_{e_1}^{-1} v x_{e_1} & v \in V_1(\varepsilon_2) \setminus V_1(\varepsilon_1); \\ x_{e_1}^{-1} x_{e_2}^{-1} v x_{e_2} x_{e_1} & v \notin V_1(\varepsilon_2), \end{cases}$$

which completes the proof. □

Corollary 3.6. *The functor $\mathcal{F} : \mathcal{G} \to \mathcal{A}$ is full.*

Proof. Every graph isomorphism and partial conjugation comes essentially from a graph isomorphism and an edge-twist or its formal inverse, which is again a morphism in $\mathcal{G}$ by definition. Hence the induced functor $\overline{\mathcal{F}} : \overline{\mathcal{G}} \to \mathcal{A}$ is full, and so is the functor $\mathcal{F}$ since $\mathcal{G}$ is the quotient category of $\overline{\mathcal{G}}$. □

Recall Proposition 2.35, which claims that the only identity map can be both a graph isomorphism and an edge-twist in $\mathcal{G}$. Indeed, we insist a bit stronger statement below which proves Proposition 2.35 as a direct consequence.

Proposition 3.7. *Let $\alpha, \mathcal{E} \in \mathrm{Hom}_{\mathcal{G}}(\Gamma, \Delta)$ such that α is a graph isomorphism and $\mathcal{E}$ is a composition of edge-twists and their inverses. If $\mathcal{F}(\alpha) = \mathcal{F}(\mathcal{E})$ in $\mathrm{Hom}_{\mathcal{A}}(A_\Gamma, A_\Delta)$, then $\Gamma = \Delta$ and both α and $\mathcal{E}$ are the identities.*

Proof. Let us assume that $\mathcal{E} = \mathcal{E}(\eta)$ for some $\eta : \mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]} \to \mathbb{Z}$. Unless α is the identity, the induced automorphism $\mathcal{F}(\alpha)$ can not be the identity. Hence, it suffices to prove that $\mathcal{E}$ is the identity.

We use the induction on $\mathrm{Diam}(\mathcal{C}\mathsf{h}_\Gamma)$, which is always even. If $\mathrm{Diam}(\mathcal{C}\mathsf{h}_\Gamma) = 0$, then there are no edges in $\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]}$ and therefore $\mathcal{E}$ must be trivial. If $\mathrm{Diam}(\mathcal{C}\mathsf{h}_\Gamma) = 2$, then the center $*_\Gamma$ must be a separating edge e since the leaves in $\mathcal{C}\mathsf{h}_\Gamma$ are chunks. As above, all edge-twists induce partial conjugations that fix both s and t. Suppose that $\eta(\varepsilon) = a \neq 0$ for some $\varepsilon = (e, C) \in \mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]}$. Since $C = (V_C, E_C)$ is triangle-free and edge-separated, we can pick an edge $f = \{v, w\} \in E_C$ such that $e \cap f = \varnothing$. Then we have

$$\mathcal{F}(\alpha)(\{v, w\}) = \{\alpha(v), \alpha(w)\} = \{x_e^{-a} v x_e^a, x_e^{-a} w x_e^a\} = \mathcal{F}(\mathcal{E})(\{v, w\}),$$

which generate subgroups $G(\alpha(f))$ and $x_e^{-a} G(f) x_e$. Unless $f = \alpha(f)$, these two subgroups can not be the same since $G(f)$ and $G(\alpha(f))$ are not conjugate to each other. Hence we have $f = \alpha(f)$ and so x_e^a is in the normalizer of $G(f)$, which is a power of x_f. This is a contradiction since $G(e) \cap G(f)$ is trivial and so a must be 0.

Suppose that the assertion holds for every Γ with $\mathrm{Diam}(\mathcal{C}\mathsf{h}_\Gamma) \leq 2N$. We assume that

$$\mathrm{Diam}(\mathcal{C}\mathsf{h}_\Gamma) = 2N + 4.$$

Let $\Gamma' \subset \Gamma$ and $\Delta' \subset \Delta$ be the subgraphs which are unions of chunks in Γ and Δ within a distance $(N+1)$ (or N[3]) from the center $*_\Gamma$ and $*_\Delta$, respectively. Then since both α and $\mathcal{E}$ induce isomorphisms on $(\mathcal{C}\mathsf{h}_\Gamma, *_\Gamma) \to (\mathcal{C}\mathsf{h}_\Delta, *_\Delta)$, their restrictions

$$\alpha| : \Gamma' \to \Delta' \quad \text{and} \quad \mathcal{E}| : \Gamma' \to \Delta'$$

are well-defined so that $\alpha|$ is a graph isomorphism and $\mathcal{E}|$ is a composition of edge-twists again. Furthermore, they induce the same maps so that $\mathcal{F}(\alpha|) = \mathcal{F}(\mathcal{E})$ in $\mathrm{Hom}_{\mathcal{A}}(A_{\Gamma'}, A_{\Delta'})$ and therefore both $\alpha|$ and $\mathcal{E}|$ must be the identity by the induction hypothesis.

Hence the only possibility for $\mathcal{E}$ is a composition of edge-twists involving chunks which are farthest from the center $*_\Gamma$. Suppose that there exists an edge $\varepsilon = (e, C) \in \mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]}$ with $\eta(\varepsilon) = a \neq 0$ involving a farthest chunk C. Then the exactly same argument as above yields a contradiction and therefore $\mathcal{E}$ must be trivial. □

[3]Since the vertices at the distance $(N+1)$ from $*_\Gamma$ correspond to separating edges which are already contained in chunks at the distance N from $*_\Gamma$.

Proof of Proposition 2.35. Let $\alpha, \mathcal{E}$ be two morphisms satisfying the hypothesis. If $\alpha = \mathcal{E}$, then $\mathcal{F}(\alpha) = \mathcal{F}(\mathcal{E})$ and so both α and $\mathcal{E}$ are the identities by Proposition 3.7. $\square$

Moreover, Proposition 3.7 also implies the faithfulness of the functor $\mathcal{F}$ as follows:

Corollary 3.8. *Let $f, g \in \mathrm{Hom}_{\mathcal{G}}(\Gamma, \Delta)$. If $\mathcal{F}(f) = \mathcal{F}(g) \in \mathrm{Hom}_{\mathcal{A}}(A_\Gamma, A_\Delta)$, then $f = g$.*

In other words, the functor $\mathcal{F} : \mathcal{G} \to \mathcal{A}$ is faithful.

Proof. Let $f, g \in \mathrm{Hom}_{\mathcal{G}}(\Gamma, \Delta)$. Then as observed in (2.7), f and g can be expressed as compositions

$$f = \mathcal{E}\alpha, \qquad \mathcal{E} = \mathcal{E}(\eta), \qquad g = \mathcal{E}'\alpha', \qquad \mathcal{E}' = \mathcal{E}(\eta')$$

for some $\eta, \eta' : \mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]} \to \mathbb{Z}$. Then since $\mathcal{F}(f) = \mathcal{F}(g)$, by pre-composition of $\mathcal{F}(\alpha)^{-1}$ and post-composition of $\mathcal{F}(\mathcal{E}')$, we have

$$\mathcal{F}(\mathcal{E}'^{-1} f \alpha^{-1}) = \mathcal{F}(\mathcal{E}')^{-1}\mathcal{F}(f)\mathcal{F}(\alpha)^{-1} = \mathcal{F}(\mathcal{E}')^{-1}\mathcal{F}(g)\mathcal{F}(\alpha)^{-1} = \mathcal{F}(\mathcal{E}'^{-1} g \alpha^{-1}).$$

However, the left hand side is the induced map of edge-twists $\mathcal{F}(\mathcal{E}'^{-1} f \alpha^{-1}) = \mathcal{F}(\mathcal{E}'^{-1}\mathcal{E})$ while the right hand side is the induced map of graph isomorphisms $\mathcal{F}(\mathcal{E}'^{-1} g \alpha^{-1}) = \mathcal{F}(\alpha'\alpha^{-1})$. Then by Proposition 3.7, we must have

$$\mathcal{E}'^{-1}\mathcal{E} = \alpha'\alpha^{-1} = \mathrm{Id} \in \mathrm{Hom}_{\mathcal{G}}(\Gamma, \Gamma),$$

and therefore $\alpha' = \alpha$ and $\mathcal{E}' = \mathcal{E}$, which implies that $f = g$. $\square$

In summary, we have the following theorem.

Theorem 3.9. *The induced functor $\mathcal{F} : \mathcal{G} \to \mathcal{A}$ is an equivalence of categories.*

In particular, for each Γ, there is a group isomorphism

$$\mathrm{Aut}_{\mathcal{G}}(\Gamma) \cong \mathrm{Aut}_{\mathcal{A}}(\Gamma).$$

Proof. In order to show that $\mathcal{F}$ is an equivalence, we will show (i) the essential surjectivity, and (ii) the fully-faithfulness.

(i) By definition, every object in $\mathcal{A}$ is an Artin group presentation for a CLTTF graph and every morphism in $\mathcal{A}$ is an isomorphism. Hence the essential surjectivity is obvious.

(ii) The fully-faithfulness comes from Corollaries 3.6 and 3.8, and we are done. $\square$

Before closing this section, we will prove the equivalence of categories between $\mathcal{G}$ and the subcategory of the groupoid defined in [2]. Let $\mathcal{I}so$ be the category of edge-separated CLTTF graphs *up to isomorphism* whose morphisms are the set of all group isomorphisms $A_\Gamma \to A'_\Gamma$.

Then by [2, Theorem 1], the morphisms in $\mathcal{I}so$ are generated by graph automorphisms, leaf and global inversions, inner automorphisms, and partial conjugations.

We define the subcategory $\mathcal{I}so_0$ of $\mathcal{I}so$ whose morphisms are generated by graph automorphisms and partial conjugations. Then we will show an equivalence between two categories $\mathcal{G}$ and $\mathcal{I}so$.

For each isomorphism class $[\Gamma]$, we fix a representative $\Gamma_0 \in [\Gamma]$. Moreover, for each $\Delta \in [\Gamma]$, we also fix a graph isomorphism $\alpha_\Delta : \Gamma_0 \to \Delta$. Obviously, $\alpha_\Delta = \mathrm{Id}$ if and only if Δ represents its isomorphism class.

Now we define a functor $[\cdot] : \mathcal{G} \to \mathcal{I}so_0$ as follows: For each CLTTF graph Γ,

$$[\cdot] : \Gamma \mapsto [\Gamma],$$

where $[\Gamma]$ is the graph isomorphism class of Γ.

For each graph isomorphism $\alpha : \Gamma \to \Delta$, we have a graph automorphism $\alpha_\Delta^{-1}\alpha\alpha_\Gamma : \Gamma_0 \to \Gamma_0$, which induces an isomorphism

$$[\alpha] \coloneqq (\alpha_\Delta^{-1}\alpha\alpha_\Gamma)_\# : A_{\Gamma_0} \to A_{\Gamma_0}.$$

Here Γ_0 is the chosen representative of $[\Gamma] = [\Delta]$.

For each edge-twist $\varepsilon : \Gamma \to \Delta$, we have a composition $\alpha_\Delta^{-1}\varepsilon\alpha_\Gamma : \Gamma_0 \to \Delta_0$, which induces an isomorphism

$$[\varepsilon] \coloneqq (\alpha_\Delta)_\#^{-1}\varepsilon_\#(\alpha_\Gamma)_\# : A_{\Gamma_0} \to A_{\Delta_0}.$$

Here Γ_0 and Δ_0 are the chosen representative of $[\Gamma]$ and $[\Delta]$, respectively.

Theorem 3.10. *The functor* $[\cdot] : \mathcal{G} \to \mathcal{I}so_0$ *is an equivalence of categories.*

Proof. By definition of $\mathcal{I}so_0$, Theorem 1 in [2] and Theorem 3.9, the functor $[\cdot]$ is well-defined, surjective, and full. The faithfulness follows obviously from Proposition 3.7 as well. □

4. The automorphism group $\mathrm{Aut}(A_\Gamma)$

In this section, we will provide finite presentations for both $\mathrm{Aut}(A_\Gamma)$ and $\mathrm{Out}(A_\Gamma)$. To this end, we first analyze the automorphism group $\mathrm{Aut}(A_\Gamma)$, which is generated by $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$, the inner automorphism group $\mathrm{Inn}(A_\Gamma)$ and the global inversion $\iota : A_\Gamma \to A_\Gamma$, and define a pairing between elements in $\mathrm{Twist}(\Gamma)$ called a *twisted intersection product*.

4.1. Preliminaries

4.1.1. *Positive automorphisms.* For each $f \in \mathrm{Aut}(A_\Gamma)$, let us consider the induced map $H_1(f)$ on the abelianization $H_1(A_\Gamma)$

$$H_1(f) : H_1(A_\Gamma) \to H_1(A_\Gamma),$$

which is either a permutation of vertices or a composition of the global inversion and a permutation of vertices. Namely, there exists $\mathrm{sgn}(f) = \pm 1$ such that for each $v \in V$,

$$H_1(f)([v]) = \mathrm{sgn}(f)[w] \in H_1(A_\Gamma)$$

for some $w \in V_\Gamma$. We say that f is *positive* if $\mathrm{sgn}(f) = 1$. Then sgn defines a group homomorphism

$$\mathrm{sgn} : \mathrm{Aut}(A_\Gamma) \to \mathbb{Z}_2,$$

whose kernel is the subgroup of positive automorphisms and denoted by $\mathrm{Aut}_+(A_\Gamma) := \ker(\mathrm{sgn})$. Then

$$\mathrm{Inn}(A_\Gamma) \lhd \mathrm{Aut}_+(A_\Gamma) \lhd \mathrm{Aut}(A_\Gamma)$$

and so we define the group $\mathrm{Out}_+(A_\Gamma)$ as the quotient

$$\mathrm{Out}_+(A_\Gamma) := \mathrm{Aut}_+(A_\Gamma) \big/ \mathrm{Inn}(A_\Gamma).$$

Obviously, the global inversion $(\iota : A_\Gamma \to A_\Gamma) \in \mathrm{Aut}(A_\Gamma)$ is not contained in $\mathrm{Aut}_+(A_\Gamma)$ and acts on inner automorphisms, partial conjugations and graph isomorphisms by conjugation. More precisely, for each isomorphism $\alpha_\#, \varepsilon_\# : A_\Gamma \to A_\Delta$ coming from a graph isomorphism α and an edge-twist ε, or inner automorphism $g_\# : A_\Gamma \to A_\Gamma$, we have

$$\alpha_\# \iota = \iota \alpha_\#, \qquad \varepsilon_\#^{-1} \iota = \iota \varepsilon_\#, \qquad \overline{g}_\#^{-1} \iota = \iota g_\#,$$

where $\overline{g} = v_k \cdots v_1$ is the reverse of $g = v_1 \cdots v_k$ with $v_i \in V$. Therefore the following lemmas are immediate consequences.

Lemma 4.1. *There is a commutative diagram with exact rows and columns*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \mathrm{Inn}(A_\Gamma) \arrow[r, equal] \arrow[d] & \mathrm{Inn}(A_\Gamma) \arrow[d] & & \\
1 \arrow[r] & \mathrm{Aut}_+(A_\Gamma) \arrow[r] \arrow[d] & \mathrm{Aut}(A_\Gamma) \arrow[r, "\mathrm{sgn}"] \arrow[d] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{Out}_+(A_\Gamma) \arrow[r] \arrow[d] & \mathrm{Out}(A_\Gamma) \arrow[r, "\mathrm{sgn}"] \arrow[d] & \mathbb{Z}_2 \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

where the group $\mathbb{Z}_2 \cong \langle \iota \mid \iota^2 \rangle$ is generated by the global inversion ι. Therefore the rows split and

$$\mathrm{Aut}(A_\Gamma) \cong \mathrm{Aut}_+(A_\Gamma) \rtimes \mathbb{Z}_2, \qquad \mathrm{Out}(A_\Gamma) \cong \mathrm{Out}_+(A_\Gamma) \rtimes \mathbb{Z}_2.$$

Lemma 4.2. *An automorphism f is in $\mathrm{Aut}_+(A_\Gamma)$ if and only if it is a composition of inner automorphisms, partial conjugations and graph isomorphisms.*

In particular, we have the subgroup $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \subset \mathrm{Aut}_+(A_\Gamma)$ consisting of compositions of partial conjugations and graph isomorphisms.

4.1.2. *The special automorphism.* Now recall the special automorphism $\Phi \in \mathrm{Aut}_{\mathcal{G}}(\Gamma)$ defined in Example 2.38. We claim that the induced map $\Phi_{\#} = \mathcal{F}(\Phi) \in \mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$ is either the identity or the inner automorphism as follows:

$$\Phi_{\#} = \begin{cases} \mathrm{Id} & \text{if } *_\Gamma \text{ is a chunk;} \\ x_{e\#} & \text{if } *_\Gamma \text{ is a separating edge } e. \end{cases}$$

If $*_\Gamma$ is a chunk, then $\Phi \in \mathrm{Aut}_{\mathcal{G}}(\Gamma)$ is the identity and so is $\Phi_{\#}$. Otherwise, assume that $*_\Gamma$ is a separating edge $e = \{s, t\}$. By definition of Φ, either

$$\Phi = \mathcal{E}_{*_\Gamma} \quad \text{or} \quad \Phi = \mathcal{E}_{*_\Gamma}\alpha_{*_\Gamma}$$

according to the parity of the label $m(e)$. Namely, if $m(e)$ is even, then

$$\Phi_{\#}(v) = \begin{cases} x_e^{-1} v x_e & v \notin \{s,t\}; \\ s & v = s; \\ t & v = t, \end{cases}$$

which is the inner automorphism $x_{e\#}$ since $s = x_e^{-1} s x_e$ and $t = x_e^{-1} t x_e$.

If $m(e)$ is odd, then since Φ is a composition with a graph isomorphism α that interchanges s and t, we have

$$\Phi_{\#}(v) = \begin{cases} x_e^{-1} v x_e & v \notin \{s,t\}; \\ t & v = s; \\ s & v = t \end{cases}$$

and therefore $\Phi_{\#} = x_{e\#}$ again since $s = x_e^{-1} t x_e$ and $t = x_e^{-1} s x_e$.

For a sake of convenience, the subgroup generated by $\Phi_{\#}$ will be denoted by Z_Γ. Unless $\Phi_{\#}$ is trivial, it is of infinite order since the Artin group A_Γ is centerless.

$$Z_\Gamma = \langle \Phi_{\#} \rangle \cong \begin{cases} 1 & *_\Gamma \text{ is a chunk;} \\ \mathbb{Z} & *_\Gamma \text{ is a separating edge.} \end{cases}$$

Proposition 4.3. *For each $A_\Gamma \in \mathcal{A}$,*

$$\mathrm{Inn}(A_\Gamma) \cap \mathrm{Aut}_{\mathcal{A}}(A_\Gamma) = Z_\Gamma.$$

Proof. By the above discussion, the subgroup $Z_\Gamma \subset \mathrm{Inn}(A_\Gamma) \cap \mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$.

Suppose that $\phi \in \mathrm{Inn}(A_\Gamma) \cap \mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$. Then by Theorems 2.36 and 3.9,

$$\phi = g_{\#} = \mathcal{E}_{\#}\alpha_{\#}$$

for some $g \in A_\Gamma$, graph isomorphism $\alpha : \Gamma \to \Delta$ and composition of edge-twists $\mathcal{E} : \Delta \to \Gamma$.

Let $*_\Gamma$ be the central vertex of the chunk tree Ch_Γ. If $*_\Gamma$ is a chunk $C = (V_C, E_C)$, then $D = (V_D, E_D) \coloneqq \alpha(C) \subset \Delta$ is isomorphic to C. Moreover, for any edge-twist $\varepsilon \in E^{\mathsf{out}}(\mathsf{Ch}_\Delta)$, the set $V_2(\varepsilon)$ contains no vertices in D. Therefore for each $v \in V_C$,

$$\phi(v) = g^{-1} v g = \mathcal{E}_{\#}(\alpha(v)) = \alpha(v) \in V_D \subset V,$$

which implies that the full subgraph defined by V_D in Γ is isomorphic to C. In other words, the graph isomorphism α on C is a graph automorphism and therefore if we restrict ϕ to the Artin group A_C, then we have

$$\phi|_{A_C} = (\alpha|_C)_* : A_C \to A_C.$$

Since the graph automorphism $\alpha|_C$ is of finite order, we have

$$\mathrm{Id} = (\alpha|_C)_\#^N = (g_\#|_{A_C})^N = (g^N)_\#|_{A_C}$$

for some $N \geq 1$. This means that g^N is contained in the centralizer $C_{A_\Gamma}(A_C)$ of A_C in A_Γ, which is trivial. Since A_Γ is torsion-free, g must be trivial.

If $*_\Gamma$ is a separating edge $e = \{s,t\}$, then $\alpha(\{s,t\}) = \{s,t\}$ is the central separating edge in $\mathcal{C}\mathsf{h}_\Delta$. As above, both vertices s and t are not contained in $V_2(\varepsilon)$ of any edge-twist $\varepsilon \in \mathcal{E}_\Delta^{\mathsf{out}}$. Therefore, for $v \in \{s,t\}$, we have

$$\phi(v) = g^{-1}vg = \mathcal{E}_\#(\alpha(v)) = \alpha(v) \in \{s,t\} \subset V,$$

and either

$$\begin{cases} \alpha(s) = s; \\ \alpha(t) = t, \end{cases} \qquad \text{or} \qquad \begin{cases} \alpha(s) = t; \\ \alpha(t) = s. \end{cases}$$

Since g normalizes $G(e)$, it is a power of x_e as proved in [6] and we are done. □

Corollary 4.4. *The group Z_Γ is contained in the center of* $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$.

Proof. Since there is nothing to prove when $*_\Gamma$ is a chunk, we assume that $*_\Gamma$ is a separating edge $e = \{s,t\}$.

For each $\phi = \mathcal{E}_\#\alpha_\# \in \mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$, it suffices to prove that $\Phi_\#\phi = \phi\Phi_\#$. As seen in the proof of Proposition 4.3, for each $v \in \{s,t\}$, we have $\phi(v) = \alpha(v) \in V$, where α on $e = \{s,t\}$ is a graph automorphism. Therefore $\phi(x_e) = \alpha_\#(x_e) = x_e$, and so for any $v \in V$,

$$(\Phi_\#\phi)(v) = x_e^{-1}\phi(v)x_e = \phi(x_e^{-1}vx_e) = (\phi\Phi_\#)(v),$$

which completes the proof. □

In particular, the group Z_Γ is a normal subgroup of $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$. Hence, there is a commutative diagram with exact rows as follows:

$$\begin{tikzcd} 1 \arrow[r] & Z_\Gamma \arrow[r] \arrow[d, hook] & \mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \arrow[r] \arrow[d, hook] & \mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma \arrow[r] \arrow[d, "\overline{\Psi}"] & 1 \\ 1 \arrow[r] & \mathrm{Inn}(A_\Gamma) \arrow[r] & \mathrm{Aut}_+(A_\Gamma) \arrow[r] & \mathrm{Out}_+(A_\Gamma) \arrow[r] & 1. \end{tikzcd} \tag{4.1}$$

The right vertical arrow $\overline{\Psi}$ is the induced map of the inclusion $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \to \mathrm{Aut}_+(A_\Gamma)$.

Theorem 4.5. *There is an isomorphism*

$$\overline{\Psi} : \mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma \cong \mathrm{Out}_+(A_\Gamma).$$

Therefore, by Lemma 4.1,

$$\mathrm{Out}(A_\Gamma) \cong \big(\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma\big) \rtimes \mathbb{Z}_2.$$

Proof. By Lemma 4.2, any automorphism in $\mathrm{Aut}_+(A_\Gamma)$ is a composition of inner automorphisms, partial conjugations and graph isomorphisms. Therefore, for each $[\phi] \in \mathrm{Out}_+(A_\Gamma)$, we have a representative $\phi \in \mathrm{Aut}_+(A_\Gamma)$ which is a composition of partial conjugations and graph isomorphisms. However, by definition of $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$, the map ϕ is also contained in $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$ and therefore $\overline{\Psi}$ is surjective.

Suppose that $\ker(\overline{\Psi})$ is nontrivial. Then there exists $\phi \in \mathrm{Inn}(A_\Gamma)\cap\mathrm{Aut}_{\mathcal{A}}(A_\Gamma) = Z_\Gamma$, which is trivial in $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma$ as desired. □

Unfortunately, the row exact sequences in (4.1) do not split in general. However, when $*_\Gamma$ is a chunk, then the group Z_Γ is trivial and therefore

$$\mathrm{Out}_+(A_\Gamma) \cong \mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \subset \mathrm{Aut}_+(A_\Gamma).$$

Corollary 4.6. *If $*_\Gamma$ is a chunk, then*

$$\mathrm{Aut}(A_\Gamma) \cong \mathrm{Inn}(A_\Gamma) \rtimes \mathrm{Out}(A_\Gamma), \qquad \mathrm{Out}(A_\Gamma) \cong \mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \rtimes \mathbb{Z}_2.$$

In particular, if Γ is furthermore discretely rigid, then we have the following corollary.

Corollary 4.7. *Let Γ be a discretely rigid* CLTTF *graph such that $*_\Gamma$ is a chunk. Then*

$$\begin{aligned}\mathrm{Aut}(A_\Gamma) &\cong \mathrm{Inn}(A_\Gamma) \rtimes \left(\left(\mathbb{Z}^{\#(\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]})} \rtimes \mathrm{Aut}(\Gamma)\right) \rtimes \mathbb{Z}_2\right),\\ \mathrm{Out}(A_\Gamma) &\cong \left(\mathbb{Z}^{\#(\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]})} \rtimes \mathrm{Aut}(\Gamma)\right) \rtimes \mathbb{Z}_2.\end{aligned}$$

Proof. This is a combination of Corollaries 2.37, 4.6 and Theorem 3.9. □

4.1.3. *Twisted intersection product.* Let us define the *twisted intersection product* $(\alpha, \beta) = \mathcal{E}(\eta_{\alpha,\beta})$ as a composition of edge-twists in $\mathcal{G}$, where

$$\eta_{\alpha,\beta}(\varepsilon) = \begin{cases}\overline{\eta}_\alpha(\varepsilon)\cdot\overline{\eta}_\beta(\varepsilon) & \varepsilon \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]};\\ 0 & \text{otherwise.}\end{cases}$$

Indeed, this product captures the carry-over of the sum of $\overline{\eta}_\alpha$ and $\alpha_*(\overline{\eta}_\beta)$ in binary arithmetic that we may lose since each edge-twist is involutive in $\mathrm{Twist}(\Gamma)$.

Note that the twisted intersection product is not necessarily commutative. Moreover, it satisfies the following properties.

Lemma 4.8. *For $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Twist}(\Gamma)$, we have*

$$(\alpha_1, \alpha_2\alpha_3)\cdot(\alpha_1)_*(\alpha_2, \alpha_3) = (\alpha_1\alpha_2, \alpha_3)\cdot(\alpha_1, \alpha_2).$$

Proof. For each $i = 1, 2, 3$, let $\alpha_i : \Gamma \to \Delta_i$ and $\overline{\mathcal{E}}_i = \overline{\mathcal{E}}(\overline{\eta}_i) : \Delta_i \to \Gamma$ be the unique composition of edge-twists. Then for each $\varepsilon \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]}$, both count the carry-over in the following sums

$$\begin{aligned}&\overline{\eta}_1(\varepsilon) + (\alpha_1)_*(\overline{\eta}_2(\varepsilon)) + (\alpha_1\alpha_2)_*(\overline{\eta}_3(\varepsilon))\\ = &\overline{\eta}_1(\varepsilon)\cdot(\alpha_1)_*(\overline{\eta}_2(\varepsilon)) + \overline{\eta}_1(\varepsilon)\cdot(\alpha_1\alpha_2)_*(\overline{\eta}_3(\varepsilon)) + (\alpha_1)_*(\overline{\eta}_2(\varepsilon))\cdot(\alpha_1\alpha_2)_*(\overline{\eta}_3(\varepsilon))\end{aligned}$$

by regarding them as binary digits. □

Lemma 4.9. *Let $\alpha_0 \in \mathrm{Aut}(\Gamma)$ and $\alpha_1, \alpha_2 \in \mathrm{Twist}(\Gamma)$. Then the following holds*:

(1) $(\alpha_0, \alpha_1) = (\alpha_1, \alpha_0) = \mathrm{Id}$,
(2) $(\alpha_1, \alpha_2\alpha_0) = (\alpha_1, \alpha_2)$,
(3) $(\alpha_0\alpha_1, \alpha_2) = (\alpha_0)_*(\alpha_1, \alpha_2)$,
(4) $(\alpha_1\alpha_0, \alpha_2) = (\alpha_1, \alpha_0\alpha_2)$.

Proof. (1) This is obvious since $\alpha_0 : \Gamma \to \Gamma$.

(2)–(4) These are immediate corollary of (1) and Lemma 4.8. □

Example 4.10. Recall the generators $\alpha_0, \dots, \alpha_4$ for $\mathrm{Twist}(\Gamma)$ as described in Example 2.29.

Let α and β be elements in $\mathrm{Twist}(\Gamma)$. By relation in (2.4), there exist two sequences $(i_0, \dots, i_4)$ and $(j_0, \dots, j_4)$ in $\{0, 1\}$ such that

$$\begin{aligned}\alpha &= \alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\alpha_4^{i_4}\alpha_0^{i_0}, & \mathcal{E} &= \varepsilon_1^{i_1}\varepsilon_2^{i_2}\varepsilon_3^{i_3}\varepsilon_4^{i_4} : \Gamma \to \alpha(\Gamma),\\ \alpha' &= \alpha_1^{j_1}\alpha_2^{j_2}\alpha_3^{j_3}\alpha_4^{j_4}\alpha_0^{j_0}, & \mathcal{E}' &= \varepsilon_1^{j_1}\varepsilon_2^{j_2}\varepsilon_3^{j_3}\varepsilon_4^{j_4} : \Gamma \to \alpha'(\Gamma).\end{aligned}$$

Then since $\alpha_1, \dots, \alpha_4$ preserve $\mathcal{C}\mathsf{h}_{[\![\Gamma]\!]}$ but α_0 interchanges ε_2 and ε_3, we have

$$\alpha_*(\mathcal{E}') = \begin{cases}\varepsilon_1^{j_1}\varepsilon_2^{j_2}\varepsilon_3^{j_3}\varepsilon_4^{j_4} & i_0 = 0;\\ \varepsilon_1^{j_1}\varepsilon_2^{j_3}\varepsilon_3^{j_2}\varepsilon_4^{j_4} & i_0 = 1,\end{cases} \quad (\alpha, \alpha') = \begin{cases}\varepsilon_1^{i_1j_1}\varepsilon_2^{i_2j_2}\varepsilon_3^{i_3j_3}\varepsilon_4^{i_4j_4} & i_0 = 0;\\ \varepsilon_1^{i_1j_1}\varepsilon_2^{i_2j_3}\varepsilon_3^{i_3j_2}\varepsilon_4^{i_4j_4} & i_0 = 1.\end{cases}$$

4.2. Group presentation for $\mathrm{Aut}(A_\Gamma)$

Let us define the sets:

$$\begin{aligned}S &= \text{even edge-twists and twist isomorphisms}\\ &= \left\{\varepsilon \;\middle|\; \varepsilon \in \mathcal{E}^{\mathsf{out},0}_{[\![\Gamma]\!]}\right\} \cup \left\{\varepsilon^2 \;\middle|\; \varepsilon \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]}\right\} \cup \{\widetilde{\alpha} \mid \alpha \in \mathrm{Twist}(\Gamma)\},\\ R_0 &= \text{relations coming from (the action on) } \mathrm{Inn}(A_\Gamma)\\ &= \{(s, t; m(e)) = (t, s; m(e)) \mid e = \{s, t\} \in E\}\\ &\quad \cup \{\varepsilon v = (\varepsilon_\#(v))\varepsilon \mid v \in V, \varepsilon \in S(\Gamma)\} \cup \{\varepsilon^2 v = (\varepsilon^2_\#(v))\varepsilon \mid v \in V, \varepsilon^2 \in S(\Gamma)\}\\ &\quad \cup \{\widetilde{\alpha} v = (\widetilde{\alpha}_\#(v))\widetilde{\alpha} \mid v \in V, \widetilde{\alpha} \in S(\Gamma)\} \cup \{\iota v = v^{-1}\iota \mid v \in V\},\\ R_1 &= \text{even edge-twist moves commute with each other}\\ &= \left\{\varepsilon\varepsilon' = \varepsilon'\varepsilon \;\middle|\; \varepsilon, \varepsilon' \in \mathcal{E}^{\mathsf{out},0}_{[\![\Gamma]\!]}\right\} \cup \left\{\varepsilon^2\varepsilon'^2 = \varepsilon'^2\varepsilon^2 \;\middle|\; \varepsilon, \varepsilon' \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]}\right\}\end{aligned}$$

$$
\begin{aligned}
&\cup \left\{ \varepsilon\varepsilon'^2 = \varepsilon'^2\varepsilon \;\middle|\; \varepsilon \in \mathcal{E}^{\mathsf{out},0}_{[\![\Gamma]\!]}, \varepsilon' \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]} \right\},\\
R_2 &= \text{twist isomorphisms conjugate even edge-twist moves to their}\\
&\quad\ \text{push-forwards}\\
&= \left\{ \widetilde{\alpha}\varepsilon = \alpha_*(\varepsilon)\widetilde{\alpha} \;\middle|\; \alpha \in \mathrm{Twist}(\Gamma), \varepsilon \in \mathcal{E}^{\mathsf{out},0}_{[\![\Gamma]\!]} \right\}\\
&\quad \cup \left\{ \widetilde{\alpha}\varepsilon^2 = \alpha_*(\varepsilon)^2\widetilde{\alpha} \;\middle|\; \alpha \in \mathrm{Twist}(\Gamma), \varepsilon \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]} \right\},\\
R_3 &= \text{twisted intersection products of products of twist isomorphisms}\\
&= \left\{ \widetilde{\alpha}\widetilde{\beta} = (\alpha,\beta)^2\widetilde{\alpha\beta} \;\middle|\; \alpha,\beta \in \mathrm{Twist}(\Gamma) \right\},\\
R_4 &= \text{actions of the global inversion}\\
&= \left\{\iota^2\right\} \cup \left\{ \widetilde{\alpha}\iota = \mathcal{E}^2_\alpha \iota\widetilde{\alpha} \;\middle|\; \alpha \in \mathrm{Twist}(\Gamma) \right\}\\
&\quad \cup \left\{ \varepsilon\iota\varepsilon\iota \mid \varepsilon \in \mathcal{E}^{\mathsf{out},0}_{[\![\Gamma]\!]} \right\} \cup \left\{ \varepsilon^2\iota\varepsilon^2\iota \mid \varepsilon \in \mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]} \right\},\\
\widetilde{R}_\Phi &= \text{the special automorphism in } \mathrm{Aut}(A_\Gamma)\\
&= \begin{cases} \varnothing & *_\Gamma \text{ is a chunk;}\\ \{\mathcal{E}_{*_\Gamma} = x_e\} & *_\Gamma \text{ is an even-labeled separating edge } e;\\ \{\widetilde{\alpha}_{*_\Gamma} = x_e\} & *_\Gamma \text{ is an odd-labeled separating edge } e, \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
R_\Phi &= \text{the special automorphism in } \mathrm{Out}(A_\Gamma)\\
&= \begin{cases} \varnothing & *_\Gamma \text{ is a chunk;}\\ \{\mathcal{E}_{*_\Gamma}\} & *_\Gamma \text{ is an even-labeled separating edge;}\\ \{\widetilde{\alpha}_{*_\Gamma}\} & *_\Gamma \text{ is an odd-labeled separating edge.} \end{cases}
\end{aligned}
$$

Recall (2.8) and (2.9) for the definitions of $\mathcal{E}_{*_\Gamma}$ and Φ, respectively. If $*_\Gamma$ is an even-labeled separating edge, then $\Phi = \mathcal{E}_{*_\Gamma} \in \mathrm{Dehn}_{\mathcal{G}}(\Gamma)$ is a word in S. Otherwise, if $*_\Gamma$ is an odd-labeled separating edge, then $\Phi = \mathcal{E}_{*_\Gamma}\alpha_{*_\Gamma}$ will be mapped to α_{*_Γ} via the map $\mathrm{Aut}_{\mathcal{G}}(\Gamma) \to \mathrm{Twist}(\Gamma)$. Hence it corresponds to $\widetilde{\alpha}_{*_\Gamma} \in S$.

Here, we have the main theorem of the paper.

Theorem 4.11. *Let $\Gamma = (V, E, m)$ be a CLTTF graph. Then the automorphism group $\mathrm{Aut}(A_\Gamma)$ and outer automorphism group admit the following finite group presentations*:

$$
\begin{aligned}
\mathrm{Aut}(A_\Gamma) &\cong \left\langle V, S, \iota \;\middle|\; R_0, R_1, R_2, R_3, R_4, \widetilde{R}_\Phi \right\rangle,\\
\mathrm{Out}(A_\Gamma) &\cong \left\langle V, S, \iota \;\middle|\; R_0, R_1, R_2, R_3, R_4, \widetilde{R}_\Phi, V \right\rangle\\
&\cong \left\langle S, \iota \mid R_1, R_2, R_3, R_4, R_\Phi \right\rangle.
\end{aligned}
$$

In order to prove this theorem, we first consider a short exact sequence

$$1 \longrightarrow \mathrm{Dehn}_{\mathcal{G}}(\Gamma) \longrightarrow \mathrm{Aut}_{\mathcal{G}}(\Gamma) \twoheadrightarrow \mathrm{Twist}(\Gamma) \longrightarrow 1, \tag{4.2}$$

where the quotient map

$$\begin{aligned} \mathrm{Aut}_{\mathcal{G}}(\Gamma) &\twoheadrightarrow \mathrm{Twist}(\Gamma) \\ \mathcal{E}\alpha &\longmapsto \alpha, \end{aligned}$$

is just a projection, which is well-defined since $\mathcal{E}:\Delta\to\Gamma$ if $\alpha:\Gamma\to\Delta$ and whose kernel is precisely $\mathrm{Dehn}_{\mathcal{G}}(\Gamma)$.

Now we want to provide a group presentation for $\mathrm{Aut}_{\mathcal{G}}(\Gamma)\cong\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$ by using the following proposition whose proof is elementary and will be omitted.

Proposition 4.12. *Let N and Q be groups admitting presentations*

$$N=\langle S_N \mid R_N\rangle,\quad Q=\langle S_Q\mid R_Q\rangle,$$

which fit into the short exact sequence

$$1\longrightarrow N\xrightarrow{\ i\ }G\xrightarrow{\ \pi\ }Q\longrightarrow 1.$$

Let $s:F(S_Q)\to F(S_N)$ be a group homomorphism between free groups $F(S_Q)$ and $F(S_N)$ on S_Q and S_N which makes the following diagram commutative:

$$\begin{tikzcd} F(S_Q) \arrow[rr, "s"] \arrow[d, two heads, "{[\cdot]}"'] & & F(S_N) \arrow[d, two heads, "{[\cdot]}"] \\ Q & G \arrow[l, "\pi"] & N. \arrow[l, "i"] \end{tikzcd}$$

Here the vertical maps are the canonical surjections. Then G admits a group presentation

$$G\cong\langle S_N\cup s(S_Q)\mid R_N\cup R_C\cup\widetilde{R}_Q\rangle,$$

where

$$\begin{aligned} R_C&=\{s(t)gs(t)^{-1}w^{-1}\mid g\in S_N, t\in S_Q, w\in N, [s(t)gs(t)^{-1}]=i(w)\in G\},\\ \widetilde{R}_Q&=\{s(r)h^{-1}\mid r\in R_Q, h\in N, [s(r)]=i(h)\in G\}. \end{aligned}$$

We want to use the short exact sequence in (4.2) and the proposition above to obtain a group presentation of $\mathrm{Aut}_{\mathcal{G}}(\Gamma)$.

For each $(\alpha:\Gamma\to\Delta)\in\mathrm{Twist}(\Gamma)$, there is a unique composition of edge-twists $\overline{\mathcal{E}}_\alpha=\overline{\mathcal{E}}(\overline{\eta}_\alpha):\Delta\to\Gamma$ for some $\overline{\eta}_\alpha:\mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]}\to\{0,1\}$ by Proposition 2.17. Let $\eta_\alpha:\mathcal{E}^{\mathsf{out}}_{[\![\Gamma]\!]}\to\mathbb{Z}$ be a function defined as

$$\eta_\alpha(\varepsilon)=\begin{cases}1 & \varepsilon\in\mathcal{E}^{\mathsf{out},1}_{[\![\Gamma]\!]},\ \overline{\eta}_\alpha(\varepsilon)=1;\\ 0 & \text{otherwise.}\end{cases}$$

Then we have a lift $\widetilde{\alpha}\coloneqq\mathcal{E}_\alpha\alpha:\Gamma\to\Gamma$ of α, where $\mathcal{E}_\alpha\coloneqq\mathcal{E}(\eta_\alpha)$.

For each $g \in \mathrm{Dehn}_{\mathcal{G}}(\Gamma)$, the conjugate of g by $\widetilde{\alpha}$ is then

$$\widetilde{\alpha} g \widetilde{\alpha}^{-1} = (\mathcal{E}_\alpha \alpha) g (\alpha^{-1} \mathcal{E}_\alpha^{-1}) = \mathcal{E}_\alpha \alpha_*(g) \mathcal{E}_\alpha^{-1} = \alpha_*(g).$$

For $\alpha, \beta \in \mathrm{Twist}(\Gamma)$, we have the following:

$$\widetilde{\alpha}\widetilde{\beta} = \mathcal{E}_\alpha \alpha \mathcal{E}_\beta \beta = \mathcal{E}_\alpha \alpha_*(\mathcal{E}_\beta) \alpha\beta \quad \text{and} \quad \widetilde{\alpha\beta} = \mathcal{E}_{\alpha\beta}(\alpha\beta),$$

which coincide in $\mathrm{Twist}(\Gamma)$ and so

$$\mathcal{E}_\alpha \alpha_*(\mathcal{E}_\beta) \mathcal{E}_{\alpha\beta}^{-1}$$

is a composition of even edge-twists. Indeed, this is exactly the same as $(\alpha, \beta)^2$ by the meaning of the twisted intersection product as mentioned earlier. Therefore we have

$$\mathcal{E}_\alpha \alpha_*(\mathcal{E}_\beta) \mathcal{E}_{\alpha\beta}^{-1} = (\alpha, \beta)^2 \in \mathrm{Dehn}_{\mathcal{G}}(\Gamma), \quad \text{or equivalently,} \quad \widetilde{\alpha}\widetilde{\beta} = (\alpha, \beta)^2 \widetilde{\alpha\beta}.$$

Proposition 4.13. *The groups* $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$ *and* $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma$ *admit the finite group presentations*

$$\mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \cong \langle S \mid R_1, R_2, R_3 \rangle, \qquad \mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma \cong \langle S \mid R_1, R_2, R_3, R_\Phi \rangle.$$

Proof. We use Proposition 4.12 on (4.2). Since $\mathrm{Dehn}_{\mathcal{G}}(\Gamma)$ is a free abelian group generated by even edge-twists, the group $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$ is generated by the set S. Moreover the sets R_1, R_2 and R_3 correspond to R_N, R_C and $\widetilde{R}_Q$ in Proposition 4.12 and the generator for Z_Γ corresponds to the element in R_Φ if $R_\Phi \neq \varnothing$. Therefore we are done. $\square$

Proof of Theorem 4.11. We first find the group presentation for $\mathrm{Out}(A_\Gamma)$, which is isomorphic to $(\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma) \rtimes \mathbb{Z}_2$ by Theorem 4.5. Therefore we need to justify relations in R_4.

Since ι is an involution, ι^2 is the identity. Let $\varepsilon = (e = \{s, t\}, C)$ be an edge in $\mathcal{E}_\Gamma^{\mathsf{out}}$. For each $v \in V$,

$$\begin{aligned} \iota\varepsilon_\#(v) &= \begin{cases} \iota(v) & v \in V_1(\varepsilon); \\ \iota(x_e^{-1} v x_e) & v \notin V_1(\varepsilon), \end{cases} \\ &= \begin{cases} v & v \in V_1(\varepsilon); \\ \overline{x_e}^{-1} v^{-1} \overline{x_e} & v \notin V_1(\varepsilon), \end{cases} \end{aligned}$$

where $\overline{x_e}$ is the reverse of x_e and identical to x_e in A_Γ. Therefore,

$$\iota\varepsilon_\#(v) = (\varepsilon_\#)^{-1}(v^{-1}) = \varepsilon_\#^{-1}\iota(v)$$

and so the second or third type of relations follows. We also note that for each $\varepsilon_1, \varepsilon_2 \in \mathcal{E}_\Gamma^{\mathsf{out}}$,

$$\iota(\varepsilon_1)_\#(\varepsilon_2)_\# = (\varepsilon_1)_\#^{-1}(\varepsilon_2)_\#^{-1}\iota. \tag{4.3}$$

In order to check if the relation $\widetilde{\alpha}\iota = \mathcal{E}_\alpha^2\iota\widetilde{\alpha}$ holds, let us regard $\widetilde{\alpha}$ as the composition $\widetilde{\alpha} = \mathcal{E}_\alpha\alpha$, where $\alpha : \Gamma \to \Delta$ is a graph isomorphism for some Δ and $\overline{\mathcal{E}}_\alpha : \Delta \to \Gamma$ is a composition of edge-twists. Then by (4.3)

$$(\iota\widetilde{\alpha})(v) = \iota(\mathcal{E}_\alpha)_\#\alpha(v) = (\mathcal{E}_\alpha)_\#^{-1}(\iota\alpha)(v) = (\mathcal{E}_\alpha)_\#^{-1}(\alpha\iota)(v)$$

for each $v \in V$. Hence $\iota\widetilde{\alpha} = (\mathcal{E}_\alpha)_\#^{-2}\widetilde{\alpha}\iota$, or equivalently, $(\mathcal{E}_\alpha)_\#^2\iota\widetilde{\alpha} = \widetilde{\alpha}\iota$.

Finally, we use Proposition 4.12 again to the short exact sequence between automorphism groups:

$$1 \longrightarrow \mathrm{Inn}(A_\Gamma) \longrightarrow \mathrm{Aut}(A_\Gamma) \longrightarrow \mathrm{Out}(A_\Gamma) \longrightarrow 1.$$

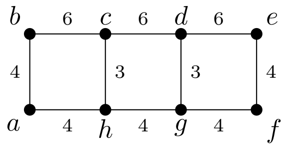
Since $\mathrm{Inn}(A_\Gamma) \cong A_\Gamma$, we may use the group presentation for A_Γ. Therefore, the group $\mathrm{Aut}(A_\Gamma)$ is generated by two sets V and S defined earlier. The relations are consisting of three types of relations such that (i) the original relations in A_Γ, (ii) the action of $\mathrm{Out}(A_\Gamma)$ on $A_\Gamma \cong \mathrm{Inn}(A_\Gamma)$, and (iii) lifts of relations in $\mathrm{Out}(A_\Gamma)$.

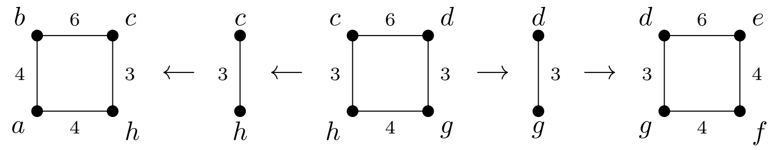
The generating set is obvious. The set R_0 consists of relations in A_Γ and relations corresponding to the action of $\mathrm{Out}(A_\Gamma)$ on A_Γ, where the latter relations are obvious by definition. Moreover, all relations but R_Φ in $\mathrm{Out}(A_\Gamma)$ hold in $\mathrm{Aut}(\Gamma)$ as well and the can be lifted without any modification. However, the relation R_Φ may not hold in $\mathrm{Aut}(\Gamma)$ when $*_\Gamma$ is a separating edge. In this case, we should identify $\mathcal{E}_{*_\Gamma}$ or $\widetilde{\alpha}_{*_\Gamma}$ with the inner automorphism $(x_e)_\#$, which is just x_e in our presentation. This completes the proof. $\square$

4.3. Examples

We will compute (outer) automorphism groups for various CLTTF graphs.

4.3.1. *Discretely rigid with the central chunk.* As seen earlier, the following CLTTF graph Γ is not rigid but discretely rigid:

$\Gamma =$ 

$\mathcal{C}\mathsf{h}_\Gamma =$ 

$$C_1 \longleftarrow e_1 \longleftarrow C = *_\Gamma \longrightarrow e_2 \longrightarrow C_2$$

Then in the chunk tree $\mathcal{C}\mathsf{h}_\Gamma$, the central vertex $*_\Gamma$ is a chunk $C = (V_C, E_C)$ with $V_C = \{c, d, g, h\}$ and there are two outward edges

$$\varepsilon_1 = (e_1, C_1) \quad \text{and} \quad \varepsilon_2 = (e_2, C_2),$$

where C_1 and C_2 are induced subgraphs of Γ with $V_{C_1} = \{a, b, c, h\}$ and $V_{C_2} = \{d, e, f, g\}$. Moreover $\mathrm{Aut}(\Gamma) = \langle \alpha \mid \alpha^2 \rangle$, where $\alpha : \Gamma \to \Gamma$ is the obvious horizontal reflection that interchanges edge-twists ε_1 and ε_2. Therefore by Corollary 4.6,

$$\mathrm{Aut}(A_\Gamma) \cong \mathrm{Inn}(A_\Gamma) \rtimes \mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \cong A_\Gamma \rtimes \left(\left(\langle \varepsilon_1, \varepsilon_2 \rangle \rtimes \langle \alpha \mid \alpha^2 \rangle \right) \rtimes \langle \iota \mid \iota^2 \rangle \right).$$

The followings are precise relations:

- Inner automorphism group relation and the action of $\mathrm{Out}(A_\Gamma)$ on $\mathrm{Inn}(A_\Gamma)$

$$\begin{aligned} R_0 = \{ (s,t;m(e)) = (t,s;m(e)) \mid e = \{s,t\} \in E \} \cup \{ \alpha v = (\alpha(v))\alpha \mid v \in V \} \\ \cup \{ \varepsilon_i v = (\varepsilon_i(v))\varepsilon_i \mid v \in V, i = 1,2 \} \cup \{ \iota v = v^{-1}\iota \mid v \in V \}. \end{aligned}$$

- Commutative relations between ε_1 and ε_2

$$R_1 = \{ \varepsilon_1 \varepsilon_2 = \varepsilon_2 \varepsilon_1 \}.$$

- The action of α on $\langle \varepsilon_1, \varepsilon_2 \rangle$ and involutivity

$$R_2 = \{ \alpha \varepsilon_1 = \varepsilon_2 \alpha \}, \quad R_3 = \{ \alpha^2 \}.$$

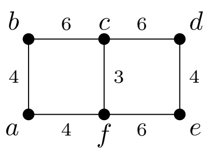
- The action of ι on $\mathrm{Aut}_{\mathcal{A}}(\Gamma)$

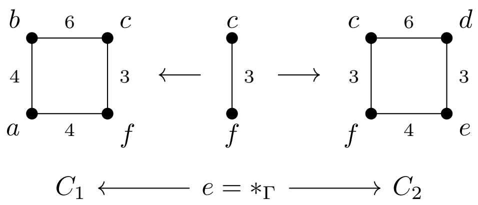
$$R_4 = \{ \iota^2, \iota\alpha = \alpha\iota \} \cup \{ \iota \varepsilon_i = \varepsilon_i^{-1} \iota \mid i = 1,2 \}.$$

Therefore we have the following group presentations:

$$\begin{aligned} \mathrm{Aut}(A_\Gamma) &= \langle V, \varepsilon_1, \varepsilon_2, \alpha, \iota \mid R_0, R_1, R_2, R_3, R_4 \rangle, \\ \mathrm{Out}(A_\Gamma) &= \langle \varepsilon_1, \varepsilon_2, \alpha, \iota \mid R_1, R_2, R_3, R_4 \rangle. \end{aligned}$$

4.3.2. *Discretely non-rigid with the central separating edge.* The CLTTF graph Γ below is rigid but not discretely rigid with the central separating edge $e = \{c, f\}$:

$\Gamma =$ 

$\mathcal{C}h_\Gamma =$ 

There are two outward edges in $\mathcal{E}_\Gamma^{\mathsf{out}}$

$$\varepsilon_1 = (e_1, C_1) \quad \text{and} \quad \varepsilon_2 = (e_2, C_2)$$

and so we have four CLTTF graphs edge-twist equivalent to Γ

$$\Gamma_0 \coloneqq \Gamma, \qquad \Gamma_1 \coloneqq \bar{\varepsilon}_1(\Gamma), \qquad \Gamma_2 \coloneqq \bar{\varepsilon}_2(\Gamma), \quad \text{and} \quad \Gamma_3 \coloneqq \bar{\varepsilon}_1\bar{\varepsilon}_2(\Gamma),$$

which are all isomorphic to Γ. Moreover, since the group of graph automorphisms is trivial, each edge-twist equivalent graph Γ_i has the unique graph isomorphism α_i, where

$$\alpha_0 = \mathrm{Id}_V,\quad \alpha_1(v) = \begin{cases} f & v = c;\\ e & v = d;\\ d & v = e;\\ c & v = f;\\ v & v \in \{a, b\},\end{cases}\qquad \alpha_2(v) = \begin{cases} e & v = d;\\ d & v = e;\\ v & v \neq d, e,\end{cases}\qquad \alpha_3 = \alpha_1\alpha_2.$$

Therefore

$$\mathrm{Twist}(\Gamma) = \left\langle \alpha_1, \alpha_2 \mid \alpha_1\alpha_2 = \alpha_2\alpha_1, \alpha_1^2, \alpha_2^2\right\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Since both α_1 and α_2 act trivially on $\mathcal{C}\mathsf{h}_\Gamma$, we have

$$(\alpha_i)_*(\varepsilon_j) = \varepsilon_j \quad \text{for all} \quad 1 \leq i, j \leq 2$$

and so the twisted intersection is then defined as follows: for $i = 1, 2$,

$$(\alpha_i, \alpha_i) = (\alpha_i, \alpha_3) = (\alpha_3, \alpha_i) = \varepsilon_i,\ (\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = \mathrm{Id},\ (\alpha_3, \alpha_3) = \varepsilon_1\varepsilon_2.$$

Now we compute the group $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$, which is generated by

$$\varepsilon_1^2, \varepsilon_2^2, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3.$$

The sets of relations are as follows:

$$\begin{aligned}
R_1 &= \{\varepsilon_1^2\varepsilon_2^2 = \varepsilon_2^2\varepsilon_1^2\},\\
R_2 &= \{\widetilde{\alpha}_i\varepsilon_j^2 = \varepsilon_j^2\widetilde{\alpha}_i \mid 1 \leq i \leq 3, 1 \leq j \leq 2\},\\
R_3 &= \{\widetilde{\alpha}_i^2 = \varepsilon_i^2 \mid 1 \leq i \leq 2\} \cup \{\widetilde{\alpha}_3^2 = \varepsilon_1^2\varepsilon_2^2\}\\
&\quad \cup\{\widetilde{\alpha}_1\widetilde{\alpha}_2 = \widetilde{\alpha}_2\widetilde{\alpha}_1 = \widetilde{\alpha}_3, \widetilde{\alpha}_1\widetilde{\alpha}_3 = \widetilde{\alpha}_3\widetilde{\alpha}_1 = \varepsilon_1^2\widetilde{\alpha}_2, \widetilde{\alpha}_2\widetilde{\alpha}_3 = \widetilde{\alpha}_3\widetilde{\alpha}_2 = \varepsilon_1^2\widetilde{\alpha}_1\}.
\end{aligned}$$

Hence one can easily see that

$$\mathrm{Aut}_{\mathcal{A}}(\Gamma) = \left\langle \varepsilon_1^2, \varepsilon_2^2, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3 \mid R_1, R_2, R_3\right\rangle = \langle \widetilde{\alpha}_1, \widetilde{\alpha}_2 \mid \widetilde{\alpha}_1\widetilde{\alpha}_2 = \widetilde{\alpha}_2\widetilde{\alpha}_1\rangle \cong \mathbb{Z}^2.$$

Moreover, since the central element $*_\Gamma$ is an odd-labeled edge, we have the nontrivial subgroup Z_Γ generated by $\mathcal{E}_{*_\Gamma}\alpha_{*_\Gamma}$, where

$$\mathcal{E}_{*_\Gamma} = \varepsilon_1\varepsilon_2 \quad \text{and} \quad \alpha_{*_\Gamma} = \alpha_1\alpha_2.$$

That is, in the above group presentation, the generator for Z_Γ corresponds to $\widetilde{\alpha}_1\widetilde{\alpha}_2$ and

$$\begin{aligned}
\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma &\cong \left\langle \varepsilon_1^2, \varepsilon_2^2, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3 \mid R_1, R_2, R_3, R_\Phi\right\rangle\\
&\cong \langle \widetilde{\alpha}_1, \widetilde{\alpha}_2 \mid \widetilde{\alpha}_1\widetilde{\alpha}_2 = \widetilde{\alpha}_2\widetilde{\alpha}_1, \widetilde{\alpha}_1\widetilde{\alpha}_2\rangle \cong \mathbb{Z},
\end{aligned}$$

where $R_\Phi = \{\varepsilon_1\varepsilon_2\}$. We have the following commutative diagram with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & Z_\Gamma \arrow[r] \arrow[d, "\cong"] & \mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \arrow[r] \arrow[d, "\cong"] & \mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma \arrow[r] \arrow[d, "\cong"] & 1 \\ 1 \arrow[r] & \langle \widetilde{\alpha}_1\widetilde{\alpha}_2 \rangle \arrow[r] & \langle \widetilde{\alpha}_1, \widetilde{\alpha}_2 \rangle \arrow[r, "\widetilde{\alpha}_2 \mapsto \widetilde{\alpha}_1^{-1}"] & \langle \widetilde{\alpha}_1 \rangle \arrow[r] & 1. \end{tikzcd} \tag{4.4}$$

Therefore by Theorem 4.5, we have

$$\mathrm{Out}(A_\Gamma) \cong (\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)/Z_\Gamma) \rtimes \mathbb{Z}_2 \cong \langle \widetilde{\alpha}_1, \iota \mid \iota\widetilde{\alpha}_1\iota = \widetilde{\alpha}_1^{-1} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}_2,$$

where the last relation comes from

$$\widetilde{\alpha}_1\iota = \varepsilon_1^2\iota\widetilde{\alpha}_1 \iff \widetilde{\alpha}_1\iota = \widetilde{\alpha}_1^2\iota\widetilde{\alpha}_1 \iff \iota\widetilde{\alpha}_1\iota = \widetilde{\alpha}_1^{-1}.$$

Moreover, the below row in (4.4) splits and so we may regard $\mathrm{Out}(A_\Gamma)$ as a subgroup of $\mathrm{Aut}(A_\Gamma)$ so that $\mathrm{Aut}(A_\Gamma) \cong \mathrm{Inn}(A_\Gamma) \rtimes \mathrm{Out}(A_\Gamma)$. Hence the automorphism group $\mathrm{Aut}(A_\Gamma)$ admits the following presentation:

$$\mathrm{Aut}(A_\Gamma) \cong \left\langle V, \widetilde{\alpha}_1, \iota \mid R_0, \iota\widetilde{\alpha}_1\iota = \widetilde{\alpha}_1^{-1} \right\rangle,$$

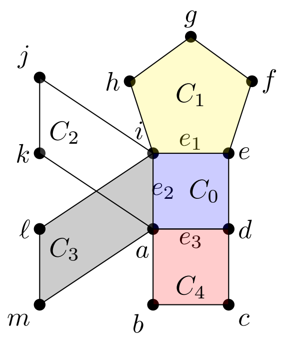
where

$$R_0 = \{(s,t;m(e)) = (t,s;m(e)) \mid e = \{s,t\} \in E\} \cup \{\widetilde{\alpha}_1 v = (\widetilde{\alpha}_1)_\#(v)\widetilde{\alpha}_1, \iota v = v^{-1}\iota\}$$

and

$$(\widetilde{\alpha}_1)_\#(v) = \begin{cases} f & v = c; \\ e & v = d; \\ d & v = e; \\ c & v = f; \\ (cfc)^{-1}v(cfc) & v \in \{a,b\}. \end{cases}$$

4.3.3. *Discretely non-rigid with the central chunk.* The CLTTF graph below is discretely non-rigid and the center $*_\Gamma$ in the chunk tree is a chunk C_0 as seen earlier.

$\Gamma =$ 

Therefore by Corollary 4.6, $\mathrm{Aut}(A_\Gamma) \cong \mathrm{Inn}(A_\Gamma) \rtimes (\mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \rtimes \mathbb{Z}_2)$. Recall the generating set $\{\alpha_0, \ldots, \alpha_4\}$ for $\mathrm{Twist}(\Gamma)$ and edge-twists $\overline{\varepsilon}_1, \ldots, \overline{\varepsilon}_4$ as described in Example 2.29 so that $\overline{\varepsilon}_i\alpha_i : \Gamma \to \Gamma$ for each $1 \le i \le 4$.

The set S of generators for $\mathrm{Aut}_{\mathcal{A}}(A_\Gamma)$ is then

$$S = \{\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_4, \varepsilon_1^2, \dots, \varepsilon_4^2\}$$

and there are three types of relations

$$\begin{aligned}
R_1 &= \{\varepsilon_i^2\varepsilon_j^2 = \varepsilon_j^2\varepsilon_i^2 \mid 1 \le i, j \le 4\},\\
R_2 &= \{\widetilde{\alpha}_i\varepsilon_j^2 = \varepsilon_j^2\widetilde{\alpha}_i \mid 1 \le i, j \le 4\} \cup \{\widetilde{\alpha}_0\varepsilon_i^2 = \varepsilon_i^2\widetilde{\alpha}_0 \mid i = 1, 4\}\\
&\quad \cup \{\widetilde{\alpha}_0\varepsilon_1^2 = \varepsilon_2^2\widetilde{\alpha}_0, \widetilde{\alpha}_0\varepsilon_2^2 = \varepsilon_1^2\widetilde{\alpha}_0\},\\
R_3 &= \{\widetilde{\alpha}_0^2\} \cup \{\widetilde{\alpha}_i^2 = \varepsilon_i^2 \mid 1 \le i \le 4\} \cup \{\widetilde{\alpha}_i\widetilde{\alpha}_j = \widetilde{\alpha}_j\widetilde{\alpha}_i \mid 1 \le i, j \le 4\}\\
&\quad \cup\{\widetilde{\alpha}_0\widetilde{\alpha}_i = \widetilde{\alpha}_i\widetilde{\alpha}_0 \mid i = 1, 4\} \cup \{\widetilde{\alpha}_0\widetilde{\alpha}_2 = \widetilde{\alpha}_3\widetilde{\alpha}_0, \widetilde{\alpha}_0\widetilde{\alpha}_3 = \widetilde{\alpha}_2\widetilde{\alpha}_0\}
\end{aligned}$$

so that

$$\mathrm{Aut}_{\mathcal{A}}(A_\Gamma) \cong \langle \widetilde{\alpha}_0, \dots, \widetilde{\alpha}_4, \varepsilon_1^2, \dots, \varepsilon_4^2 \mid R_1, R_2, R_3\rangle.$$

Now the action of ι gives us relations in $\mathrm{Out}(A_\Gamma)$ as follows:

$$R_4 = \{\iota^2\} \cup \{\iota\widetilde{\alpha}_0 = \widetilde{\alpha}_0\iota\} \cup \{\widetilde{\alpha}_i\iota = \varepsilon_i^2\iota\widetilde{\alpha}_i \mid 1 \le i \le 4\} \cup \{\varepsilon_i\iota\varepsilon_i\iota \mid 1 \le i \le 4\}$$

and we have a group presentation

$$\mathrm{Out}(A_\Gamma) = \left\langle \widetilde{\alpha}_0, \dots, \widetilde{\alpha}_4, \varepsilon_1^2, \dots, \varepsilon_4^2, \iota \;\middle|\; R_1, R_2, R_3, R_4 \right\rangle.$$

One can reduce this presentation so that

$$\begin{aligned}
\mathrm{Aut}_{\mathcal{A}}(A_\Gamma) &\cong (\langle\widetilde{\alpha}_1\rangle \times \langle\widetilde{\alpha}_2\rangle \times \langle\widetilde{\alpha}_3\rangle \times \langle\widetilde{\alpha}_4\rangle) \rtimes \langle\widetilde{\alpha}_0 \mid \widetilde{\alpha}_0^2\rangle \cong \mathbb{Z}^4 \rtimes \mathbb{Z}_2,\\
\mathrm{Out}(A_\Gamma) &\cong (\langle\widetilde{\alpha}_1\rangle \times \langle\widetilde{\alpha}_2\rangle \times \langle\widetilde{\alpha}_3\rangle \times \langle\widetilde{\alpha}_4\rangle) \rtimes \left(\langle\widetilde{\alpha}_0 \mid \widetilde{\alpha}_0^2\rangle \times \langle\iota \mid \iota^2\rangle\right) \cong \mathbb{Z}^4 \rtimes \mathbb{Z}_2^2.
\end{aligned}$$

Here ι acts on $\widetilde{\alpha}_0$ trivially and on $\widetilde{\alpha}_i$ for $1 \le i \le 4$ as

$$\iota\widetilde{\alpha}_i\iota = \widetilde{\alpha}_i^{-1}.$$

Finally, the group presentation for $\mathrm{Aut}(A_\Gamma)$ is given as

$$\begin{aligned}
\mathrm{Aut}(A_\Gamma) &\cong \left\langle V, \widetilde{\alpha}_0, \dots, \widetilde{\alpha}_4, \varepsilon_1^2, \dots, \varepsilon_4^2, \iota \;\middle|\; R_0, R_1, R_2, R_3, R_4 \right\rangle\\
&\cong A_\Gamma \rtimes \left(\mathbb{Z}^4 \rtimes \mathbb{Z}_2^2\right),
\end{aligned}$$

where

$$\begin{aligned}
R_0 = \{&(s, t; m(e)) = (t, s; m(e)) \mid e = \{s, t\} \in E\}\\
&\cup\{\widetilde{\alpha}_i v = (\widetilde{\alpha}_i)_\#(v)\widetilde{\alpha}_i \mid 0 \le i \le 4, v \in V\}\\
&\cup\{\varepsilon_i^2 v = (\varepsilon_i)_\#^2(v)\varepsilon_i^2 \mid 1 \le i \le 4, v \in V\}\\
&\cup\{\iota v = v^{-1}\iota \mid v \in V\}.
\end{aligned}$$

Remark 4.14. The above presentation can be reduced further but we omit the detail.

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