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# Utilizing a unit Gompertz distorted copula to model dependence in anthropometric data

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**Abstract**

In this research, a conversion function and a distortion associated with the conversion function are defined and used to derive a unit power Gompertz distortion. A new family of copulas is built using the global distorted function. Four base copulas, namely Clayton, Gumbel, Frank, and Gaussian, are distorted into the family. Some properties including tail dependence coefficients and tail order are examined. Kendall's tau formula is derived for new copulas when the base copula is Clayton, Gumbel, or Frank. The maximum pseudo-likelihood estimation method is employed, and a simulation study was performed. The log-likelihood and AIC are reported to compare the performance of the fitted copulas. According to the applied data, the results indicate that new distorted copulas with additional parameters improve the fit.

Keywords: distortion, copula, Gompertz, tail dependence, Kendall's tau

## 1. Introduction

Dependence occurs between different events or co-movements in a variety of real-world situations, including finance, medicine, and insurance (McNeil *et al.*, 2006; Cherubini *et al.*, 2004; Klugman and Parsa, 1999). One of the functions commonly used to investigate this dependence is known as copula. Sklar (1959) explores the first notation for a copula, and he defines the copula as a multivariate distribution function with uniform margins; for further details, see Joe (2015). Dependence can be represented by one or more variables, and the choice of a fitted copula model for the dependence between random variables can be carried out independently from the selection of margins (Jaworski *et al.*, 2010; Frees and Valdes, 1998).

Sklar's theorem has been considered a main result for the copula function recognized in the literature, and from the result of this theorem, the copula method (or sometimes called the inversion method, e.g., see Nelsen, 2006), which plays a key role in deriving numerous copulas such as Clayton, Frank, Gumbel, and Gaussian from their distributions (Nelsen, 2006), is established. In recent work, the focus has been on generating new copulas with extra parameters to make copula models more flexible, thereby potentially providing a better fit (Aldhufairi and Sepanski, 2020; Xie *et al.*, 2019).

The motivation of this study is to highlight an alternative approach for deriving a new distortion, generalizing some results of distorted copulas, and constructing a new family of distorted copulas with three parameters via a unit power Gompertz distortion, and that family can be utilized to model

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several phenomena, such as in financial studies. This is robust, as new copula models indicate a better fit for real data.

The remainder of this paper is organized as follows: In Section 2, the family of the UPG copulas and their corresponding conditional copula and copula density are formulated, and some examples are offered after limiting cases in parameters are provided. Section 3 presents the conversion function and the unit power Gompertz (UPG) distortion, and it studies the admissibility conditions on the parameters. Section 4 examines tail behaviors, and the Kendall's tau measure is detailed in Section 5. To assess the performance of the new UPG-distorted copula models, we apply the data set in Section 6, while the concluding remarks are provided in Section 7.

## 2. Copula models

Let $X_1, X_2, \ldots, X_n$ be continuous random variables. Consider $F_i(x_i) = P(X_i \leq x_i)$ corresponds to a univariate margin. Let $H$ be a multivariate distribution function.

### 2.1. Copula

Two properties hold for any base copula $C$: 1) $C$ is grounded, that is, $C(u_1, \ldots, u_n) = 0$ if $u_i = 0$ for at least one index $i \in \{1, 2, \ldots, n\}$ and $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ if all the coordinates of $(u_1, \ldots, u_i, \ldots, u_n)$ are 1 except $u_i$; and 2) $C$ is $n-$increasing, that is, for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0, 1] \times \cdots \times [0, 1]$ ($n$ times) such that $a_i \leq b_i$ for all $1 \leq i \leq n$, we have $\Delta^{(n)}_{(a_n,b_n)} \Delta^{(n-1)}_{(a_{n-1},b_{n-1})} \cdots \Delta^{(1)}_{(a_1,b_1)} C(u_1, u_2, \ldots, u_n) \geq 0$, where $\Delta^{(i)}_{(a_i,b_i)} C(u_1, u_2, \ldots, u_n) = C(u_1, \ldots, u_{i-1}, b_i, u_{i+1}, \ldots, u_n) - C(u_1, \ldots, u_{i-1}, a_i, u_{i+1}, \ldots, u_n)$. From Sklar's theorem (see Nelsen, 2006), there exists a unique copula $C$ such that

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

If $U_i = F_i(X_i) \overset{\text{iid}}{\sim} \text{Unif}[0, 1]$, for $i = 1, 2, \ldots, n$, we can write

$$C(u_1, u_2, \ldots, u_n) = H\left(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\right).$$

If $H$ is $n$ times differentiable function, the joint probability density function $h$ can be obtained by the following

$$h(x_1, x_2, \ldots, x_n) = \partial^n H(x_1, x_2, \ldots, x_n) / \partial x_n \cdots \partial x_2 \partial x_1.$$

The conditional copula and copula density can be, respectively, formulated by $C(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n | u_i) = \partial C(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_n)/\partial u_i$ and $c(u_1, u_2, \ldots, u_n) = \partial^n C(u_1, u_2, \ldots, u_n)/\partial u_n \cdots \partial u_2 \partial u_1$.

### 2.2. Distortion

A distortion $T$, which maps from $[0, 1]$ to $[0, 1]$, is defined as an increasing and continuous function satisfying $T(0) = 0$ and $T(1) = 1$. It is described as the simultaneous distortion of the margins and copula in Valdez and Xiao's (2011) work, see Durante and Sempi (2005). For $i = 1, 2, 3, \ldots, n$, if the function $T_i : [0, 1] \to [0, 1]$ is increasing and continuous with $T_i(0) = 0$ and $T_i(1) = 1$, Di Bernardino and Rulliere (2013) state that the global distorted distribution function $\tilde{H}$ of $H$ is defined as

$$\tilde{H}(x_1, x_2, \ldots, x_n) = T \circ C\left(T_1^{-1}(F_1(x_1)), \ldots, T_n^{-1}(F_n(x_n))\right). \tag{2.1}$$

The marginal distribution of $\tilde{H}$ is given by $\tilde{F}_i = T \circ T_i^{-1} \circ F_i$, for $i = 1, 2, \ldots, n$. Charpentier (2008) and Valdez and Xiao (2011) dealt with a particular situation where $T = T_1 = \cdots = T_i$, and their study involved the bivariate case.

Let $\mathbb{R}$ be a set of real numbers. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be any bijective and increasing function, then $\zeta$ is said to be a conversion function, and by Di Bernardino and Rulliere (2013), the associate distortion to $\zeta$ is given by $T_\zeta : [0,1] \to [0,1]$ such that

$$T_\zeta(u) = \text{logit}^{-1}\left(\zeta\left(\text{logit}(u)\right)\right) = \frac{e^{\zeta(\log(u/(1-u)))}}{1 + e^{\zeta\left(\log(u/(1-u))\right)}}. \tag{2.2}$$

For all $u \in [0,1]$, $T_\zeta$ is increasing because $T'_\zeta(u) \geq 0$, and $T_\zeta(0) = 0$, $T_\zeta(1) = 1$. Furthermore, the inverse function of $T_\zeta$ satisfies $T_\zeta^{-1} = T_{\zeta^{-1}}$.

Denote the distorted function by $\tilde{C}$, which does not have necessarily to be a copula, such that $\tilde{H}(x_1, x_2, \ldots, x_n) = \tilde{C}(\tilde{F}_1(x_1), \ldots, \tilde{F}_n(x_n))$. According to Proposition 2.5, as derived from the work of Di Bernardino and Rulliere (2013), the function $\tilde{C}$ is only affected by the external distortion $T$, that is,

$$\tilde{C}\left(u_1, \ldots, u_n\right) = T\left(C\left(T^{-1}(u_1), \ldots, T^{-1}(u_n)\right)\right). \tag{2.3}$$

$T$ is deemed to be an admissible distortion if (2.3) is a copula. Given that $s_1, s_2, r_1$, and $r_2$ are real numbers with two intervals $I_1$ and $I_2$ in $\mathbb{R}$ such that $r_1 \leq r_2$ and $s_1 \leq s_2$, it is said that the function $L : I_1 \times I_2 \to \mathbb{R}$ is totally positive of order 2, denoted by $\text{TP}_2$, if

$$L\left(r_1, s_1\right) L\left(r_2, s_2\right) \geq L\left(r_1, s_2\right) L\left(r_2, s_1\right).$$

In general, if the base bivariate copula $C$ is $\text{TP}_2$ and $T \circ \exp$ is log-convex, then the bivariate distorted function $\tilde{C}$ is $\text{TP}_2$, and thus, it satisfies the 2-increasing property based on Lemma 3.1 stated by Durante *et al.* (2010). In addition to the aforementioned property, because $C$ is a base copula, then the bivariate distorted function $\tilde{C}$ is a copula because the grounded property is preserved under the distortion $T$.

## 3. Conversion function and distortion

From (2.2), set $w = \text{logit}(u) = \log(u/(1-u))$. For $u \in [0,1]$, we have $w \in [-\infty, \infty]$, and the inverse transform is $u = e^w/(1+e^w)$.

### 3.1. Examples of distortions

Here, in the following examples, we attempt to define the conversion function $\zeta$ such that the associate distortion in (2.2) satisfies the definition of distortion and is convex. The associate distortion functions we derive below are found in the work of Sepanski (2020).

- If $\zeta_1(u) = -\log\left(e^{(-\log(e^u/(1+e^u)))^{\delta_1}} - 1\right), \delta_1 \leq 1$, the associate distortion $T_{\zeta_1}$ to $\zeta_1$ produces the Weibull-log distortion given by $T_{\zeta_1}(u) = e^{-(-\log u)^{\delta_1}}$, where the inverse is given by $T_{\zeta_1}^{-1}(u) = e^{-(-\log u)^{1/\delta_1}}$.

- If $\zeta_2(u) = -\log\left(1/(1-(1+e^u)^{-\delta_1}) - 1\right), \delta_2 \leq 1$, the associate distortion $T_{\zeta_2}$ to $\zeta_2$ produces the dual-power distortion given by $T_{\zeta_2}(u) = 1-(1-u)^{\delta_2}$, where the inverse is given by $T_{\zeta_2}^{-1}(u) = 1-(1-u)^{1/\delta_2}$.

- If $\zeta_3(u) = -\log\left((e^u/(1+e^u))^{-\delta_3} - 1\right), \delta_3 \geq 1$, the associate distortion $T_{\zeta_3}$ to $\zeta_3$ produces the power distortion given by $T_{\zeta_3}(u) = u^{\delta_3}$, where the inverse is given by $T_{\zeta_3}^{-1}(u) = u^{1/\delta_3}$.

- If $\zeta_4(u) = -\log\left([-\delta_4/\log((1+e^{\delta_4+u})/(1+e^u))] - 1\right), \delta_4 > 0$, the associate distortion $T_{\zeta_4}$ to $\zeta_4$ produces the logarithmic distortion given by $T_{\zeta_4}(u) = -\log(1-u(1-e^{-\delta_4}))/\delta_4$, where the inverse is given by $T_{\zeta_4}^{-1}(u) = (1-e^{-\delta_4 u})/(1-e^{-\delta_4})$.

- If $\zeta_5(u) = -\log\left([1 - \log(e^u/(1+e^u))]^{\delta_5} - 1\right), \delta_5 > 0$, the associate distortion $T_{\zeta_5}$ to $\zeta$ produces the Lomax-log distortion given by $T_{\zeta_5} = (1 - \log\ u)^{-\delta_5}$, where the inverse is given by $T_{\zeta_5}^{-1}(u) = e^{(1-u^{-1/\delta_5})}$.

Note that the functions $\zeta_1, \zeta_2, \zeta_3, \zeta_4$, and $\zeta_5$ are bijective and increasing, thereby satisfying $\zeta(-\infty) = -\infty$ and $\zeta(\infty) = \infty$. If $\delta_1 = \delta_2 = \delta_3 = 1$, from (2.1),

$$C\left(T_{\zeta_1}^{-1}(u_1), T_{\zeta_2}^{-1}(u_2), T_{\zeta_3}^{-1}(u_3)\right) = C\left(u_1, u_2, u_3\right). \tag{3.1}$$

### 3.2. Gompertz distortion

The Gompertz distribution was derived by Gompertz (1825), and it has several real applications; see Ahuja and Nash (1967) and references therein. It is one of the distributions mentioned by Durante *et al.* (2010), and hence, we intend to use this distribution to construct our new copula models.

Define

$$\zeta(v) = -\log\left(\frac{1}{1 - G\left(\log\ (1 - \log\ (e^v/(1+e^v)))\right)} - 1\right), \tag{3.2}$$

where $G$ is a distribution function. If $\zeta_0(v) = 1 - G(\log(1 - \log(e^v/(1+e^v))))$, then the first derivative of $\zeta_0$ with respect to $v$ is given by $\zeta_0^{'}(v) = g(\log(1 - \log(e^v/(1+e^v))))/[(1+e^v)(1 - \log(e^v/(1+e^v)))]$, where $G^{'} = g$. Thus, $\zeta^{'}(v) = \zeta_0^{'}(v)/[(1 - \zeta_0(v))\zeta_0(v)]$. $\zeta^{'}$ is increasing. Let $G$ be a power Gompertz (PG) distribution defined as $G(z) = 1 - \exp[-ba^{-1}(e^{az} - 1)],\ z > 0$, for $a, b > 0$. Applying (3.2), we obtain $\zeta_{PG}(v) = -\log\left(e^{b[(1-\log(e^v/(1+e^v)))^a - 1]/a} - 1\right)$. As a result, from (2.2), the associate UPG distortion to $\zeta_{PG}$ is

$$T_{\zeta_{PG}}(v) = e^{\frac{-b\left[(1-\log\ v)^a - 1\right]}{a}}. \tag{3.3}$$

The probability density function and quantile function of $V$ are defined as

$$t_{\zeta_{PG}}(v) = T^{'}_{\zeta_{PG}}(v) = \frac{b}{v}\,(1 - \log\ v)^{a-1}\, e^{\frac{-b\left[(1-\log\ v)^a - 1\right]}{a}},$$

and

$$T_{\zeta_{PG}}^{-1}(v) = e^{\left[\left(\frac{1-(1-(a\log\ v)}{b}\right)^{\frac{1}{a}}\right]}, \tag{3.4}$$

respectively.

When $a = 1$ in (3.3), the UPG distortion is transformed into the power distortion, which has the form $T_{\zeta_{PG}}(u) = u^b,\ \ b > 0$. The power distortion is a distortion function because it is increasing on $[0, 1]$ with $T(0) = 0$ and $T(1) = 1$ for all $b > 0$.

For any base copula $C$, the family of the UPG distorted distributional function (shortly, the UPG distorted function) is overall given by

$$\tilde{C}_{T_{\zeta_{PG}}}(u_1, \ldots, u_n) = \exp\left(-ba^{-1}\left[\left(1 - \log\ C\left(e^{[1-(1-(a\log\ u_1)/b)^{1/a}]}, \ldots, e^{[1-(1-(a\log\ u_n)/b)^{1/a}]}\right)\right)^a - 1\right]\right). \tag{3.5}$$

If $a = 1$ and $b = 1$, the UPG distorted function $\tilde{C}_{T_{\zeta_{PG}}}$ becomes equal to a base copula $C$, and thus, $\tilde{C}_{T_{\zeta_{PG}}}$ is a copula function.

From (2.1) and (3.1), the following distorted function

$$\tilde{C}_{T_{123}}(u_1,u_2,u_3) = T_{\zeta_{PG}} \circ C\left(T_{\zeta_1}^{-1}(u_1), T_{\zeta_2}^{-1}(u_2), T_{\zeta_3}^{-1}(u_3)\right)$$

is equal to a base copula $C(u_1,u_2,u_3)$ when $a = b = \delta_1 = \delta_2 = \delta_3 = 1$. It is worth to notice, that $\tilde{C}_{T_{123}}(u_1,u_2,u_3) = C(T_{\zeta_1}^{-1}(u_1), T_{\zeta_2}^{-1}(u_2), T_{\zeta_3}^{-1}(u_3))$ when $a = b = 1$.

General bivariate forms of the conditional cdf and copula pdf of $\tilde{C}$ can be found in Aldhufairi *et al.* (2020).

According to Theorem 3.2, as derived by Durante *et al.* (2010), we show that $T_{\zeta_{PG}} \circ \exp : (-\infty, 0] \to [0,1]$ is log-convex under the admissibility mentioned in the following corollary.

**Corollary 3.1.** Let $T_{\zeta_{PG}}(u)$ be the UPG distortion for $u \in [0,1]$. $T_{\zeta_{PG}}$ is a log-convex function if $0 < a \leq 1$ and $b > 0$.

**Proof:** Define $B(x) = \log \circ T_{\zeta_{PG}} \circ \exp(x)$. Then, the function $B(x) = -b[(1-x)^a - 1]/a$ has its first and second derivatives with respect to $x$ as follows: $B^{'}(x) = b(1-x)^{a-1}$ and $B^{''}(x) = b(1-a)(1-x)^{a-2}$. Thus, the second derivative $B^{''}$ is non-negative if $0 < a \leq 1$ and $b > 0$ for $x \in (-\infty, 0]$.

### 3.3. Examples of distorted copulas

We focus on the bivariate case in the examples of proposed copula models offered in this section. According to Joe (1997), the base copulas are $TP_2$ in any of the following examples, which are constructed using (3.3), (3.4), and (2.3).

One popular class of copulas is Archimedean, defined as if a generator function $\phi : [0,1] \to [1,\infty)$ is continuous, strictly decreasing, and convex with $\phi(1) = 0$ exists and generates the copula via $C(u_1,\ldots,u_n) = \phi^{[-1]}(\phi(u_1) + \cdots + \phi(u_n))$. If $\phi(0) = \infty$, then $\phi^{[-1]} = \phi^{-1}$. If a base copula $C$ with a generator $\phi$ belongs to the Archimedean class, then the distorted copula does too. Its distorted generator is given by $\tilde{\phi} = \phi \circ T^{-1}$, as demonstrated by Aldhufairi *et al.* (2020). One can then rewrite (3.5) as follows:

$$\tilde{C}_{T_{\zeta_{PG}}}(u_1,\ldots,u_n) = \tilde{\phi}^{-1}\left(\tilde{\phi}(u_1) + \tilde{\phi}(u_2) + \cdots + \tilde{\phi}(u_n)\right), \quad \tilde{\phi}(u) = \phi\left(e^{[1-(1-(a\log v)/b)^{1/a}]}\right).$$

Di Bernardino and Rulliere (2013) report on page 7 that $T$ is an admissible distortion if and only if $\tilde{\phi}$ is a $n-$monotone function. This result allows for the generalization of the distorted copulas. It can be accomplished by expanding the findings of Theorem 3.2 and Lemma 3.1 from the work of Durante *et al.* (2010). Furthermore, there is a need to carefully investigate which of the copulas are multivariate and totally positive of order 2. This may create a gap for future research.

Another popular class of copulas is referred to as extreme-value, defined as if a convex function $A : [0,1] \to [1/2, 1]$, satisfying $A(0) = A(1) = 1$, and $\max\{t, 1-t\} \leq A(t) \leq 1$ exists and produces the copula via $C(u,v) = \exp[\log(uv)A(\log(v)/\log(uv))], u, v \in [0,1]$ (Gudendorf and Segers, 2010).

Unlike Clayton and Frank that belong to the Archimedean class only, Gumbel belongs to the Archimedean and extreme-value classes. Some copulas have no closed form; for instance, the Gaussian copula that is proposed from the Gaussian distribution.

**Example 3.1. (UPG-Clayton copula)** The Clayton copula with its generator is given by

$$C(u_1,u_2;\theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{\frac{-1}{\theta}}, \quad \theta > 0, \ \phi(u) = \frac{\left(u^{-\theta} - 1\right)}{\theta}.$$

The UPG-distorted Clayton copula (shortly, UPG-Clayton copula) can be expressed as

$$C_{T_{\zeta_{PG}}}(u_1,u_2) = \exp\left(-\frac{b}{a}\left[\left(1+\theta^{-1}\log\left(\sum_{i=1}^{2} e^{-\theta[1-(1-ab^{-1}\log u_i)^{\frac{1}{a}}]}-1\right)\right)^{a}-1\right]\right),$$

and its generator is $\tilde{\phi}(u) = \{e^{-\theta[1-(1-ab^{-1}\log u)^{1/a}]}-1\}/\theta$.

**Example 3.2. (UPG-Gumbel copula)** The Gumbel copula with its generator is given by

$$C(u_1,u_2;\theta) = \exp\left\{-\left[(-\log u_1)^{\theta}+(-\log u_2)^{\theta}\right]^{\frac{1}{\theta}}\right\},\quad \theta\geq 1,\ \phi(t) = (-\log t)^{\theta}.$$

The UPG-distorted Gumbel copula (shortly, UPG-Gumbel copula) is

$$C_{T_{\zeta_{PG}}}(u_1,u_2) = \exp\left(-\frac{b}{a}\left[\left(1+\left(\sum_{i=1}^{2}\left(\left(1-ab^{-1}\log u_i\right)^{\frac{1}{a}}-1\right)^{\theta}\right)^{\frac{1}{\theta}}\right)^{a}-1\right]\right),$$

and its generator is $\tilde{\phi}(u) = [(1-ab^{-1}\log u)^{1/a}-1]^{\theta}$.

**Example 3.3. (UPG-Frank copula)** The Frank copula with its generator is given by

$$C(u_1,u_2;\theta) = -\theta^{-1}\log\left(1+\frac{\left[\left(e^{-\theta u_1}-1\right)\left(e^{-\theta u_2}-1\right)\right]}{e^{-\theta}-1}\right),\quad \theta\geq 0,\ \phi(u) = -\log\left(\frac{e^{-\theta u}-1}{e^{-\theta}-1}\right).$$

The UPG-distorted Frank copula (shortly, UPG-Frank copula) is

$$C_{T_{\zeta_{PG}}}(u_1,u_2) = \exp\left(-\frac{b}{a}\left[\left(1-\log\left(-\theta^{-1}\log\left(1+\frac{\Pi_{i=1}^{2}\left(e^{-\theta e^{[1-(1-ab^{-1}\log u_i)^{1/a}]}}-1\right)}{e^{-\theta}-1}\right)\right)\right)^{a}-1\right]\right),$$

and its generator is $\tilde{\phi}(u) = -\log[(e^{-\theta e^{1-(1-ab^{-1}\log u)^{1/a}}}-1)/(e^{-\theta}-1)]$.

**Example 3.4. (UPG-Gaussian copula)** The Gaussian copula is given by

$$C(u_1,u_2;\theta) = \Phi_2\left(\Phi^{-1}(u_1),\Phi^{-1}(u_2)\right)$$

with its parameter $\theta\in[0,1]$, where $\Phi^{-1}(s)$ is the quantile function of the univariate standard Gaussian distribution

$$\Phi(s) = \int_{-\infty}^{s}\frac{1}{\sqrt{2\pi}}e^{-(x^2/2)}\mathrm{dx}.$$

The bivariate standard Gaussian distribution function $\Phi_2$ is given by

$$\Phi_2(s_1,s_2) = \int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\frac{1}{2\pi\sqrt{1-\theta^2}}\exp\left(-\frac{x^2-2\theta xy+y^2}{2\sqrt{1-\theta^2}}\right)\mathrm{dxdy}.$$

Thus, the UPG-distorted Gaussian copula (shortly, UPG-Gaussian copula) is a direct application of this expression $T_{\zeta_{PG}}(C(T^{-1}_{\zeta_{PG}}(u_1),T^{-1}_{\zeta_{PG}}(u_2)))$. That is,

$$C_{T_{\zeta_{PG}}}(u_1,u_2) = T_{\zeta_{PG}}\left[\Phi_2\left(\Phi^{-1}\left(T^{-1}_{\zeta_{PG}}(u_1)\right),\Phi^{-1}\left(T^{-1}_{\zeta_{PG}}(u_2)\right)\right)\right]. \tag{3.6}$$

### 3.4. Limiting case

The following proposition investigates the limit of the UPG copulas from a given copula $C$ when one or both parameters go to a boundary.

**Proposition 3.1.** Consider the UPG-distorted function in (3.5), where $0 < a \leq 1$ and $b > 0$. Then, for any base copula $C$, if this exponent of $C_{T_{\zeta_{PG}}}$, $b(1 - \log C(T^{-1}_{\zeta_{PG}}(u_1), \ldots, T^{-1}_{\zeta_{PG}}(u_n)))^a - 1$ goes to $\log u_1 + \cdots + \log u_n$ whenever $b \to \infty$, we have that $C_{T_{\zeta_{PG}}}$ approaches the independence copula.

**Proof:** Let $x_i = e^{[1-(1-ab^{-1}\log u_i)^{1/a}]}$, for $i = 1, \ldots, n$. Set $m = 1/b$ and consider $I_m = C(x_1, \ldots, x_n)$. As $C$ is a base copula, $I_m \to 1$ as $m \to 0$. By chain rule, the derivative of $I_m$ with respect to $m$ is given by

$$\frac{dI_m}{dm} = \frac{\partial C(x_1, \ldots, x_n)}{\partial x_1}\frac{dx_1}{dm} + \cdots + \frac{\partial C\,(x_1, \ldots, x_n)}{\partial x_n}\frac{dx_n}{dm}.$$

For $i = 1, 2, \ldots, n$, we have

$$\frac{dx_i}{dm} = (\log\, u_i)\,(1 - am\log\, u_i)^{\frac{1}{a-1}}\exp\left[1 - (1 - am\log\, u_i)^{\frac{1}{a}}\right].$$

The limit of the exponent term in (3.5) as $b \to \infty$ exists, by L'Hopital's rule,

$$\lim_{m\to 0}\frac{(1 - \log\, I_m)^a - 1}{am} = -\lim_{m\to 0}\frac{(1 - \log\, I_m)^{a-1}}{I_m}\frac{dI_m}{dm}.$$

As $m \to 0$, we have $dx_i/dm \to \log\, u_i$, for $i = 1, 2, \ldots, n$. Thus, we obtain the following $\lim_{b\to\infty} C_{T_{\zeta_{PG}}}$ $(u_1, \ldots, u_n) = u_1 \cdots u_n$, which is the independent copula.

The limit of the UPG-distorted copula in the parameter obtained from a base copula $C$ can be calculated by finding the limit of the base copula in the parameter. The limits of the chosen base copulas were evaluated in Joe (2015). For example, as $\theta \to 0^+$, the bivariate Clayton copula $C$ approaches the independent copula. As a result, we derive the following: $C(T^{-1}(u_1), T^{-1}(u_2)) \to T^{-1}(u_1)T^{-1}(u_2)$, as $\theta \to 0^+$. Thus, the distorted bivariate copula $\tilde{C}_{T_{\zeta_{PG}}}$ approaches the copula $\exp(-ba^{-1}[((1-\sum_{i=1}^{2}(a\log\, u_i)$ $/b)^{1/a} - 1)^a - 1])$ whenever $\theta \to 0^+$.

### 3.5. Frèchet bounds and independent case

Here, we look at three common cases that show the impact of choosing the initial copula $C$ in $\tilde{C}$.

- Assume $C$ is counter-monotonic, which is the lower Frèchet, in bivariate case $n = 2$. If $C(T^{-1}(u_1), T^{-1}$ $(u_2)) = \max\{T^{-1}(u_1) + T^{-1}(u_2) - 1, 0\}$, then we have $\tilde{C}(u_1, u_2) = T \circ C(T^{-1}(u_1), T^{-1}(u_2)) = \max\{T(T^{-1}(u_1) + T^{-1}(u_2) - 1), 0\}$ since $T$ is monotonically increasing. For example, when $T_{\zeta_{PG}}(u) = u^b$, with $b \geq 1$, then $\tilde{C}(u_1, u_2) = \max\{(u_1^{-b_1} + u_2^{-b_1} - 1)^{-1/b_1}, 0\}$, where $b_1 = -1/b \in [-1, 0)$, which is the Clayton copula.
- If $C$ is comonotonic, which is upper Frèchet, given by $C(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n)$, then because $T_{\zeta_{PG}}$ increases, we have $\tilde{C}(u_1, \ldots, u_n) = T_{\zeta_{PG}}(\min(T^{-1}_{\zeta_{PG}}(u_1), \ldots, T^{-1}_{\zeta_{PG}}(u_n))) = \min(u_1, \ldots, u_n)$. Thus, $\tilde{C}$ is also comonotonic. In this case, we can see that $T_{PG}$ has no effect on the result of the base copula $C$. This implies that $T_{PG}$ plays the role of an identity distorter, $T(u) = u$.
- Consider that $C$ is an independent copula. If $C(T^{-1}(u_1), \ldots, T^{-1}(u_n)) = T^{-1}(u_1)\times\cdots\times T^{-1}(u_n)$, then $\tilde{C}(u_1, \ldots, u_n) = T \circ C(T^{-1}(u_1), \ldots, T^{-1}(u_n)) = T(T^{-1}(u_1)\cdots T^{-1}(u_n))$. For example, if $T_{\zeta_{PG}}(u) = u^b, b > 0$, we have $\tilde{C}(u_1, \ldots, u_n) = u_1 \cdots u_n$. Thus, $\tilde{C}$ is the independent copula.

Table 1: Tail orders and dependence coefficients for copulas

| Initial Copula | $\kappa_L$ or $\lambda_L$ | $\kappa_U$ or $\lambda_U$ |
|---|---|---|
| Gumbel | $\kappa_L = 2^{1/\theta}$ | $\lambda_U = 2 - 2^{1/\theta}$ |
| Clayton | $\lambda_L = 2^{-1/\theta}$ | $\kappa_U = 2$ |
| Frank | $\kappa_L = 2$ | $\kappa_U = 2$ |
| Gaussian | $\kappa_L = 2/(1+\theta)$ | $\kappa_U = 2/(1+\theta)$ |
| UPG-Copula | $\kappa_{L_{T_{\zeta_{PG}}}}$ or $\lambda_{L_{T_{\zeta_{PG}}}}$ | $\kappa_{U_{T_{\zeta_{PG}}}}$ or $\lambda_{U_{T_{\zeta_{PG}}}}$ |
| UPG-Gumbel | $\kappa_{L_{T_{\zeta_{PG}}}} = 2^{1/\theta}$ | $\lambda_U = 2 - 2^{1/\theta}$ |
| UPG-Clayton | $\lambda_{L_{T_{\zeta_{PG}}}} = 2^{-b/\theta}$ if $a = 1$ | $\kappa_{U_{T_{\zeta_{PG}}}} = 2$ |
| UPG-Frank | $\kappa_{L_{T_{\zeta_{PG}}}} = 2$ | $\kappa_{U_{T_{\zeta_{PG}}}} = 2$ |
| UPG-Gaussian | $\kappa_{L_{T_{\zeta_{PG}}}} = 2/(1+\theta)$ | $\kappa_{U_{T_{\zeta_{PG}}}} = 2/(1+\theta)$ |

## 4. Tail orders and dependence coefficients

This section explores the lower and upper tail behaviors of the UPG copulas from a given base copula $C$. One possible thought beyond the addition of new parameters is to produce new models of copulas that adapt to the different behaviors of tail dependence to estimate risky and extreme events.

The survival copula is defined as $\hat{C}(u_1, u_2, \ldots, u_n) = P(1 - U_1 \le u_1, 1 - U_2 \le u_2, \ldots, 1 - U_n \le u_n) = \bar{C}(1 - u_1, 1 - u_2, \ldots, 1 - u_n)$, where $\bar{C}$ is the joint survival function of $C$. The survival bivariate copula can be written as $\hat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2)$. A regularly varying function $\ell$ with index $\xi$ is defined by $\lim_{u \to 0^+} \ell(\gamma u)/\ell(u) = \gamma^{\xi}$ for all $\gamma > 0$. If $\xi = 0$, $\ell$ is said to be slowly varying, see Hua and Joe (2013). If a lower tail dependence coefficient ($\lambda_L$) of a base copula $C$ exists, the lower tail dependence coefficient ($\lambda_{T,L}$) of the distorted copula $C_T$ is given by

$$\lambda_{T,L} = \lim_{u \to 0^+} \frac{T\left(C\left(T^{-1}(u), T^{-1}(u)\right)\right)}{u} = \lim_{u \to 0^+} \frac{T\left(C(u,u)\right)}{T(u)}.$$

If an upper tail dependence coefficient ($\lambda_U$) of $C$ exists, the upper tail dependence coefficient ($\lambda_{T,U}$) of $C_T$ is given as follows:

$$\lambda_{T,U} = 2 - \lim_{u \to 1^-} \frac{1 - T\left(C\left(T^{-1}(u), T^{-1}(u)\right)\right)}{1 - u} = 2 - \lim_{u \to 0^+} \frac{1 - T\left(C(u,u)\right)}{1 - T(u)}.$$

If we consider the following expansions

$$(1 + u)^a \sim 1 + au, \quad \log(1 - u) \sim -u, \quad e^u \sim 1 + u, \quad \text{as } u \to 0,$$

we obtain $T_{\zeta_{PG}}^{-1}(1 - u) \sim 1 - u/b$, as $u \to 0^+$. For $C(u, v) \sim u^{\kappa_L} \ell(u)$, the UPG-distorted copula has a lower tail order of $\kappa_L$ because

$$\begin{aligned} T_{\zeta_{PG}}\left(C\left(T_{\zeta_{PG}}^{-1}(u), T_{\zeta_{PG}}^{-1}(v)\right)\right) &= e^{b\left[\left(1 - \log C(T_{\zeta_{PG}}^{-1}(u), T_{\zeta_{PG}}^{-1}(v))\right)^a - 1\right]/a} \sim e^{b\left[\left(1 - a\log(T_{\zeta_{PG}}^{-1}(u))^{\kappa_L} \ell(T_{\zeta_{PG}}^{-1}(u))\right) - 1\right]/a} \\ &\sim e^{b\left[\kappa_L\left(1 - \left(1 - ab^{-1}\log u\right)^{1/a}\right) + \log \ell\left(T_{\zeta_{PG}}^{-1}(u)\right)\right]} \sim e^{b\left[\kappa_L b^{-1}\log u + \log \ell\left(T_{\zeta_{PG}}^{-1}(u)\right)\right]} \\ &\sim u^{\kappa_L}\left[\ell\left(T_{\zeta_{PG}}^{-1}(u)\right)\right]^b. \end{aligned}$$

**Proposition 4.1.** Let $\ell$ and $\ell_*$ be two slowly varying functions. As $u \to 0^+$, assume $C(u, u) \sim u^{\kappa_L} \ell(u)$ and $\overline{C}(1 - u, 1 - u) \sim u^{\kappa_U} \ell_*(u)$ at $0^+$. The UPG copula $C_{T_{\zeta_{PG}}}$ satisfies the following: $\kappa_{L_{T_{\zeta_{PG}}}} = \kappa_L$, $\lambda_{T_{\zeta_{PG}},L} = (\lambda_L)^b$ if $a = 1$, $\kappa_{T_{\zeta_{PG}},U} = \kappa_U$, and $\lambda_{T_{\zeta_{PG}},U} = \lambda_U$.

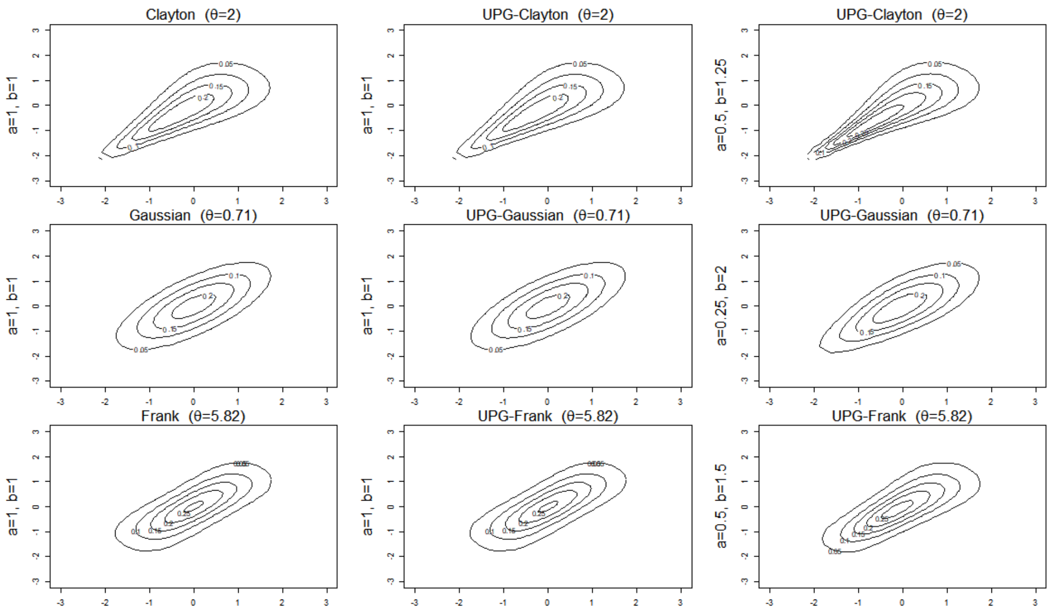


Figure 1: *Density contour plots with standard normal margins. The parameter is chosen so that the base copula has Kendall's tau of* $1/2$. *The first and second columns show contour plots of the base and UPG-distorted copulas with* $(\theta, a, b) = (2,1,1), (0.71,1,1)$, *and* $(5.82,1,1)$. *The contour plots for the UPG-Clayton* $(2, 0.5, 1.25)$, *UPG-Gaussian* $(0.71, 0.25, 2)$, *and UPG-Frank* $(5.82, 0.5, 1.5)$ *are shown in the third column.*

**Proof:** The lower tail dependence coefficient of $C_{T_{\zeta_{PG}}}$ is given by

$$\lambda_{T_{\zeta_{PG}},L} = \lim_{u\to 0^+} \frac{T_{\zeta_{PG}}(C(u,u))}{T_{\zeta_{PG}}(u)} = \lim_{u\to 0^+} \frac{e^{-b[(1-\log C(u,u))^a - 1]/a}}{e^{-b[(1-\log u)^a - 1]/a}}$$
$$= \lim_{u\to 0^+} e^{-ba^{-1}(1-\log u)^a\left[\left(\frac{1-\log C(u,u)}{1-\log u}\right)^a - 1\right]}. \tag{4.1}$$

We can see some possibilities for (4.1). If $a < 1$, then $\lambda_{T_{\zeta_{PG}},L} = 0$, which means there is no lower tail dependence. If $a = 1$, then $\lambda_{T_{\zeta_{PG}},L} = \lim_{u\to 0^+} e^{-b[\log(u/C(u,u))]} = (\lambda_L)^b$. However, the following can be obtained from (3.5) as

$$\left[1 - \log C\left(T^{-1}_{\zeta_{PG}}(1-u), T^{-1}_{\zeta_{PG}}(1-u)\right)\right]^a \sim [1 - \log C(1-u/b, 1-u/b)]^a$$
$$\sim \left[1 - \log\left(1 - 2u/b + \overline{C}(1-u/b, 1-u/b)\right)\right]^a \sim [1 - \log(1 - 2u/b + (u/b)^{\kappa_U}\ell_*(u/b))]^a$$
$$\sim [1 + 2u/b - (u/b)^{\kappa_U}\ell_*(u/b)]^a \sim [1 + a(2u/b - (u/b)^{\kappa_U}\ell_*(u/b))] = K_{PG}.$$

It follows that

$$\hat{C}_{T_{\zeta_{PG}}}(u,v) = 2u - 1 + e^{-ba^{-1}\left[\left(1-\log C(T^{-1}_{\zeta_{PG}}(1-u), T^{-1}_{\zeta_{PG}}(1-u))\right)^a - 1\right]} \sim 2u - 1 + 1 - ba^{-1}(K_{PG} - 1)$$
$$\sim 2u - ba^{-1}[2au/b - a(u/b)^{\kappa_U}\ell_*(u/b)] \sim b^{1-\kappa_U}u^{\kappa_U}\ell_*(u/b).$$

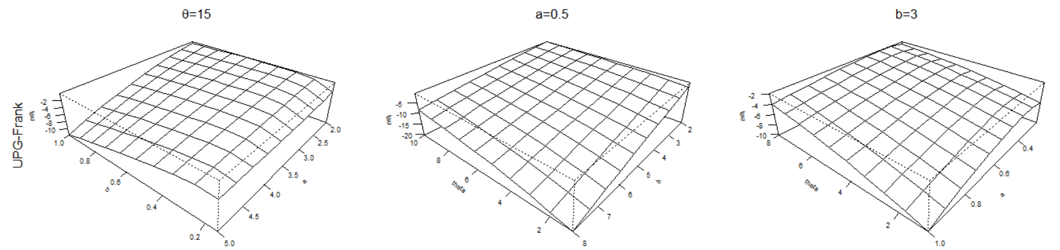


Figure 2: *Kendall's tau surface plot for the UPG-Frank copula, which displays tau values at fixed one of the parameters* $\theta = 15, a = 0.5$, *or* $b = 3$ *for the other two parameters.*

As a result, the upper tail order $\kappa_{T_{\zeta_{PG}},U}$ of $C_{T_{\zeta_{PG}}}$ is $\kappa_U$. The upper tail dependence coefficient of $C_{T_{\zeta_{PG}}}$ is given by

$$\begin{aligned}\lambda_{T_{\zeta_{PG}},U} &= 2 - \lim_{u \to 1^-} \frac{1 - e^{-b[(1-\log\, C(u,u))^a - 1]/a}}{1 - e^{-b[(1-\log\, u)^a - 1]/a}} \\ &= 2 - \lim_{u \to 1^-} \left\{ \frac{e^{-b[(1-\log\, C(u,u))^a - 1]/a}}{e^{-b[(1-\log\, u)^a - 1]/a}} \left[ \frac{1 - \log\, C(u,u)}{1 - \log\, u} \right]^{a-1} \frac{u}{C(u,u)} \frac{dC(u,u)}{du} \right\} \\ &= \lambda_U.\end{aligned}$$

Table 1 provides a summary of tail order and dependence for the base and new copulas. The new UPG-distorted copulas can accommodate extra parameters in the lower tail dependence when the base copula has the lower tail dependence.

Figure 1 displays contour plots of a bivariate pdf, $h$, for three new copulas distorted from Clayton, Gaussian, and Frank. The parameters in all base copulas have been chosen when Kendall's tau has a value of $1/2$.

A bivariate copula $C$ is said to be symmetric when $C(u_1, u_2) = C(u_2, u_1)$ for $u_1, u_2 \in [0, 1]$. As shown in Figure 1, the resulting new copulas from the given symmetric Frank and Gaussian copulas are asymmetric when $a$ or $b$ differ from a value of 1.

## 5. Kendall's tau

The section derives the Kendall's tau formulas for three copulas, namely, UPG-Clayton, UPG-Gumbel, and UPG-Frank. Then, it studies the ordering of concordance based on the formulas we derive.

Copulas offer a natural approach for measuring the dependence between two random variables, and one of these measures is called Kendall's tau, which is a non-parametric measure.

The general and Archimedean formulas of Kendall's tau for the bivariate distortion $T$ can be found in Aldhufairi *et al*. (2020).

Note that the values computed from the Kendall's tau formula can either be an increase or decrease in one parameter while the remaining parameters are held constant. If the values obtained from the Kendall's tau formula increase but never decrease, or decrease but never increase, in such a parameter $r$ , then the function $\tilde{C}$ is ordered by $r$.

**Example 5.1.** Let $\phi$ be a generator function for Clayton. The following can be formulated as $\phi(u)/\phi'(u)$

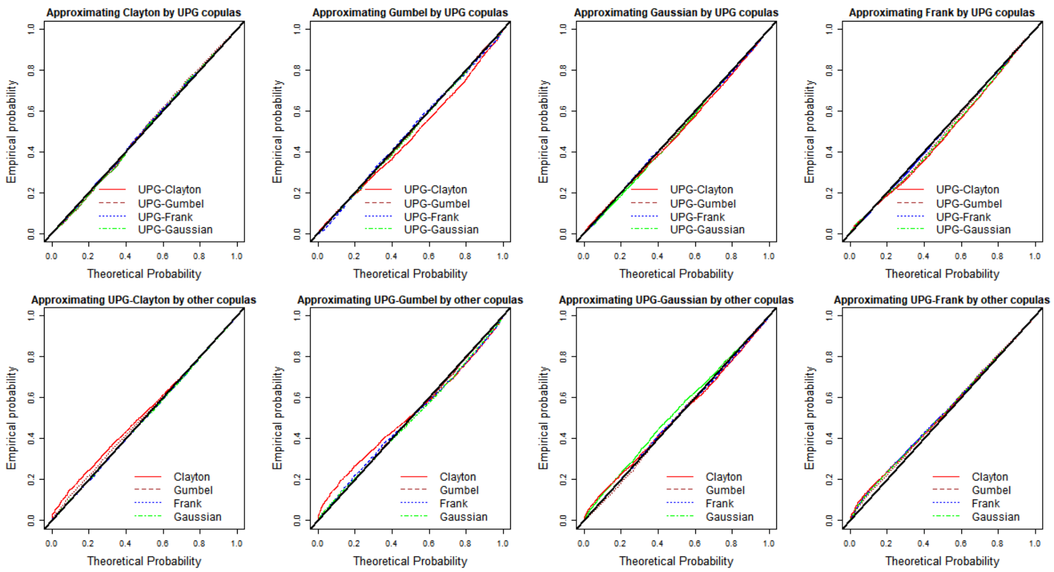


Figure 3: *P-P plots show the estimated and empirical cumulative distributions for different theoretical models. Results from fitting data obtained from the Clayton, Gumbel, Frank, and Gaussian copulas are shown in the first row. Data fitting results generated using the UPG-Clayton, UPG-Gumbel, UPG-Frank, and UPG-Gaussian copulas place in the second row.*

$= -\theta^{-1}u(1-u^{\theta})$. Kendall's tau of the UPG-Clayton is given by

$$\tau_{\phi_{T_{\zeta_{PG}}}} = 1 - \frac{4b^2}{\theta}\int_0^1 \frac{\left(1-u^{\theta}\right)}{u}(1-\log\, u)^{2a-2} e^{-2b((1-\log\, u)^a-1)/a}\text{du}.$$

The UPG-Clayton copula is ordered in $a$ and $b$ because checking the first derivative with respect to $a$ and $b$ reveals that the UPG-Clayton copula is increasing in $a$ and $b$.

**Example 5.2.** Let $\phi$ be a generator function for Gumbel. Then, $\phi(u)/\phi'(u) = -u(-\log(u))/\theta$, and thus, Kendall's tau for the UPG-Gumbel copula is given by

$$\tau_{\phi_{T_{\zeta_{PG}}}} = 1 - \frac{4b^2}{\theta}\int_0^1 \frac{(-\log\, u)}{u}\,(1-\log\, u)^{2a-2}\, e^{-2b[(1-\log\, u)^a-1]/a}\text{du}.$$

The UPG-Gumbel copula is also ordered in $a$ and $b$ because it is increasing in $a$ and $b$.

**Example 5.3.** Let $\phi$ be a generator function for Frank. Then, $\phi(u)/\phi'(u) = -\theta^{-1}(1-e^{\theta u})\log\left[(e^{-\theta u}-1)/(e^{-\theta}-1)\right]$. Thus, Kendall's tau of the UPG-Frank with setting $v = \theta u$ is given by

$$\tau_{\phi_{T_{\zeta_{PG}}}} = 1 - 4b^2\int_0^{\theta} \frac{(1-e^{v})\,(1-\log\,(v/\theta))^{2a-2}}{v^2 e^{2b((1-\log\,(v/\theta))^a-1)/a}}\log\left[\frac{e^{-v}-1}{e^{-\theta}-1}\right]\text{dv}.$$

As shown in Figure 2, the concordance ordering in $a$ or $b$ can fail to hold. Nonmonotonic curves can be seen with parameters $a$ and $b$ in the plots titled with fixed $\theta = 15$ and $b = 3$ for the UPG-Frank copula.

Table 2: Summary statistics of BMI, BAI, BFP, and WC variables

| | BMI | BAI | BFP | WC |
|---|---|---|---|---|
| Minimum | 15.73 | −6.68 | 10.83 | 0.24 |
| Maximum | 37.29 | 38.13 | 47.84 | 1.09 |
| Median | 23.24 | 22.02 | 27.25 | 0.81 |
| $1^{st}$ quartile | 21.15 | 18.71 | 21.82 | 0.75 |
| $3^{st}$ quartile | 25.72 | 25.19 | 32.16 | 0.88 |
| Standard deviation | 3.33 | 5.26 | 7.19 | 0.10 |

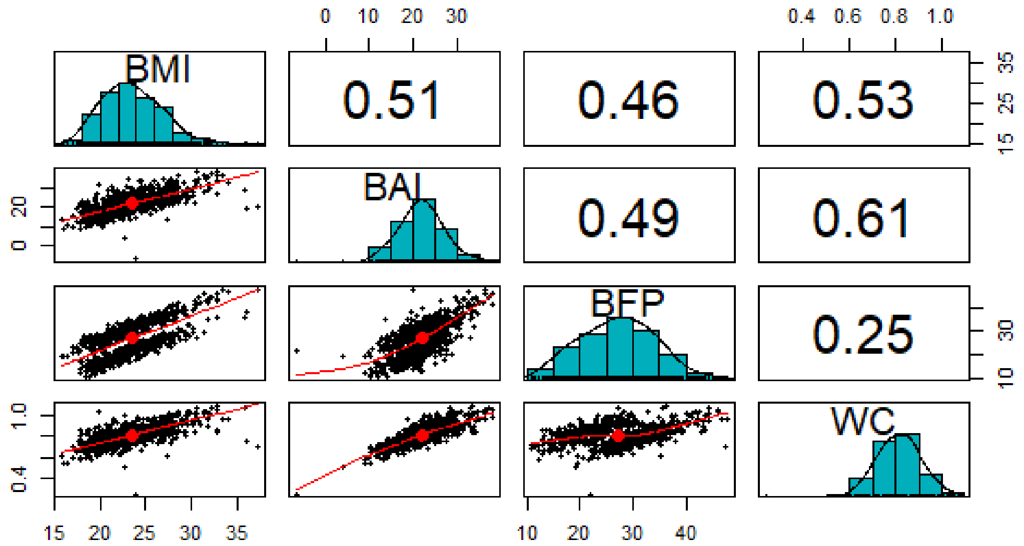


Figure 4: *The correction matrix. Scatter plots of Kendall's tau correlation between anthropometric measurements (BMI, BAI, BFP, and WC) with a fitted line. Distribution of each variable in the data is diagonal in the matrix.*

## 6. Application

Here, the data set is analyzed utilizing the R programming language to assess the performance of the new UPG-distorted copula models in this section. Based on the chosen base copulas, namely, Clayton, Gumbel, Frank, and Gaussian, the Akaike's information criterion (AIC) statistics are used to determine the best copula model. We perform the Cramer-von Mises (CvM) goodness-of-fit test (Genest *et al.*, 2009) and compare the performance of the statistics with the base copulas, where the CvM test statistics measure the sum of square deviations between the empirical cdf and an estimated copula cdf. Larger CvM values are less desired. The null hypothesis of the CvM is that a candidate copula models bivariate data.

The data are analyzed using the maximum pseudo-likelihood estimation (MPLE), see Joe (2015) for the bivariate case. The MPLE maximizes

$$\sum_{i=1}^{m} \log\left[c_T\left(u_{1,i}, u_{2,i}, \ldots, u_{n,i}; \theta, a, b\right)\right], \tag{6.1}$$

where $u_{i,j} = F_i(x_{i,j}), i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$, are the pseudo-observations (shortly, pseudo-obs) and $c_T$ is the copula pdf.

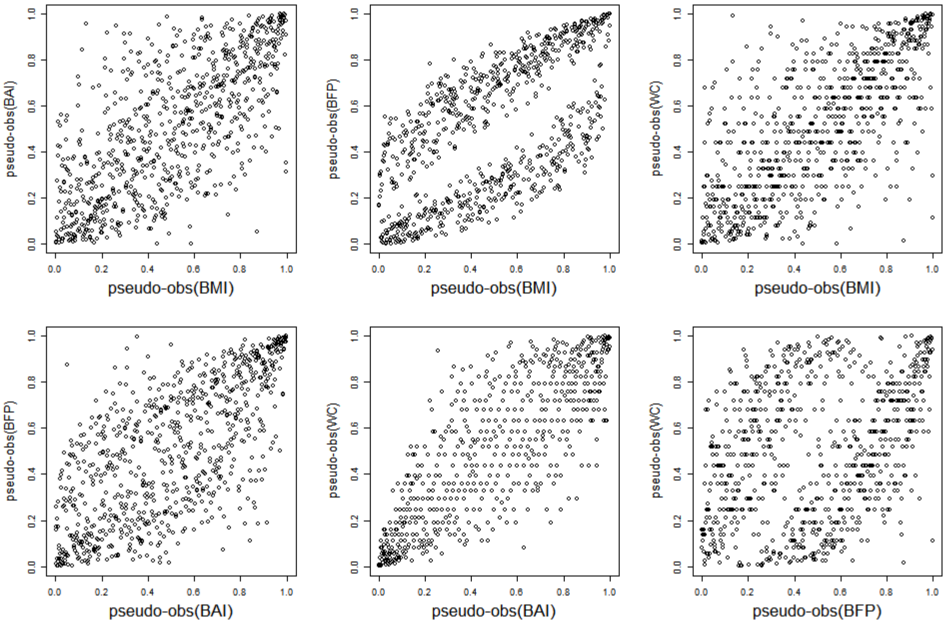


Figure 5: *Scatter plots between pseudo anthropometric measurements observations (pseudo-obs (BMI), pseudo-obs (BAI), pseudo-obs (BFP), and pseudo-obs (WC)).*

## 6.1. A simulation study

In this subsection, a simulation study is carried out to examine the flexibility of the new copulas as a result of unit Gompertz distortion. For simulating bivariate data from copulas, see Aldhufairi *et al.* (2020) for details about the conditional method they describe.

First, we simulated four bivariate data sets from Clayton, Gumbel, Frank, and Gaussian copulas, each with 1500 observations, using parameter values of $2, 2, 5.82$, and $0.71$, respectively. Then, using UPG-Clayton, UPG-Gumbel, UPG-Frank, and UPG-Gaussian copulas with parameter values of $(2, 0.5, 1.25), (2, .5, 2), (0.71, 0.25, 2)$, and $(5.82, 0.5, 1.5)$, four data sets of 1500 each were respectively produced. The parameters are estimated using the pseudo-likelihood estimation method in (6.1). Following that, each data set produced from the UPG-distorted copulas was fitted to using the base copula models, and vice versa.

In the first and second rows of the P-P plots of the empirical cdf and the estimated cdf in Figure 3, the black curve is from the true copula model, while the curves that are not black are from fitting the estimated copula models. We can identify which copula model has the worst fits by looking at the departure curves from the black curve. For instance, it appears that the Clayton copula has the worst models since the red curve strongly deviates from the black curve. Compared to the second row of data, the first row of data generated from the base copulas appears to be well approximated by various UPG-distorted copula models. It would assume that UPG-distorted copulas with additional parameters will be more adaptable and enhance fit.

## 6.2. Data of anthropometric measurements

In this subsection, we use the data of 783 observations for anthropometric measurements on four variables, body mass index (BMI), body adiposity index (BAI), body fat percentage (BFP), and widest

Table 3: MPLE, AIC, $\hat{\theta}$ $\hat{a}$, $\hat{b}$, and parameter estimates (standard errors) for the UPG-distorted (base) and UL-distorted copula models

| | Family | MPLE | AIC | $\widehat{\theta}$ | $\widehat{a}$ | $\widehat{b}$ |
|---|---|---|---|---|---|---|
| BMI & BAI | UPG-Clayton | 220.80(139.20) | −435.59(−276.33) | 0.017(0.002)(2.092(0.084)) | 0.713(0.002) | 0.068(0.001) |
| | UPG-Gumbel | 271.08(256.40) | −536.17(−510.82) | 1.399(0.001)(1.919(0.053)) | 0.518(0.001) | 0.190(0.001) |
| | UPG-Frank | 259.64(257.70) | −513.29(−513.42) | 2.824(0.284)(5.920(0.308)) | 0.306(0.041) | 0.441(0.041) |
| | UPG-Gaussian | 260.26(258.70 ) | −514.52(−515.35 ) | 0.831(0.051)(0.699(0.012)) | 0.991(1.023) | 5.756(0.237) |
| | UL-Clayton | 205.25 | −404.51 | 2.845(0.630) | 2.702(0.252) | 0.550(0.021) |
| | UL-Gumbel | 270.67 | −535.33 | 1.569(0.065) | 1.001(0.104) | 4.236(1.211) |
| | UL-Frank | 236.33 | −466.66 | 4.993(0.308) | 1.001(0.046) | 2.721(0.092) |
| | UL-Gaussian | 259.96 | −513.93 | 0.784(0.012) | 2.717(0.126) | 1.005(0.001) |
| BMI & BFP | UPG-Clayton | 170.81( 77.19) | −335.61(−152.37) | 0.001(0.003)(1.717(0.066) ) | 0.437(0.262) | 0.261(0.262) |
| | UPG-Gumbel | 226.75(224.90) | −447.50(−447.80 ) | 1.668(0.070)( 1.762(0.079)) | 0.114(1.447) | 6.825(2.798) |
| | UPG-Frank | 206.57(177.20) | −407.14(−352.44 ) | 2.609(0.327)(4.411(0.378)) | 0.505(0.002) | 0.184(0.002) |
| | UPG-Gaussian | 251.16(227.30) | −496.31(−452.64) | 0.417(0.014)(0.668(0.025)) | 0.998(0.001) | 0.023(0.001) |
| | UL-Clayton | 137.18 | −268.36 | 2.536(0.291) | 2.730(1.072) | 1.172(0.002) |
| | UL-Gumbel | 226.77 | −447.55 | 1.661(0.075) | 4.913(1.632) | 3.547(1.025) |
| | UL-Frank | 164.36 | −322.73 | 3.381(0.277) | 1.001(0.030) | 2.720(0.063) |
| | UL-Gaussian | 227.84 | −449.73 | 2.758(0.014) | 2.717(0.103) | 1.003(0.006) |
| BMI & WC | UPG-Clayton | 241.48(116.70) | −476.96(−231.35) | 0.007(0.001)(2.271(0.092)) | 0.412(0.002) | 0.059(0.001) |
| | UPG-Gumbel | 293.13(285.40) | −580.25(−568.86 ) | 1.592(0.047)(2.019(0.052)) | 0.669(0.001) | 0.102(0.001) |
| | UPG-Frank | 281.61(273.80) | −557.22(−545.59) | 3.489(0.308)(6.271(0.292)) | 0.359(0.040) | 0.374(0.040) |
| | UPG-Gaussian | 268.63(263.40) | −531.26(−524.71) | 0.885(0.003)(0.704(0.011)) | 0.999(0.051) | 18.73(0.246) |
| | UL-Clayton | 205.11 | −404.22 | 2.781(0.229) | 2.598(0.431) | 0.550(0.051) |
| | UL-Gumbel | 291.90 | −577.80 | 1.741(0.080) | 1.000(0.001) | 2.599(0.699) |
| | UL-Frank | 238.44 | −470.88 | 5.124(0.314) | 1.001(0.051) | 2.721(0.085) |
| | UL-Gaussian | 266.37 | −526.73 | 0.789(0.012) | 2.718(0.005) | 1.001(0.001) |
| BAI & BFP | UPG-Clayton | 205.35( 110.6) | −404.71(−219.12) | 0.003(0.001)(1.953(0.080)) | 0.380(0.001) | 0.247(0.001) |
| | UPG-Gumbel | 285.08(284.30) | −564.16(−566.52) | 1.886(0.076)( 1.968(0.072)) | 0.0003(2.15) | 1.026(2.869) |
| | UPG-Frank | 268.82(234.00) | −531.64(−465.93) | 4.064(0.362)(5.445(0.342)) | 0.519(0.001) | 0.188(0.001) |
| | UPG-Gaussian | 277.35(268.40) | −548.70(−534.71) | 0.889(0.007)(0.708(0.017)) | 0.992(0.040) | 18.50(0.016) |
| | UL-Clayton | 180.81 | −355.62 | 2.564(0.196) | 2.697(0.023) | 0.550(0.001) |
| | UL-Gumbel | 285.01 | −564.03 | 1.902(0.078) | 7.616(0.909) | 2.718(0.001) |
| | UL-Frank | 201.85 | −397.69 | 4.244(0.292) | 1.001(0.102) | 2.721(0.081) |
| | UL-Gaussian | 270.89 | −535.78 | 0.793(0.012) | 2.717(0.036) | 1.001(0.002) |
| BAI & WC | UPG-Clayton | 369.83(333.20) | −733.67(−664.36) | 0.042(0.001)(3.066(0.183)) | 0.895(0.001) | 0.033(0.002) |
| | UPG-Gumbel | 410.60(348.00) | −815.20(−693.97) | 1.600(0.046)(2.212(0.082)) | 0.005(0.001) | 1.204(0.001) |
| | UPG-Frank | 386.81(370.60 ) | −767.61(−739.24) | 3.743(0.279)(7.660(0.420)) | 0.014(0.001) | 1.106(0.001) |
| | UPG-Gaussian | 425.21(411.20) | −844.42(−820.33) | 0.688(0.097)(0.809(0.013)) | 0.431(0.392) | 0.870(0.554) |
| | UL-Clayton | 379.13 | −752.26 | 3.641(0.276) | 1.909(0.081) | 0.690(0.002) |
| | UL-Gumbel | 379.71 | −753.41 | 1.945(0.062) | 1.098(0.351) | 2.718(0.002) |
| | UL-Frank | 387.54 | −769.08 | 8.249(0.628) | 1.145(0.203) | 2.719(0.016) |
| | UL-Gaussian | 423.92 | −841.84 | 0.819(0.038) | 1.171(0.297) | 4.757(0.231) |
| BFP & WC | UPG-Clayton | 55.14( 17.65) | −104.27(−33.29) | 0.003(0.001)(0.670(0.049)) | 0.784(0.002) | 0.058(0.003) |
| | UPG-Gumbel | 64.59(64.15) | −123.18(−126.30) | 1.266(0.043)( 1.293(0.045)) | 0.001(1.077) | 19.51(1.459) |
| | UPG-Frank | 64.52(46.38 ) | −123.04(−90.76) | 1.332(0.342)( 2.062(0.275)) | 0.765(0.008) | 0.070(0.005) |
| | UPG-Gaussian | 68.53(66.07) | −131.06(−130.13) | 0.780(0.018)(0.399(0.037)) | 0.981(0.180) | 97.50(0.002) |
| | UL-Clayton | 33.10 | −60.20 | 1.014(0.139) | 2.719(0.001) | 2.718(0.002) |
| | UL-Gumbel | 64.58 | −123.15 | 1.266(0.044) | 11.717(0.594) | 2.710(0.159) |
| | UL-Frank | 35.51 | −65.02 | 2.183(0.419) | 2.747(0.260) | 2.749(0.292) |
| | UL-Gaussian | 64.92 | −123.85 | 0.503(0.030) | 2.718(0.007) | 1.000(0.001) |

circumference (WC). A source can be assessed at the website: https://figshare.com/articles/dataset. The summary statistics for the four measurements are presented in Table 2. Data were gathered from 437 women and 346 men between the ages 25 and 80 years, and their average age was approximately 49 years.

The widest circumference was in centimeters, and it has been converted to meters. Age is in years, and BMI is in kg/m$^2$. We calculated the BFP measurement as BFP = (1.39*BMI) + (0.16*Age) − (10.34*S) −9, where S = 0 for men and S = 1 for women, and the BAI measurement was based on a formula placed on the website: https://www.omnicalculator.com/health/bai. The four variables consider the risk to human health, as mentioned by Sapporo and Gongs in (2020), who carry out

Table 4: Cramer-von Mises statistics ($p$-value) for the chosen copula models

| | Clayton | Gumbel | Frank | Gaussian |
|---|---|---|---|---|
| BMI & BAI | 0.55(0.025) | 0.09(0.609) | 0.03(0.998) | 0.04(0.986) |
| BMI & BFP | 0.77(0.006) | 0.15(0.335) | 0.24(0.171) | 0.14(0.401) |
| BMI & WC | 1.07(0.001) | 0.15(0.321) | 0.11(0.531) | 0.19(0.253) |
| BAI & BFP | 0.70(0.005) | 0.03(0.999) | 0.12(0.876) | 0.07(0.890) |
| BAI & WC | 0.43(0.071) | 0.34(0.080) | 0.14(0.413) | 0.11(0.580) |
| BFP & WC | 0.48(0.019) | 0.60(0.006) | 0.71(0.020) | 0.69(0.013) |
| | UPG-Clayton | UPG-Gumbel | UPG-Frank | UPG-Gaussian |
| BMI & BAI | 0.45(0.542) | 0.02(0.941) | 0.04(0.929) | 0.03(0.932) |
| BMI & BFP | 0.51(0.494) | 0.14(0.765) | 0.16(0.758) | 0.11(0.829) |
| BMI & WC | 0.73(0.257) | 0.09(0.827) | 0.13(0.768) | 0.18(0.720) |
| BAI & BFP | 0.35(0.685) | 0.03(0.923) | 0.07(0.821) | 0.04(0.896) |
| BAI & WC | 0.50(0.480) | 0.20(0.708) | 0.23(0.699) | 0.08(0.843) |
| BFP & WC | 0.38(0.713) | 0.28(0.741) | 0.29(0.755) | 0.24(0.782) |
| | UL-Clayton | UL-Gumbel | UL-Frank | UL-Gaussian |
| BMI & BAI | 0.37(0.578) | 0.02(0.916) | 0.15(0.799) | 0.04(0.927) |
| BMI & BFP | 0.85(0.168) | 0.14(0.754) | 0.42(0.551) | 0.13(0.804) |
| BMI & WC | 0.79(0.276) | 0.11(0.824) | 0.41(0.537) | 0.18(0.707) |
| BAI & BFP | 0.54(0.508) | 0.03(0.912) | 0.34(0.606) | 0.06(0.870) |
| BAI & WC | 0.34(0.564) | 0.20(0.690) | 0.82(0.193) | 0.09(0.830) |
| BFP & WC | 0.68(0.440) | 0.28(0.730) | 0.62(0.451) | 0.26(0.762) |

similar work.

The scatter plots in Figure 4 show how anthropocentric variables are related. Here, the Kendall's tau values of the samples among each of the two selected variables are shown in Figure 4, as well. The correlation matrix, for instance, shows that the highest tau value is 0.51 between BAI and WC and the lowest tau value is 0.25 between BFP and WC.

Figure 5 gives an important sign for any lower or upper tail dependence that may help in assigning a suitable copula. There appear to be upper and lower tail dependencies, as demonstrated in Figure 5, and as a result, the UPG-distorted copula models look appropriate due to additional parameters and may be flexible to improve the fit. To illustrate more, the UPG-Gaussian copula model will perform better than the Clayton, Gumbel, and Frank copula models if there is upper and lower tail dependence between BAI and WC.

The results of the distorted copula models are presented in Table 3, which include MPLE, AIC, $\hat{\theta}$, $\hat{a}$, and $\hat{b}$. Here, we simulate 783 Clayton, Gumbel, Frank and Gaussian observations, and then, we calculate the CvM test based on 1500 replicates. The Clayton copula model was not suitable for all situations in Table 3, so its performance is the worst. Gumbel, Frank, and Gaussian provide a good fit in terms of MPLE and AIC. The UPG-distorted copula models improve the fit in terms of MPLE and AIC. They have the ability to improve parameter estimations; note that all standard errors are small.

Based on the applied data, though the UL-distorted models may not perform as well as the UPG-distorted models, the UL-distorted copulas are anticipated to improve the model fit in terms of MPLE and AIC more than the base copula models.

The new distorted model of the Clayton copula successfully fits the data for all the dependent situations in Table 3, though UPG-Gumbel, UPG-Frank, and UPG-Gaussian continue to maintain their best overall performance model positions. As shown in Table 4, no distorted copula model would be rejected, as far as the CvM statistical test and its $p$-values are concerned. Table 5 summarizes the best copula models for each situation among two anthropometric measurements. Additionally, the values of Kendall's tau and coefficients of the upper tail dependence are calculated and reported in Table 5. Furthermore, the values and coefficients between the base and distorted models are close to

Table 5: Summary of the best base or distorted copula model with its Kendall's tau value and its coefficient of upper tail dependence

| | Good distorted (or base) model | $\hat{\tau}_T(\hat{\tau})$ | $\hat{\lambda}_{T,U}(\hat{\lambda}_U)$ |
|---|---|---|---|
| BMI & BAI | UPG-Gumbel(Frank) | 0.510(0.292) | 0.722(–) |
| BMI & BFP | UPG-Gaussian(Gaussian) | 0.516(0.466) | –(–) |
| BMI & WC | UPG-Gumbel(Gumbel) | 0.527(0.505) | 0.454(0.590) |
| BAI & BFP | UPG-Gumbel(Gumbel) | 0.644(0.492) | 0.556(0.578) |
| BAI & WC | UPG-Gaussian(Gaussian) | 0.586(0.510) | –(–) |
| BFP & WC | UPG-Gaussian(None) | 0.220(–) | –(–) |

each other, and they reflect the values of Kendall's tau shown in Figure 4.

## 7. Concluding remarks

Herein, we define the associate distortion that is linked to the conversion function, and then, it is used to construct a UPG distortion with two parameters. Additionally, a global distorted distribution function is used to construct a new family of the UPG-distorted function. New distorted bivariate copulas, namely, UPG-Gumbel, UPG-Clayton, UPG-Frank, and UPG-Gaussian, are explicitly expressed and given on the basis that the base copulas are, respectively, Gumbel, Clayton, Frank, and Gaussian. We look at the effect of the countermonotonic or comonotonic copulas on the distorted copula. For any base copula, the limiting cases in parameters for the family of the UPG-distorted copulas are carefully examined. The tail behaviors are investigated for the UPG-distorted copula. We derive Kendall's tau formulas for UPG-Clayton, UPG-Gumbel, and UPG-Frank. Furthermore, these formulas are used to measure dependence in proposed copula models and examine the order of concordance. Based on the results of the application, the new distorted bivariate copula models are the best overall relative to their corresponding base bivariate copula models. In future research, we will attempt to extend our base and distorted copula models in the Application section to a high- or 4-dimensional base, and then, we can compare their performances. This paper would allow for a reasonable extension of this study to n-dimensional distorted copulas.

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