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On Extended Hurwitz-Lerch Zeta Function

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Abstract. This paper investigates an extended form Hurwitz-Lerch zeta function, as well as related integral images, ordinary and fractional derivatives, and series expansions, using the term extended beta function. We establish a connection between the extended Hurwitz-Lerch zeta function and the Laguerre polynomials. Furthermore, we present a probability distribution application of the extended Hurwitz-Lerch zeta function$\zeta_{\nu,\lambda}^{\delta,\mu}$. Several results, both known and new, are shown to follow as special cases of our findings.

1. Introduction

The Hurwitz zeta function is classically defined by the formula [10]:

$$\zeta(z,a)=\sum_{n=0}^{\infty}(a+n)^{-z}\quad (a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\Re(z)>1), \tag{1.1}$$

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which is a generalization of the Riemann zeta function

$$\zeta(z)=\sum_{n=0}^{\infty} n^{-z} \quad (\Re(z)>1). \tag{1.2}$$

Here and elsewhere, let $\mathbb{C}, \mathbb{R}, \mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}_0^-$ denote, respectively the sets of the complex numbers, real numbers, positive integers, non-negative integers, and the non-positive integers.
The Hurwitz-Lerch zeta function [10,p.27(1)]:

$$\Phi(y,z,a)=\sum_{n=0}^{\infty} \frac{y^n}{(a+n)^z}, \tag{1.3}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; z \in \mathbb{C} \text{ when } |y|<1; \Re(z)>1 \text{ when } |y|=1),$$

is a special function that generalizes the Hurwitz zeta function $\zeta(z,a)$. Φ is an analytic function in both variables y and z in a suitable region and it reduces to the ordinary Lerch zeta function when $y=e^{2\pi i\lambda}$:

$$\Phi(e^{2\pi i\lambda},z,a)=\phi(\lambda,z,a)=\sum_{n=0}^{\infty} \frac{e^{2\pi in\lambda}}{(a+n)^z}. \tag{1.4}$$

The zeta function $\Phi(y,z,a)$ has since been extended by Goyal and Laddha [15,p. 100 (1.5)] in the form:

$$\Phi_{\mu}^{*}(x,z,a)=\sum_{n=0}^{\infty} \frac{(\mu)_n x^n}{(a+n)^z n!}, \tag{1.5}$$

where $(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}=\alpha(\alpha+1)...(\alpha+n-1)$ denotes the Pochhammer's symbol, $\mu \in \mathbb{C}, a \neq \mathbb{Z}_0^-$ and $|x|<1$. Obviously, when $\mu=1$, (1.5) reduces to (1.3).
Bin-Saad introduced the hypergeometric type generating function of the Hurwitz-Lerch zeta function in [5], which is defined by (1.3) and (1.5) as follows:

$$\zeta_{\lambda}^{\mu}(x,y;z,a)=\sum_{m=0}^{\infty} (\mu)_m \Phi(y,z,a+\lambda m)\frac{x^m}{m!} \tag{1.6}$$

where $|x|<1, |y|<1; \mu \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda \in \mathbb{C} \setminus \{0\}; a \in \mathbb{C} \setminus \{-(n+\lambda m)\}, \{n,m\} \in \mathbb{N}_0$ and Φ is the Hurwitz-Lerch zeta function defined by (1.3).
The case when $y=0$ of the definition (1.6) gives us the following further generalization of the zeta function defined by (1.5) [5]:

$$\zeta_{\lambda}^{\mu}(x,0;z,a)=\Phi_{\mu,\lambda}^{*}(x,z,a)=\sum_{m=0}^{\infty} \frac{(\mu)_m x^m}{m!(a+\lambda m)^z}, \tag{1.7}$$

where $|x| < 1; \mu \in \mathbb{C} \setminus \mathbb{Z}_0^-; a \in \mathbb{C} \setminus \{-(\lambda m)\}, m \in \mathbb{N}_0$.

A further generalization of the Hurwitz-Lerch zeta function Φ_μ^* (see Eq. (1.5)):

$$\Phi_{\lambda,\mu;\nu}(x,z,a) = \sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n \; x^n}{(\nu)_n(a+n)^z}, \tag{1.8}$$

where $\lambda, \mu \in \mathbb{C}; \nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; x \in \mathbb{C}$ when $|x| < 1; \Re(z+\nu-\lambda-\mu) > 1$ when $|x| = 1$, was investigated earlier by Garg et al. [14,p.313, Eq(1.7)]. The Beta function $B(x,y)$ is a function of two complex variables x and y and defined by

$$B(x,y) = \begin{cases} \int_0^1 t^{x-1}(1-t)^{y-1}dt, & \Re(x) > 0, \Re(y) > 0, \\ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, & x, y \in \mathbb{C} \setminus \mathbb{Z}_0^-. \end{cases} \tag{1.9}$$

The integrals in (1.9) are known as first-order Eulerian integrals. Several authors have recently considered extensions of some well-known special functions (see [6-9]). Chaudhry et al. [5] presented the following extension of the Beta function in 1997:

$$B(x,y;p) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt; \Re(p) > 0, \Re(x) > 0, \Re(y) > 0, \tag{1.10}$$

and it has been proved that this extension has a connection with Macdonald, Error, and Whittaker's functions. Subsequently, Chaudhry et al. [8] used the definition of the function $B(x,y;p)$ to provide an extension of the Gaussian hypergeometric function ${}_2F_1$ and the confluent hypergeometric function ${}_1F_1$ as follows

$$F_p\left(a,b;c;x\right) = \sum_{n=0}^{\infty} \frac{B(b+n, c-b;p)(a)_n}{B(b,c-b)} \frac{x^n}{n!}; p \geq 0, |x| < 1, \Re(c) > \Re(b) > 0, \tag{1.11}$$

and

$$\Phi_p\left(a,b;c;x\right) = \sum_{n=0}^{\infty} \frac{B(b+n, c-b;p)}{B(b,c-b)} \frac{x^n}{n!}; p \geq 0, |x| < 1, \Re(c) > \Re(b) > 0. \tag{1.12}$$

The special cases of (1.10), (1.11), and (1.12) when $p = 0$ reduce immediately to the classical beta function $B(x,y)$, the Gaussian hypergeometric function ${}_2F_1$ and confluent hypergeometric function ${}_1F_1$ respectively. Various integral representations [6] of the extended beta function are available, two of which are shown below:

$$B(x,y;p) = 2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1} \exp(-p \; \sec^2\theta \csc^2\theta) d\theta, \tag{1.13}$$

$$B(a,b;p) = 2^{1-a-b} \int_{-\infty}^{\infty} \exp\left[(a-b)x - 4\; p\cosh^2 x)\right] \frac{dx}{(\cosh x)^{a+b}} dx. \tag{1.14}$$

Formulas (1.13) and (1.14) can be obtained by using $t = \cos^2\theta$ and $t = \tanh x$ in formula (1.10) respectively. By using the definition (1.5) and the extended Beta function (1.10), the authors in [25] introduced an extension of the generalized Hurwitz-Lerch zeta function as follows:

$$\Phi_{\lambda,\mu,\nu}(z,s,a;p) = \sum_{n=0}^{\infty} \frac{B(\mu+n,\nu-\mu;p)}{B(\mu,\nu-\mu)} \frac{(\lambda)_n z^n}{n!(a+n)^s}. \tag{1.15}$$

In [1] Barnes introduced double zeta function in the form:

$$\zeta_2(z;a,\omega) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(a+n+m\omega)^z}, \tag{1.16}$$

where $a > 0, \omega$ is a non-zero complex number with $|\arg(\omega)| < \pi$.
Note that the definition (1.16) is direct generalization of Hurwitz zeta function (1.1) and also we observe that in (1.16) the product of an arbitrary parameter ω with the summation index m. Again, for $n = 0$, equation (1.16) reduces to the zeta funtion:

$$\zeta(z;a,\omega) = \sum_{m=0}^{\infty} \frac{1}{(a+\omega m)^z}, \tag{1.17}$$

which further for $\omega = 1$ reduces to Hurwitz zeta function (1.1) and for $\omega = 1,\ a = 0$ gives us the Riemann zeta function:

$$\zeta(z) = \sum_{m=0}^{\infty} m^{-z}. \tag{1.18}$$

Barnes [2] then proceeded to develop a theory of more general functions and he introduced the multiple zeta function defined by

$$\zeta_r(s;\alpha,(\omega_1,\cdots,\omega_r)) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty} \frac{1}{(\alpha+m_1\omega_1+\cdots+m_r\omega_r)^z}. \tag{1.19}$$

Motivated by the works of Barnes (1.16) and (1.19), recently many researchers adopted the approach of Bernes in [1] and [2] and proposed new extended (and generalized) Hurwitz zeta functions. For instance, Bin-Saad [5] proposed the definition (1.7) as an analogy and generalization of the definition in (1.17). Further, in [21] Matsumoto introduced a generalization of Hurwitz zeta function in the form

$$\begin{aligned} &\zeta_r((s_1,\cdots,s_r);(\alpha_1,\cdots,\alpha_r),(\omega_1,\cdots,\omega_r)) \\ &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty} (\alpha_1+m_1\omega_1)^{-s_1}\times(\alpha_1+m_1\omega_1+m_2\omega_2)^{-s_2}\times\cdots\times(\alpha_1+m_1\omega_1+\cdots+m_r\omega_r)^{-s_r}. \end{aligned} \tag{1.20}$$

Furthermore, a generalized Hurwitz-Lerch zeta function is proposed in the following manner (see [19]):

$$\phi_{\mu}^{\alpha,\beta}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n z^n}{(a+\alpha z^{\beta}n)^s n!}. \tag{1.21}$$

It is worth noting that adding a complex parameter, say λ multiplied by the summation index m to the denominator, as shown in equations (1.7), (1.16), (1.17), and (1.19)-(1.21), makes the factor λm arbitrary, and various interesting special cases can be obtained by appropriately specializing the relevant parameter λ. In this paper, we use the same approach as in [1, 2, 4, 21, 25] to introduce a new extended Hurwitz-Lerch zeta function $\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)$. This class of zeta function has a richer mathematical structure and many related useful properties that cannot be deduced from known definitions. The rest of this paper is organized as follows. In Section 2 we introduce and describe some properties and relationships for the function $\zeta_{\nu,\lambda}^{\delta,\mu}$. Relevant connections of the function $\zeta_{\nu,\lambda}^{\delta,\mu}$ with those considered in [5] are also indicated. In Section 3, we establish several integral representations for the function $\zeta_{\nu,\lambda}^{\delta,\mu}$ involving integral representations of Eulerian integral of the second kind, contour integral and Mellin-Barnes integral. Section 4 is devoted to the differentiation of the function $\zeta_{\nu,\lambda}^{\delta,\mu}$ with respect to arguments x,z,λ,δ and a. In Section 5, we present some series expansions for the function $\zeta_{\nu,\lambda}^{\delta,\mu}$ involving Appell's function of two variables F_2, the extended Gauss hypergeometric function F_p, the extended confluent hypergeometric function Φ_p and the generalized hypergeometric function ${}_3F_2$. Also, we present a connection between our new extended zeta function and the Laguerre polynomials. Finally, in Section 6, we present an application of the extended Hurwitz-Lerch zeta function $\zeta_{\nu,\lambda}^{\delta,\mu}$ to probability distributions.

2. The Extended Hurwitz-Lerch Zeta Function $\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)$

Based on the extension of Euler's beta function (1.10) and Hurwitz-Lerch zeta function (1.7), we propose and investigate an extended Hurwitz-Lerch zeta function of the form

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)=\sum_{m=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}\frac{(\delta)_m x^m}{m!(a+\lambda m)^z}, \tag{2.1}$$

$\left(|x|<1;\{\delta,\mu,\nu\}\in\mathbb{C}\setminus\mathbb{Z}_0^-,\lambda\in\mathbb{C}\setminus\{0\};a\in\mathbb{C}\setminus\{-(\lambda m)\},\Re(p)\geq 0,m\in\mathbb{N}_0\right).$

Remark 2.1. By letting $\lambda=1$ in (2.1), we obtain the definition (1.15) of Parmar and Raina [25]. Hence definition (2.1) is a unification and generalization of (1.15). Consequently, if we let $\lambda=1$ in any result of our findings in this work, we get new formulas for the definition (1.15).
Clearly, we have

$$\zeta_{\mu,1}^{1,\mu}(1; z, a; 0) = \zeta(z, a), \tag{2.2}$$

$$\zeta_{\mu,1}^{1,\mu}(x; z, a; 0) = \Phi(x, z, a), \tag{2.3}$$

$$\zeta_{\nu,1}^{\mu,\nu}(x; z, a; 0) = \Phi_{\mu}^{*}(x, z, a), \tag{2.4}$$

$$\zeta_{\nu,1}^{\delta,\mu}(x; z, a; 0) = \Phi_{\delta,\mu;\nu}(x, z, a). \tag{2.5}$$

The case when $p = 0$ of the assertion (2.1) gives us a new generalization of the zeta function $\Phi_{\lambda,\mu;\nu}$ in the form:

$$\Phi_{\nu,\lambda}^{\delta,\mu}(x; z, a) = \sum_{m=0}^{\infty} \frac{(\delta)_m (\mu)_m x^m}{(\nu)_m m! (a + \lambda m)^z}, \tag{2.6}$$

$\left(|x| < 1; \{\delta, \mu, \nu\} \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda \in \mathbb{C} \setminus \{0\}; a \in \mathbb{C} \setminus \{-\lambda m\}, m \in \mathbb{N} \cup \{0\}\right)$.
In the case when $\lambda = p = 0$, we have simply

$$\zeta_{\nu,1}^{\delta,\mu}(x; 1, a; 0) = a^{-1} \sum_{m=0}^{\infty} \frac{(a)_m (\delta)_m (\mu)_m x^m}{(\nu)_m (a + 1)_m m!}, \tag{2.7}$$

which implies the next result.

Corollary 2.1. *Let* $|x| < 1, \Re(a) > 0$. *Then*

$$\zeta_{\nu,1}^{\delta,\mu}(x; 1, a; 0) = a^{-1} {}_3F_2[a, \delta, \mu; a + 1, \nu; x], \tag{2.8}$$

where ${}_3F_2$ is the generalized Gaussian hypergeometric function of one variable (cf. [29]).

Corollary 2.2. *Let* $\lambda = 0, |x| < 1$. *Then*

$$\zeta_{\nu,0}^{\delta,\mu}(x; z, a; p) = a^{-z} F_p\left(\delta, \mu; \nu; x\right). \tag{2.9}$$

Proof. We have

$$\zeta_{\nu,0}^{\delta,\mu}(x; z, a; p) = \sum_{m=0}^{\infty} \frac{B(\mu + m, \nu - \mu; p)}{B(\mu, \nu - \mu)} \frac{(\delta)_m x^m}{m! a^z}.$$

Then by considering the extended Gaussian hypergeometric function F_p (see Eq.

(1.11)), we get (2.9). □

Further, we recall the definition of the derivative operator D_x^m (see e.g. [22]):

$$D_x^m x^{\delta+m-1} = \frac{\Gamma(\delta+m)}{\Gamma(\delta)} x^{\delta-1} = (\delta)_m x^{\delta-1}, m \in \mathbb{N}_0. \tag{2.10}$$

Now, from (2.10) it is not difficult to infer the following interesting relation

Corollary 2.3. *Let* $\Re(\delta) > 0, \lambda = 0, |x| < 1$. *Then*

$$\zeta_{\nu,0}^{\delta,\mu}(x; z, a; p) = x^{1-\delta} \Phi_p(\mu; \nu; D_x x) x^{\delta-1}, \tag{2.11}$$

where Φ_p is the extended confluent hypergeometric function defined by (1.12).
Proof. We refer to the proof of (2.9). □

Further, with the help of Euler's integral [10]

$$a^{-z}\Gamma(z) = \int_0^\infty e^{-at} t^{z-1} dt, \tag{2.12}$$

and Hankel's integral [13,p.32(1.5.1.5)]:

$$\frac{1}{2\pi i} \int_c e^t t^{-(\delta+m)} dt = \frac{1}{\Gamma(\delta+m)}, m \in \mathbb{N}_0, \delta \neq \mathbb{Z}_0^-, \tag{2.13}$$

we can derive the following connections of the extended Hurwitz-Lerch zeta function $\zeta_{\nu,\lambda}^{\delta,\mu}$.

Corollary 2.4. *Let* $\Re(\delta) > 0, \Re(z) > 0, |x| < 1$. *Then*

$$\frac{1}{\Gamma(z)} \int_0^\infty e^{-at} t^{z-1} \zeta_{\nu,\lambda}^{\delta,\mu}(xe^{-\lambda t}; -z, a; p) dt = F_p(\delta, \mu; \nu; x), \tag{2.14}$$

$$\frac{a^z \Gamma(\delta)}{2\pi i} \int_c e^t t^{-\delta} \zeta_{\nu,0}^{\delta,\mu}(xt^{-1}; z, a; p) dt = \Phi_p(\mu; \nu; x). \tag{2.15}$$

Proof. The results (2.14) and (2.15) follow directly from the formulas (2.12) and (2.13) respectively. □

Furthermore, putting $\delta = \alpha + \beta$ in (2.1) and using the classical formula of Nörlund for the Pochhammer symbol (cf. [3, Section 1, Chapter 3]):

$$(a+b)_k = \sum_{m=0}^{k} \binom{k}{m} (a)_{k-m} (b)_m, \tag{2.16}$$

we find the form (2.1) that

$$\zeta_{\nu,\lambda}^{\alpha+\beta,\mu}(x;z,a;p)=\sum_{m=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}\frac{(\alpha)_m}{(a+\lambda m)^z}$$

$$\times\ {}_2F_1\left[\begin{array}{cc}-m,\beta; & \\ & 1\\ 1-\alpha-m; & \end{array}\right]\frac{x^m}{m!}. \tag{2.17}$$

Since

$$\lim_{|\lambda|\to\infty}\left\{(\lambda)_m\frac{x^m}{\lambda}\right\}=x^m,$$

for bounded x and $m\in\mathbb{N}_0$, we infer the following confluent form of the function $\zeta_{\nu,\lambda}^{\delta,\mu}$:

(2.18)

$$\lim_{|\delta|\to\infty}\left\{\zeta_{\nu,\lambda}^{\delta,\mu}\left(\frac{x}{\delta};z,a;p\right)\right\}=\zeta_{\nu,\lambda}^{\mu}\left(x;z,a;p\right)=\sum_{m=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}\frac{x^m}{m!(a+\lambda m)^z}.$$

3. Integral Images of $\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)$

First, by using the results

$$(a+\lambda m)^{-z}x^m=\frac{1}{\Gamma(z)}\int_0^{\infty}\left(xe^{-\lambda}\right)^m t^{z-1}e^{-at}dt, \tag{3.1}$$

and

$$(\delta)_m=\frac{1}{\Gamma(\delta)}\int_0^{\infty}u^{\delta+m-1}e^{u}du, \tag{3.2}$$

which follow from the Eulerian integral(2.12) and exploiting the integral representation of the extended beta function $B(x,y;p)$[6]:

$$B(x,y;p)=e^{-2p}\int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x+y}}\exp\left[-p(u+u^{-1})\right]du, \tag{3.3}$$

we can derive the following triple integral image of $\zeta_{\nu,\lambda}^{\delta,\mu}$:

Theorem 3.1. *Let* $\min\{\Re(a),\Re(z),\Re(\delta),\Re(\mu)\}>0$. *Then*

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)=\frac{e^{-2p}}{B(\mu,\nu-\mu)\Gamma(\delta)\Gamma(z)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{t^{z-1}s^{\mu-1}u^{\delta-1}}{(1+u)^{\nu}}e^{-(at+s)}$$

$$\times e^{-p(u+u^{-1})}\times e^{\left(\frac{uxse^{-\lambda}}{1+u}\right)}dudsdt, \tag{3.4}$$

Proof. Given the definitions (2.1) and (3.3), it is easily seen that

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \frac{e^{-2p}}{B(\mu,\nu-\mu)} \sum_{m=0}^{\infty} \int_0^{\infty} \frac{u^{\mu+m-1}}{(1+u)^{\nu+m}} \exp\left[-p(u+u^{-1})\right]$$

$$\times \frac{(\delta)_m x^m}{m!(a+\lambda m)^z} du. \tag{3.5}$$

Now, with the aid of the results (3.1) and (3.2) and by interchanging the order of summation and integration, equation (3.5) gives us the right-hand side of the assertion (3.4). □

Now, we processed to derive integral connections between $\zeta_{\nu,\lambda}^{\delta,\mu}$ and the functions F_p and $\Phi_{\mu,\lambda}^{*}$.

Theorem 3.2. *Let* $\min\{\Re(a),\Re(\delta),\Re(\mu)\} > 0$. *Then*

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \frac{1}{\Gamma(z)} \int_0^{\infty} e^{-at} t^{z-1} F_p\left(\delta,\mu;\nu;xe^{-\lambda t}\right) dt, \tag{3.6}$$

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \frac{\Gamma(\nu)e^{-2p}}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^{\infty} \frac{s^{\mu-1}}{(1+s)^{\nu}} \exp\left[-p(s+s^{-1})\right]$$

$$\times \Phi_{\delta,\lambda}^{*}\left(\frac{xs}{1+s};z,a\right) ds. \tag{3.7}$$

Proof. To prove the formulas (3.6) and (3.7), we employ the integral relations (2.12) and (3.3) respectively, and exploit the same procedure leading to (3.4).□

Next, utilizing the integral images of the generalized extended Euler's beta function (1.10),(1.13), and (1.14) we can derive the following results.

Theorem 3.3. *Let* $\Re(a) > 0$ *and* $\Re(z) > 0$. *Then*

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \frac{1}{B(\mu,\nu-mu)} \int_0^{1} t^{\mu-1}(1-t)^{\nu-\mu-1} \exp\left[\frac{-p}{t(1-t)}\right] \Phi_{\lambda}^{*}(xt,z,a)dt, \tag{3.8}$$

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \frac{1}{B(\mu,\nu-\mu)} \times \int_0^{\frac{\pi}{2}} (\cos\theta)^{2\mu-1}(\sin\theta)^{2(nu-\mu)-1} \exp\left(-p\sec^2\theta\csc^2\theta\right)$$

$$\times \Phi_{\delta,\lambda}^{*}(x\cos^2\theta,z,a)dt, \tag{3.9}$$

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \frac{2^{1-\nu}}{B(\mu,\nu-\mu)} \int_{-\infty}^{\infty} (\cosh x)^{\nu} \exp\left[-(x\nu + 4p\cosh^2 x)\right]$$

$$\times \Phi_{\delta,\lambda}^{*}(x\cosh x\ e^{-\nu}, z, a)dx. \tag{3.10}$$

Proof. The results follow directly from the formulas (1.10), (1.13), and (1.14) respectively. □

Further, we shall prove $\zeta_{\nu,\lambda}^{\delta,\mu}$ as an application of the Mellin-Barnes type integral. Our starting point is the formula (see [30, section 14.51,p.289, Corollary]; also see [17]):

$$(1-\omega)^{-z} = \frac{1}{2\pi i} \int_c \frac{\Gamma(z+\nu)\Gamma(-\nu)(-\omega)^{\nu}}{\Gamma(z)} d\nu, \tag{3.11}$$

where z and ω are complex with $\Re(z) > 0, |arg(\omega)| < \pi, \omega \neq 0$, and the path is the vertical line from $c - i\infty$ to $c + i\infty$. This formula in [30] is stated with $c = 0$, (with suitable modification of the path near the point $z = 0$), but it is clear that the formula is also valid for $-\Re(z) < c < 0$.

Theorem 3.4. *Let* $\Re(z) > 0, \Re(a-b) > 0, \Re(b) > 0, |x| < 1$. *Then*

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \frac{b^{z+\nu}}{2\pi i} \int_c \frac{(\Gamma(\nu+z)\Gamma(-\nu)}{\Gamma(z)} \zeta_{\mu,\nu}^{\delta,\lambda}(x;-\nu,a-b;p)d\nu, \tag{3.12}$$

Proof. Let $\omega = (b-a-\lambda m)/b$ in (3.11) and multiply both the sides by

$$\frac{B(\mu+m,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m}{m!}, (m \in \mathbb{N}_0),$$

to obtain

$$\left(\frac{B(\mu+m,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m}{m!}\right) (a+\lambda m)^{-z}$$

$$= \int_c \frac{(\Gamma(\nu+z)\Gamma(-\nu)}{\Gamma(z)} \times \left(\frac{B(\mu+m,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m}{m!(a-b+\lambda m)^{-\nu}}\right) d\nu, m \geq 0.$$

Therefore, if we assume $(1-\Re(\nu) < c < -1$, then from (2.1), we get (3.12)□.

Furthermore, by using the Mellin transform representation of the extended beta function $B(x,y;p)$ in terms of Mellin-Barnes type contour integral [9]:

$$B(x,y;z,a;p) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(\Gamma(s)\Gamma(x+s)\Gamma(y+s)}{\Gamma(x+y+2s)} p^{-s} ds, \tag{3.13}$$

we have the following complex integral representation for $\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)$.

Theorem 3.5. *Let* $\Re(p) \geq 0, m > 0$ *and* $\gamma > 0$. *Then*

(3.14)
$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(\Gamma(s)\Gamma(\nu-\mu+s)\Gamma(\mu+s)\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)\Gamma(nu+2s)} p^{-s}\Phi_{\nu+2s,\lambda}^{\delta,\mu+s}(x;z,a)ds.$$

where $\Phi_{\nu,\lambda}^{\delta,\mu}$ is zeta function defined by (2.6).

Proof. Using the formula (3.13) in the definition (2.1), interchanging the order of summation and integration, and considering the definition (2.6), we led to the desired result (3.14). □

Now, with help of the limiting case (2.18), we establish the next integral image.

Theorem 3.6. *Let* $\Re(p) \geq 0, |x| < 1$. *Then*

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \frac{1}{\Gamma(\delta)}\int_0^\infty u^{\delta-1}e^u \zeta_{\nu,\lambda}^{\mu}(xu;z,a;p)du. \tag{3.15}$$

Proof. The result follows directly from the definition (2.1) and the formula (3.2).□

4. Differentiation Formulas of $\zeta_{\nu,\lambda}^{\delta,\mu}$

The extended Hurwitz-Lerch zeta function $\zeta_{\nu,\lambda}^{(\delta,\mu)}$, as a function satisfies several differential relations. Fortunately these properties of $\zeta_{\nu,\lambda}^{\delta,\mu}$ can be developed directly from the definition (2.1). In this section, we will find the differentiation of the function $\zeta_{\nu,\lambda}^{\delta,\mu}$ concerning arguments x, z, λ, δ, a and p. Firstly, we recall the following result [22]

$$D_x^m x^n = \frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m}, n-m \geq 0, D_x = \frac{d}{dx}. \tag{4.1}$$

Theorem 4.1. *Let* $k \in \mathbb{N}$. *Then*

$$D_x^k\left\{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)\right\} = \frac{(\delta)_k(\mu)_k}{(\nu)_k}\zeta_{\nu+k,\lambda}^{\delta+k,\mu+k}(x;z,a+\lambda k;p), \tag{4.2}$$

$$D_a^k\left\{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)\right\} = (-1)^k(z)_k\ \zeta_{\nu,\lambda}^{\delta,\mu}(x;z+k,a;p). \tag{4.3}$$

Proof. Using (4.1), we get

$$D_x^k\left\{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)\right\} = \sum_{m=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)}\frac{(\delta)_m x^{m-k}}{(m-k)!(a+\lambda m)^z}. \tag{4.4}$$

Now, letting $m \to m+k$ in (4.4) and considering the definition (2.1), we get the right-hand side of the formula (4.2). Similarly, one can prove the formula (4.3). $\square$

According to the relation (2.2), formula (4.3) reduces to the result

$$D_a^k \zeta(z,a) = (-1)^k (z)_k \zeta(z+k,a), \tag{4.5}$$

which is a known result (see e.g. [12,p.2(1.8)]). Given the relationship (2.5), we find from equation (4.3) that

$$D_a^k \left\{\Phi_{\delta,\mu;\nu}(x,z,a)\right\} = (-1)^k (z)_k \Phi_{\delta,\mu;\nu}(x,z+k,a). \tag{4.6}$$

Secondly, we show that the extended Hurwitz-Lerch zeta function $\zeta_{\mu,\nu}^{\delta,\lambda}$ satisfies the following differential relation.

Theorem 4.2. *Let* $\{(\delta-k),(\mu-k),(\nu-k)\} \neq \mathbb{Z}_0^-, k \in \mathbb{N}_0$. *Then*

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = (-1)^k \frac{(1-\nu)_k}{(1-\delta)_k(1-\mu)_k} D_x^k \left[\zeta_{\nu-k,\lambda}^{\delta-k,\mu-k}(x;z,a-\lambda k;p)\right]. \tag{4.7}$$

Proof. Let I denote the right-hand side of the assertion (4.7). Then given (2.1) and (4.1) we have

$$I = \frac{(-1)^k(1-\nu)_k}{(1-\mu)_k(1-\delta)_k} \sum_{m=0}^{\infty} \frac{B(\mu+m-k,\nu-\mu;p)(\delta-k)_m x^{m-k}}{B(\mu,\nu-\mu)(a-\lambda k+\lambda m)^z (m-k)!}. \tag{4.8}$$

Now, by letting $m = m+k$ and using the identities

$$B(\mu-k,\nu-\mu) = \frac{(1-\nu)_k}{(1-\mu)_k} B(\mu,\nu-\mu), \tag{4.9}$$

and

$$(a)_{-n} = \frac{(-1)^n}{(1-a)_n}, \tag{4.10}$$

we led to the left-hand side of (4.7) $\square$

Next, we establish the derivative of the function $\zeta_{\nu,\lambda}^{\delta,\mu}$ with respect to the argument λ.

Theorem 4.3. *Let* $b \in \mathbb{R}$. *Then*

$$\frac{\partial}{\partial\lambda} \zeta_{\nu,\lambda}^{\delta,\mu}(x;z-1,a+\lambda b;p)$$

$$= (1-z)\left[\left(\frac{x\delta\mu}{\nu}\right)\zeta^{\delta+1,\mu+1}_{\nu+1,\lambda}(x;z,a+\lambda(b+1);p) + b\zeta^{\delta,\mu}_{\nu,\lambda}(x;z,a+\lambda b;p)\right]. \tag{4.11}$$

Proof. We have
$\frac{\partial}{\partial\lambda}\zeta^{\delta,\mu)}_{\nu,\lambda}(x;z-1,a+\lambda b;p)$

$$= (1-z)\left[\sum_{m=1}^{\infty}\frac{B(\mu+m,\nu-\mu;p)(\delta)_m x^m}{B(\mu,\nu-\mu)(m-1)!(a+\lambda(m+b))^z}\right.$$

$$\left. +b\sum_{m=0}^{\infty}\frac{B(\mu+m,\nu-\mu;p)(\delta)_m x^m}{B(\mu,\nu-\mu)m!(a+\lambda(m+b))^z}\right]. \tag{4.12}$$

Now , let $m \to m+1$ in the (4.13) and then use the identity

$$(\mu)_{m+k} = (\mu)_k(\mu+k)_m,$$

to obtain (4.11). □

The same type of differentiation gives the next result.

Theorem 4.4. *Let* $q \in \mathbb{R}$. *Then*

$$\frac{\partial}{\partial q}\zeta^{\delta,\mu}_{\nu,\lambda}(x;z-1,a+bq;p) = b(1-z)\zeta^{\delta,\mu}_{\nu,\lambda}(x;z,a+bq;p). \tag{4.13}$$

Proof. We refer to the proof of assertion (4.11). □

Because of relation (2.5), the assertion (4.13) gives the result

$$\frac{\partial}{\partial q}\Phi_{\delta,\mu;\nu}(x;z-1,a+bq) = b(1-z)\Phi_{\delta,\mu;\nu}(x;z,a+bq). \tag{4.14}$$

Also, it is easily observed that the differential relation (4.13) is a generalization of the known result (see e.g. [11,p.451(2.2)]):

$$\frac{\partial}{\partial q}\zeta(z-1,a+qb) = b(1-z)\zeta(z,a+qb). \tag{4.15}$$

Closely associated with the derivative of the gamma function is the digamma function defined by (see e.g.[20])

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma^{'}(x)}{\Gamma(x)}, x \neq 0,-1,-2,\cdots. \tag{4.16}$$

Now, we establish the derivative of the function $\zeta_{\nu,\lambda}^{\delta,\mu}$ with respect to the parameter δ.

Theorem 4.5. *Let* $\delta \in \mathbb{C} \setminus \mathbb{Z}_0^-$. *Then*

$$\frac{\partial}{\partial\delta}\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)$$

$$= \sum_{m=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m}{(a+\lambda m)^z m!}\psi(\delta+m) - \psi(\delta)\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p). \tag{4.17}$$

Proof. By noting that

$$\frac{\partial}{\partial\delta}\left[(\delta)_m\right] = \frac{\partial}{\partial\delta}\left[\frac{\Gamma(\delta+m)}{\Gamma(\delta)}\right] = (\delta)_m\left[\psi(\delta+m) - \psi(\delta)\right], \tag{4.18}$$

we obtain the result (4.17)□.
Further, let us recall the definition of the Weyl fractional derivative of the exponential function $e^{-at}, a > 0$ of order ν in the form (see [22,p.248(7.4)]):

$$D^{\nu}\left\{e^{-at}\right\} = a^{\nu}e^{-at} \ (\nu \text{ not restricted to be positive integer}). \tag{4.19}$$

We now proceed to find the fractional derivative of the function $\zeta_{\nu,\lambda}^{\delta,\mu}$ concerning z.

Theorem 4.6. *Let* $\nu > 0$. *Then*

$$D_z^{\nu}\left\{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)\right\}$$

$$= \sum_{m=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m}{m!(a+\lambda m)^z} \times \left[\log(a+\lambda m)\right]^{\nu}. \tag{4.20}$$

Proof. Since

$$(a+\lambda m)^{-z} = e^{-z\log(a+\lambda m)},$$

we have

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \sum_{m=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m}{m!} e^{-z\log(a+\lambda m)}.$$

The desired result now follows by applying the formula (4.19) to the above identity.
□

Finally, it is interesting to note that the $k-th$ derivative of the extended zeta function $\zeta_{\nu,\lambda}^{\delta,\mu}$ concerning the parameter p is given by the following theorem.

Theorem 4.7. *Let $k \in \mathbb{N}$. Then*

$$\hat{D}_p^k \left\{ \zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) \right\} = (-1)^k \frac{B(\mu-k,\nu-\mu-k)}{B(\mu,\nu-\mu)} \zeta_{\nu-2k,\lambda}^{\delta,\mu-k}(x;z,a;p). \tag{4.21}$$

Proof. From definitions (2.1) and (1.10) and the formula (4.1), we can state that

$$\hat{D}_p^k \left\{ \zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) \right\} = (-1)^k \sum_{m=0}^{\infty} \frac{B(\mu+m-k,\nu-\mu-k;p)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m}{m!(a+\lambda m)^z}. \tag{4.22}$$

Now, by interpreting the above series in the form of the definition (2.1), we obtain the desired result (4.21). □

5. Series Involving $\zeta_{\nu,\lambda}^{\delta,\mu}$

The purpose of this section is to establish some series relations for the extended bivariate series zeta function $\zeta_{\nu,\lambda}^{\delta,\mu}$. First, based on the two forms of Taylor's theorem for the deduction of addition and multiplication theorems for the confluent hypergeometric function (cf.[13,p.63, Equations(2.8.8) and (2.8.9)] or [28,p.21-22]):

$$f(x+y) = \sum_{m=0}^{\infty} f^{(m)}(x) \frac{y^m}{m!}, \tag{5.1}$$

and

$$f(xy) = \sum_{m=0}^{\infty} f^{(m)}(x) \frac{[(y-1)x]^m}{m!}, \tag{5.2}$$

where $|y| < \rho$, ρ being the radius of convergence of the analytic function $f(x)$, we aim to discuss certain addition and multiplication theorems of the extended bivariate zeta function $\zeta_{\nu,\lambda}^{\delta,\mu}$.

Theorem 5.1. *Let $|\omega| < 1$. Then*

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x+\omega;z,a;p) = \sum_{k=0}^{\infty} \frac{(\delta)_k(\mu)_k}{(\nu)_k} \zeta_{\nu+k,\lambda}^{\delta+k,\mu+k}(x+\omega;z,a+\lambda k;p) \frac{\omega^k}{k!}, \tag{5.3}$$

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x\omega;z,a;p) = \sum_{k=0}^{\infty} \frac{(\delta)_k(\mu)_k}{(\nu)_k} \zeta_{\nu+k,\lambda}^{\delta+k,\mu+k}(x(\omega-1);z,a+\lambda k;p) \frac{x^k}{k!}, \tag{5.4}$$

Proof. The proof is a direct application of the formulas (5.1),(5.2), and the first result of Theorem 4.1. □

Next, we derive the Taylor expansion of $\zeta_{\nu,\lambda}^{\delta,\mu}$ in the fourth variable a.

Theorem 5.2. *Let* $|\omega| < \Re(a)$. *Then*

$$\sum_{k=0}^{\infty} \zeta_{\nu,\lambda}^{\delta,\mu}(x; z, a+k+\omega; p) y^k = \sum_{k=0}^{\infty} (-1)^k \Phi(y, z+k, a) \times \zeta_{\mu,\nu}^{\delta,\lambda}(x, -k, \omega) \frac{(z)_k}{k!}. \tag{5.5}$$

Proof. We have

(5.6)
$$\sum_{k=0}^{\infty} \zeta_{\nu,\lambda}^{\delta,\mu}(x; z, a+k+\omega; p) y^k = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{B(\mu+m, \nu-\mu; p)}{B(\mu, \nu-\mu)} \frac{(\delta)_m x^m y^k}{m!} (a+k)^{-z} \left(1 + \frac{\omega+\lambda m}{a+k}\right)^{-z}.$$

The result now follows from the binomial expansion and the definitions (1.3) and (2.1). □

Equation (5.5) gives several known and new series expansions as particular cases. For instance, because of the relation (2.3), we find from (5.5) that

$$\Phi(y, z, a+\omega) = \sum_{k=0}^{\infty} (z)_k \Phi(y, z+k, a) \frac{\omega^k}{k!}, (z \neq 1, |\omega| < |a), \tag{5.7}$$

which is a known result due to Raina and Chhajed [27,p.93(3.3)]. Moreover according to the relationship (2.2), equation (5.5) yields

$$\zeta(z, a+\omega) = \sum_{k=0}^{\infty} (-1)^k (z)_k \zeta(z+k, a) \frac{\omega^k}{k!}. \tag{5.8}$$

Note that, formula (5.8) is a known result due to Kanemitsu et al. [16,p.5(2.6)]. Furthermore, if in (5.7) we let $y = e^{2\pi i\alpha}$(in conjunction with (1.4)), formula (5.7) reduces to a known power series expansion due to Klusch [18]:

$$\phi(\alpha, a+\omega, z) = \sum_{k=0}^{\infty} (-1)^k (z)_k \phi(\alpha, a, z+k) \omega^k, |\omega| < a. \tag{5.9}$$

Another expansion function for $\zeta_{\nu,\lambda}^{*(\delta,\mu)}$ can be derived by using the result [20,p.374, Exercise 9.4(7)]:

$$ {}_2F_1\left[a, a+\frac{1}{2}; \frac{1}{2}; x\right] = \frac{1}{2}\left(1+\sqrt{x}\right)^{-2a} + \frac{1}{2}\left(1-\sqrt{x}\right)^{-2a}. \tag{5.10}$$

Theorem 5.3. *Let $\mu \geq 1, Re(a) > 0, |x| < 1, |y| < 1$ and $|\omega| < |a|$. Then*

$$\sum_{k=0}^{\infty} \binom{z+2k-1}{2k} \zeta_{\nu,\lambda}^{\delta,\mu}(x; z+2k, a; p)\omega^{2k}$$

$$= \frac{1}{2}\left[\zeta_{\nu,\lambda}^{\delta,\mu}(x; z, a+\omega; p) + \zeta_{\nu,\lambda}^{\delta,\mu}(x; z, a-\omega; p)\right]. \tag{5.11}$$

*Proof.*We have

$$\sum_{k=0}^{\infty} \binom{z+2k-1}{2k} \zeta_{\nu,\lambda}^{\delta,\mu}(x; z+2k, a; p)\omega^{2k}$$

$$= \sum_{m=0}^{\infty} \frac{B(\mu+m, \nu-\mu; p)}{B(\mu, \nu-\mu)} \frac{(\delta)_m x^m}{m!(a+\lambda m)^z} \sum_{k=0}^{\infty} \frac{(z)_{2k}\omega^{2k}}{(2k)!(a+\lambda m)^{2k}}. \tag{5.12}$$

By applying the formula (5.10) to the last summation on the right-hand side of the equation (5.12), we led to the result (5.11). □

Next, we derive a series expansion for the extended zeta function $\zeta_{\nu,\lambda}^{\delta,\mu}$ involving Apple's hypergeometric function of two variables F_2 defined by the series (see e.g. [29]):

$$F_2\left[a, b_1, b_2; c_1, c_2; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n x^m y^n}{(c_1)_m(c_2)_n m!n!}, \tag{5.13}$$

$$max\{|x|, |y|\} < 1.$$

Theorem 5.4. *Let $|b| < \Re(a)$ and $\lambda \neq 0$. Then*

$$\sum_{k=0}^{\infty} (z)_k \zeta_{\nu,\lambda}^{\delta,\mu}(x; z+k, a+b; p)\frac{\omega^k}{k!} = \sum_{m=0}^{\infty} \frac{B(\mu+m, \nu-\mu; p)}{B(\mu, \nu-\mu)} \frac{(\delta)_m x^m}{m!}$$

$$\times F_2\left[z, \nu, 1; z, 1; \frac{-\lambda m}{a+b}, \frac{\omega}{a+b}\right](a+b)^{-z}. \tag{5.14}$$

Proof. Since

$$(a+b+\lambda m)^{-(z+k)} = (a+b)^{-(z+k)} \left(1 + \frac{\lambda m}{a+b}\right)^{-(z+k)},$$

it follows that

$$\sum_{k=0}^{\infty} (z)_k \zeta_{\nu,\lambda}^{\delta,\mu}(x; z+k, a+b; p)\frac{\omega^k}{k!}$$

$$= \sum_{m=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m}{m!(a+b)^z}$$

$$\times \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(z)_{k+n}(\nu)_k}{k!n!(z)_k} \times \left(\frac{\omega}{a+b}\right)^k \left(\frac{-\lambda m}{a+b}\right)^n. \tag{5.15}$$

The result (5.14) now follows from the definition (5.13). □

Indeed, for $x = p = b = 0$ equation (5.14) reduces to the well-known result of Ramanujan (see [26] or [24,p.396(6)]):

$$\zeta(z, a-\omega) = \sum_{k=0}^{\infty} \frac{(z)_k}{k!} \zeta(z+k, a)\omega^n (|\omega| < |a|, z \neq 1).$$

Moreover, we give a representation of the extended zeta function $\zeta_{\nu,\lambda}^{\delta,\mu}$ in terms of Laguerre polynomials. We start by recalling the useful identity used in [23]

$$e^{\left(\frac{-p}{t(1-t)}\right)} = e^{-2p} \sum_{m,n=0}^{\infty} L(p)_m L(p)_n t^{m+1}(1-t)^{n+1}; |t| < 1. \tag{5.16}$$

Theorem 5.5. *Let* $\{\delta,\mu,\nu\} \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda \in \mathbb{C} \setminus \{0\}; a \in \mathbb{C} \setminus \{-(\lambda m)\}, \Re(p) \geq 0.$ *Then*

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p) = \frac{e^{-2p}}{B(\mu,\nu-\mu)} \sum_{m=0}^{\infty}\sum_{s,r=0}^{\infty} \frac{B(\mu+\delta+m+1,\nu-\mu+r+1)}{m!(a+\lambda m)^z}$$

$$\times (\delta)_s L_s(p) L_r(p) x^m. \tag{5.17}$$

Proof. Using (5.16) in (3.8), employing the series expansion of the zeta function ζ_λ^δ and interchange the order of integration and summation, we obtain the result (5.17). □

According to the definition of beta function (see Eq. (1.9)), the assertion (5.17) can be rewritten in the following alternative form:

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)$$

$$= \frac{e^{-2p}\mu(\nu-\mu)}{\nu} \sum_{s,r=0}^{\infty} \frac{(\mu+1)_s(\nu-\mu+1)_r L_s(p)L_r(p)}{(\nu+1)_{s+r}} \zeta_{\nu+s+r+1,\lambda}^{\delta,\mu+s+1}(x;z,a). \tag{5.18}$$

Now, we turn to derive two generating functions for the function $\zeta_{\nu,\lambda}^{\delta,\mu}$.

Theorem 5.6. *Let* $|\omega| < 1, \Re(p) \geq 0$. *Then*

$$\sum_{n=0}^{\infty} (\delta)_n \zeta_{\nu,\lambda}^{\delta+n,\mu}(x;z,a;p) \frac{\omega^n}{n!} = (1-\omega)^{-\delta} \zeta_{\nu,\lambda}^{\delta,\mu} \left(\frac{x}{1-\omega}\ z, a; p \right), \tag{5.19}$$

$$\sum_{n=0}^{\infty} (z)_n \zeta_{\nu,\lambda}^{\delta,\mu}(x;z+n,a;p) \frac{\omega^n}{n!} = \zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a-\omega;p). \tag{5.20}$$

Proof. We have

$$\sum_{n=0}^{\infty} (\delta)_n \zeta_{\nu,\lambda}^{\delta+n,\mu}(x;z,a;p) \frac{\omega^n}{n!}$$
$$= \sum_{m=0}^{\infty} (\delta)_m \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)} \left\{ \sum_{n=0}^{\infty} (\delta+m)_n \frac{\omega^n}{n!} \right\} \frac{x^m}{m!(a+\lambda m)^z}.$$

Now, on applying the result

$$(1-t)^{-\lambda} = \sum_{m=0}^{\infty} (\lambda)_m \frac{t^m}{m!} (|t| < 1),$$

we obtain the generating function (5.19). Again, starting from the right-hand side of the assertion (5.20), we can state that

$$\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a-\omega;p) = \sum_{m=0}^{\infty} \frac{B(\mu+m,\nu-\mu;p)}{B(\mu,\nu-\mu)} \frac{(\delta)_m x^m}{m!} \left(1 - \frac{\omega}{a+\lambda m}\right)^{-z} (a+\lambda m)^{-z}.$$

Now, by exploiting the same procedure leading to (5.19) we obtain the assertion (5.20). □

6. Probability Distribution of $\zeta_{\nu,\lambda}^{\delta,\mu}$

As in the theory of probability, we introduce the following definition.

Definition 6.1. A random variable ξ is said to be extended generalized Hurwitz distributed if its probability density function is given by

$$f_\xi(a) = \begin{cases} \frac{z\zeta_{\nu,\lambda}^{\delta,\mu}(x;z+1,a;p)}{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,1;p)}, & (a \geq 1) \\ 0 & (\text{otherwise}), \end{cases} \tag{6.1}$$

where it is tacitly assumed that the arguments x, z and p and the parameters δ, μ, ν, and λ are fixed and suitably constrained so that the probability density function $f_\xi(a)$ remains nonnegative.

Theorem 6.1. *Suppose that ξ is a continuous random variable with its probability density function defined by (6.1). Then the moment generating function $M(t)$ of the random variable ξ is given by*

$$M(t) = \mathbb{E}_z\left[e^{t\xi}\right] = \sum_{n=0}^{\infty} \mathbb{E}_z\left[\xi^n\right]\frac{t^n}{n!}$$

with the moment $\mathbb{E}_z\left[\xi^n\right]$ of order n given by

$$\mathbb{E}_z\left[\xi^n\right] = \sum_{k=0}^{n} \frac{n!}{(n-k)!}\frac{\Gamma(z-k)}{\Gamma(z)}\frac{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z-k,1;p)}{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,1;p)}. \tag{6.2}$$

Proof. The assertion in (6.1) can be derived easily by using the exponential series for $e^{t\xi}$. On the other hand, since

$$\frac{d}{da}\left\{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)\right\} = -z\zeta_{\nu,\lambda}^{\delta,\mu}(x;z+1,a;p), \tag{6.3}$$

which follows readily from the assertion (4.3), if we make use of integration by parts, we find from the definition of the moment $E_z[\xi^n]$ that

$$\begin{aligned}
E_z[\xi^n] = \int_1^\infty a^n f_\xi(a)da &= \frac{z}{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,1;p)}\int_1^\infty a^n\zeta_{\nu,\lambda}^{\delta,\mu}(x;z+1,a;p)\,da \\
&= -\frac{1}{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,1;p)}\int_1^\infty a^n\frac{d}{da}\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)\,da \\
&= -\frac{a^n\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)}{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,1;p)}\Bigg|_{a=1}^{\infty} + \frac{n}{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,1;p)} \\
&\times\int_1^\infty a^{n-1}\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)\,da \\
&= 1 - \lim_{a\to\infty}\left\{\frac{a^n\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)}{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,1;p)}\right\} + \frac{n}{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,1;p)} \\
&\times\int_1^\infty a^{n-1}\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)\,da
\end{aligned}$$

$$= 1 + \frac{n}{\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,1;p)}\times\int_1^\infty a^{n-1}\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)\,da \qquad (n\in\mathbb{N}), \tag{6.4}$$

where, in addition to the derivative property (6.3), we have used the following limit formula (see Eq. (3.6)):

$$\lim_{a\to\infty}\left\{a^n\zeta_{\nu,\lambda}^{\delta,\mu}(x;z,a;p)\right\}$$

$$= \lim_{a\to\infty} \left\{ \frac{a^n}{\Gamma(z)} \int_1^\infty e^{-at} t^{z-1} F_p \left(\delta, \mu; \nu; xe^{-\lambda t}\right) dt \right\}$$

$$= \frac{1}{\Gamma(z)} \int_1^\infty \lim_{a\to\infty} \left\{a^n e^{-at}\right\} t^{z-1} F_p \left(\delta, \mu; \nu; xe^{-\lambda t}\right) dt = 0. \tag{6.5}$$

Consequently, we have the following reduction formula for $E_z[\xi^n]$:

$$E_z[\xi^n] = 1 + \frac{\zeta_{\nu,\lambda}^{\delta,\mu}(x; z-1, 1; p)}{\zeta_{\nu,\lambda}^{\delta,\mu}(x; z, 1; p)} \frac{n}{z-1} E_{z-1}[\xi^{n-1}]. \tag{6.6}$$

By iterating the recurrence (6.4), we arrive at the desired result (6.2) asserted by Theorem 6.1. Note that, a special case of Theorem 4.1 when $\lambda = 0$ was considered by Parmar and Raina [25]. □

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