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Some Results on Generalized Asymptotically Nonexpansive Mappings in p-Hadamard Spaces

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Abstract. In this paper, we study the fixed point property for generalized asymptotically nonexpansive mappings in the setting of p-Hadamard spaces, with $p \geq 2$. We prove the strong convergence of the sequence generated by the modified two-step iterative sequence for finding a fixed point of a generalized asymptotically nonexpansive mapping in p-Hadamard spaces.

1. Introduction

Let (X, d) be a metric space and $x, y \in X$. A *geodesic* joining x to y is a map γ from the closed interval $[0, d(x, y)] \subset \mathbb{R}$ to X such that $\gamma(0) = x, \gamma(d(x, y)) = y$ and $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, d(x, y)]$. The image of γ is called a *geodesic segment* joining x and y. When it is unique, this geodesic segment is denoted by $[x, y]$ and we write $\alpha x \oplus (1 - \alpha)y$ for the unique point z in the geodesic segment joining from x to y such that $d(x, z) = (1-\alpha)d(x, y)$ and $d(y, z) = \alpha d(x, y)$ for $\alpha \in [0, 1]$. The space (X, d) is said to be a *geodesic metric space* [2] if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X and a geodesic segment between each pair of vertices. A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in X is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Let $\triangle$ be a geodesic triangle in X and $\overline{\triangle}$ be a comparison triangle for $\triangle$ in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for any $x, y \in \triangle$ and their

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comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$, the following holds,

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

A geodesic metric space (X,d) is said to be a CAT(0) *space* if all geodesic triangles satisfy the CAT(0) inequality. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space; see more details in [2]. Other examples include Euclidean spaces, Hilbert spaces, the Hilbert ball [7], $\mathbb{R}$-trees [17], and many others.

In 2017, Khamsi and Shukri [9] introduced the concept of $\mathrm{CAT}_p(0)$ spaces based on the idea that comparison triangles belong to a general Banach space instead of the Euclidean plane as follows:

Definition 1.1. Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space. A geodesic metric space (X,d) is said to be a $\mathrm{CAT}_{\mathbb{E}}(0)$ *space* if for any geodesic triangle $\triangle$ in X, there exists a comparison triangle $\overline{\triangle}$ in $\mathbb{E}$ such that the comparison axiom is satisfied, i.e., for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$, we have

$$d(x,y) \leq \|\bar{x} - \bar{y}\|.$$

If $\mathbb{E} = l_p$, for $p \geq 1$, we say X is a $\mathrm{CAT}_p(0)$ space.

It is obvious that $\mathrm{CAT}_2(0)$ space is exactly the classical CAT(0) space, which has been extensively studied.

Let x, y, z be in a $\mathrm{CAT}_p(0)$ space X, with $p \geq 2$, and $\frac{x \oplus y}{2}$ is the midpoint of the geodesic $[x,y]$. Then the comparison axiom implies

$$d\left(z, \frac{x \oplus y}{2}\right)^p \leq \frac{1}{2} d(z,x)^p + \frac{1}{2} d(z,y)^p - \frac{1}{2^p} d(x,y)^p. \tag{1.1}$$

This inequality is the (CN_p) inequality of Khamsi and Shukri [9]. Note that the (CN_p) inequality coincides with the classical (CN) inequality [4] if $p = 2$. Below are some strong inequalities in a $\mathrm{CAT}_p(0)$ space.

Lemma 1.2.([1, 5]) *Let (X,d) be a $\mathrm{CAT}_p(0)$ space, with $p \geq 2$. Then, for any x, y, z in X and $\alpha \in [0,1]$, we have*

(i) $d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z,x) + (1-\alpha) d(z,y)$;

(ii) $d(z, \alpha x \oplus (1-\alpha)y)^p \leq \alpha d(z,x)^p + (1-\alpha) d(z,y)^p - \frac{1}{2^{p-1}} \alpha(1-\alpha) d(x,y)^p$.

Let C be a nonempty subset of a $\mathrm{CAT}_p(0)$ space (X,d). A subset C of X is said to be *convex* if C includes every geodesic segment joining any two of its points, that is, for any $x, y \in C$, we have $[x,y] \subset C$.

A complete $\mathrm{CAT}_p(0)$ space is called a *p-Hadamard space*. Throughout our work, we mainly focus on p-Hadamard spaces for $p \geq 2$.

Let T be a mapping of C into itself. An element $x \in C$ is called a *fixed point* of T if $x = Tx$. The set of all fixed points of T is denoted by $F(T)$, that is,

$F(T) = \{x \in C : x = Tx\}$. A sequence $\{x_n\}$ in C is called *approximate fixed point sequence* for T (AFPS in short) if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

The famous fixed point theorem for nonexpansive mappings in Banach spaces have first studied by Browder [3]and Göhde [8] in 1965 as follows:

Theorem 1.3. *Let X be a uniformly convex Banach space and C be a nonempty bounded closed convex subset of X. Then, each nonexpansive mapping $T : C \to C$ has a fixed point.*

In 1972, Geobel and Kirk [6] extended their result to asymptotically nonexpansive mappings. Later in 2004, Kirk [10] obtained a similar result for complete CAT(0) spaces. In 2013, Phuengrattana and Suantai [12] extended those result to generalized asymptotically nonexpansive mappings and to complete uniformly convex metric spaces as follows:

Theorem 1.4. *Let (X, d) be a complete uniformly convex metric space. Let C be a nonempty bounded closed convex subset of X and $T : C \to C$ be a generalized asymptotically nonexpansive mapping whose graph $G(T) = \{(x, y) \in C \times C : y = Tx\}$ is closed. Then, T has a fixed point.*

In this work, we extend some known existence and convergence results for generalized asymptotically nonexpansive mappings in Hadamard spaces to the case of p-Hadamard spaces, for $p \geq 2$.

2. Preliminaries

Let C be a nonempty subset of a $\mathrm{CAT}_p(0)$ space (X, d) and $T : C \to C$ be a mapping. A mapping T is said to be *generalized asymptotically nonexpansive* [12, 15] if there exist sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} s_n = 0$ such that

$$d\left(T^n x, T^n y\right) \leq k_n d(x, y) + s_n,$$

for all $x, y \in C$ and $n \in \mathbb{N}$. In the case of $s_n = 0$ for all $n \in \mathbb{N}$, a mapping T is called *an asymptotically nonexpansive mapping.* In particular, if $k_n = 1$ and $s_n = 0$ for all $n \in \mathbb{N}$, a mapping T reduce to a *nonexpansive mapping.*

Remark 2.1. If T is a generalized asymptotically nonexpansive mapping, it is know that $F(T)$ is not necessarily closed; see [13].

Recall that $\phi : X \to [0, \infty)$ is called a *type function* if there exists a bounded sequence $\{x_n\}$ in X such that

$$\phi(x) = \limsup_{n\to\infty} d(x_n, x),$$

for any $x \in X$. A sequence $\{z_n\}$ in X is said to be a *minimizing sequence* of ϕ whenever

$$\lim_{n\to\infty} \phi(z_n) = \inf\{\phi(x) : x \in X\}.$$

Lemma 2.2.([9]) *Let (X, d) be a p-Hadamard space, with $p \geq 2$. Let C be a nonempty bounded closed convex subset of X and ϕ be a type function defined on C. Then any minimizing sequence of ϕ is convergent and its limit z is the unique minimum point of ϕ, i.e., $\phi(z) = \inf\{\phi(x) : x \in C\}$.*

The following results are needed for proving our convergence results.

Definition 2.3.([11]) A mapping $T : C \to C$ is said to be *semi-compact* if C is closed and each bounded AFPS for T in C has a convergent subsequence.

Definition 2.4.([14]) A mapping $T : C \to C$ is said to satisfy the *condition (I)* if there exists a nondecreasing function $\rho : [0, \infty) \to [0, \infty)$ with $\rho(0) = 0$ and $\rho(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Tx) = \rho(\text{dist}(x, F(T)))$$

for all $x \in C$, where $\text{dist}(x, F(T)) = \inf\{d(x, z) : z \in F(T)\}$.

Definition 2.5.([13]) Let $\{x_n\}$ be a sequence in a metric space (X, d) and $F \subset X$. We say that $\{x_n\}$ is of *monotone type (I) with respect to F* if there exists sequences $\{\delta_n\}$ and $\{\gamma_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, and

$$d(x_{n+1}, z) \leq (1 + \delta_n) d(x_n, z) + \gamma_n$$

for all $n \in \mathbb{N}$ and $z \in F$.

Lemma 2.6.([13]) *Let $\{x_n\}$ be a sequence in a complete metric space (X, d) and $F \subset X$. If $\{x_n\}$ is of monotone type (I) with respect to F and $\liminf_{n\to\infty}$ dist$(x_n, F) = 0$, then $\lim_{n\to\infty} x_n = z$ for some $z \in X$ satisfying dist(x_n, F). In particular, if F is closed, then $z \in F$.*

Lemma 2.7.([18]) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + b_n) a_n + c_n, \; n \in \mathbb{N},$$

where $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

3. Existence Theorems

In this section, we study the existence theorems for a generalized asymptotically nonexpansive mapping in p-Hadamard spaces.

We now state and prove our existence results.

Theorem 3.1. *Let (X, d) be a p-Hadamard space, with $p \geq 2$. Let C be a nonempty bounded closed convex subset of X and $T : C \to C$ be a generalized asymptotically nonexpansive mapping whose graph $G(T) = \{(x, y) \in C \times C : y = Tx\}$ is closed. Then, T has a fixed point.*

Proof. Fix $x \in C$. Consider the type function ϕ generated by the bounded sequence $\{T^n x\}$. Let z be the minimum point of ϕ which exists by using Lemma 2.2. Therefore,

$$d\left(T^{n+m}x, T^m z\right) \leq k_m d(T^n x, z) + s_m,$$

for any $n, m \in \mathbb{N}$. Taking $n \to \infty$, we get

$$\phi(T^m z) \leq k_m \phi(z) + s_m = k_m \inf\{\phi(x) : x \in C\} + s_m,$$

for any $m \in \mathbb{N}$. Taking $m \to \infty$, we get

$$\lim_{m\to\infty} \phi(T^m z) \leq \inf\{\phi(x) : x \in C\}.$$

Since z is the minimum point of ϕ, it implies that $\lim_{m\to\infty} \phi(T^m z) = \inf\{\phi(x) : x \in C\}$. Thus, $\{T^m z\}$ is a minimizing sequence of ϕ. By Lemma 2.2, we obtain that $T^m z \to z$ as $m \to \infty$, and so $T(T^m z) = T^{m+1} z \to z$ as $m \to \infty$. By the closedness of $G(T)$, we have $Tz = z$. This completes the proof. □

Theorem 3.2. *Let (X, d) be a p-Hadamard space, with $p \geq 2$. Let C be a nonempty bounded closed convex subset of X and $T : C \to C$ be a generalized asymptotically nonexpansive mapping whose graph $G(T) = \{(x, y) \in C \times C : y = Tx\}$ is closed. Then, $F(T)$ is nonempty closed and convex.*

Proof. By Theorem 3.1, $F(T)$ is nonempty. To show that $F(T)$ is closed, we let $\{x_n\}$ be a sequence in $F(T)$ such that $\lim_{n\to\infty} x_n = x$. By the definition of T, we have

$$d(T^n x, x) \leq d(T^n x, x_n) + d(x_n, x) \leq (1 + k_n) d(x_n, x) + s_n.$$

Since $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$, we get $\lim_{n\to\infty} d(T^n x, x) = 0$. That is $T^n x \to x$ as $n \to \infty$, and so $T(T^n x) = T^{n+1} x \to x$ as $n \to \infty$. By the closedness of $G(T)$, we have $Tx = x$. Hence $x \in F(T)$ so that $F(T)$ is closed.

In order to prove $F(T)$ is convex, it is enough to prove that $\frac{x \oplus y}{2} \in F(T)$ whenever $x, y \in F(T)$ with $x \neq y$. Set $z = \frac{x\oplus y}{2}$. By the (CN_p) inequality and the definition of T, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(T^n z, z)^p &= d\left(T^n z, \frac{x \oplus y}{2}\right)^p \\
&\leq \frac{1}{2} d(T^n z, x)^p + \frac{1}{2} d(T^n z, y)^p - \frac{1}{2^p} d(x, y)^p \\
&= \frac{1}{2} d(T^n z, T^n x)^p + \frac{1}{2} d(T^n z, T^n y)^p - \frac{1}{2^p} d(x, y)^p \\
&\leq \frac{1}{2} \left(k_n d(z, x) + s_n\right)^p + \frac{1}{2} \left(k_n d(z, y) + s_n\right)^p - \frac{1}{2^p} d(x, y)^p.
\end{aligned}$$

Since $z = \frac{x\oplus y}{2}$, we get that $d(z, x) = \frac{1}{2} d(x, y)$ and $d(y, x) = \frac{1}{2} d(x, y)$. So, we have

$$d(T^n z, z)^p \leq \left(\frac{k_n}{2} d(x, y) + s_n\right)^p - \frac{1}{2^p} d(x, y)^p.$$

By $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$, we get $\lim_{n\to\infty} d(T^n z, z)^p = 0$. This implies that $T^n z \to z$ as $n \to \infty$, and so $T(T^n z) = T^{n+1} z \to z$ as $n \to \infty$. From the closedness of $G(T)$, we have $Tz = z$. Therefore, $F(T)$ is convex. This completes the proof. □

Next, we show that the existence of a fixed point of a generalized asymptotically nonexpansive mapping in a p-Hadamard space is equivalent to the existence of a bounded orbit at a point.

Theorem 3.3. *Let (X, d) be a p-Hadamard space, with $p \geq 2$. Let C be a nonempty closed convex subset of X and $T : C \to C$ be a generalized asymptotically nonexpansive mapping whose graph $G(T) = \{(x, y) \in C \times C : y = Tx\}$ is closed. Then $F(T) \neq \emptyset$ if and only if there exists an $x \in C$ such that $\{T^n x\}$ is bounded.*

Proof. The necessity is obvious. Conversely, assume that x is an element in C such that $\{T^n x\}$ is bounded. Consider the type function ϕ generated by $\{T^n x\}$. By the same step of the proof as in Theorem 3.1, we can conclude that $F(T) \neq \emptyset$. This completes the proof. □

Remark 3.4.

(i) If T is generalized asymptotically nonexpansive, $F(T)$ is not necessarily closed. However, if $G(T)$ is also closed, Theorem 3.2 guarantee that $F(T)$ is always closed.

(ii) If T is continuous, then $G(T)$ is always closed. Therefore, Theorems 3.1, 3.2 and 3.3 are obtained for a class of continuous generalized asymptotically nonexpansive mappings.

4. Convergence Theorems

In this section, we study the strong convergence theorems for a generalized asymptotically nonexpansive mapping by the modified two-step iterative sequence for finding fixed points of such mapping in p-Hadamard spaces. We now introduce *the modified two-step iterative sequence* [16] as below:

Let C be a nonempty closed convex subset of a p-Hadamard space (X, d) and $T : C \to C$ be a generalized asymptotically nonexpansive mapping. We generate the sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$\begin{cases} y_n = \beta_n x_n \oplus (1 - \beta_n) T^n x_n, \\ x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n) T^n y_n, \ n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

Now we prove the strong convergence results.

Lemma 4.1. *Let (X, d) be a p-Hadamard space, with $p \geq 2$. Let C be a nonempty bounded closed convex subset of X and $T : C \to C$ be a uniformly continuous* *generalized asymptotically nonexpansive mapping with sequences* $\{k_n\} \subset [1, \infty)$ *and* $\{s_n\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $\sum_{n=1}^{\infty} s_n < \infty$. *Let* $x_1 \in C$ *and* $\{x_n\}$ *be a sequence in* C *defined by* (4.1) *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$ *such that* $0 < a \leq \alpha_n, \beta_n \leq b < 1$. *Then, we have the following:*

- *(i) there exit two sequences* $\{\delta_n\}$ *and* $\{\varepsilon_n\}$ *of nonnegative real numbers such that* $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, *and* $d(x_{n+1}, z) \leq (1 + \delta_n)d(x_n, z) + \varepsilon_n$ *for all* $n \in \mathbb{N}$ *and* $z \in F(T)$;
- *(ii)* $\lim_{n\to\infty} d(x_n, z)$ *exists for all* $z \in F(T)$;
- *(iii)* $\{x_n\}$ *is an AFPS for* T.

Proof. (i) : By the uniform continuity of T, we have $G(T)$ is closed. It implies from Theorem 3.1 that $F(T) \neq \emptyset$. Let $z \in F(T)$. Since T is generalized asymptotically nonexpansive, by Lemma 1.2(i), we have

$$\begin{aligned} d(y_n, z) &\leq \beta_n d(x_n, z) + (1 - \beta_n)d(T^n x_n, z) \\ &= \beta_n d(x_n, z) + (1 - \beta_n)d(T^n x_n, T^n z) \\ &\leq \beta_n d(x_n, z) + (1 - \beta_n)(k_n d(x_n, z) + s_n) \\ &\leq (\beta_n + (1 - \beta_n)k_n)d(x_n, z) + s_n \\ &= \left(\frac{\beta_n}{k_n} + (1 - \beta_n)\right) k_n d(x_n, z) + s_n. \end{aligned}$$

Since $0 \leq \frac{\beta_n}{k_n} + (1 - \beta_n) \leq 1$, we obtain

$$d(y_n, z) \leq k_n d(x_n, z) + s_n. \tag{4.2}$$

This implies that

$$\begin{aligned} d(x_{n+1}, z) &\leq \alpha_n d(y_n, z) + (1 - \alpha_n)d(T^n y_n, z) \\ &= \alpha_n d(y_n, z) + (1 - \alpha_n)d(T^n y_n, T^n z) \\ &\leq \alpha_n d(y_n, z) + (1 - \alpha_n)(k_n d(y_n, z) + s_n) \\ &\leq (\alpha_n + (1 - \alpha_n)k_n)d(y_n, z) + s_n \\ &= \left(\frac{\alpha_n}{k_n} + (1 - \alpha_n)\right) k_n d(y_n, z) + s_n. \end{aligned}$$

Since $0 \leq \frac{\beta_n}{k_n} + (1 - \beta_n) \leq 1$, by (4.2), we have

$$\begin{aligned} d(x_{n+1}, z) &\leq k_n d(y_n, z) + s_n \\ &\leq k_n(k_n d(x_n, z) + s_n) + s_n \\ &\leq k_n^2 d(x_n, z) + k_n s_n + s_n \\ &= (1 + (k_n - 1))^2 d(x_n, z) + (1 + (k_n - 1))s_n + s_n \\ &= (1 + 2(k_n - 1) + (k_n - 1)^2)d(x_n, z) + (k_n - 1)s_n + 2s_n \\ &= (1 + \delta_n)d(x_n, z) + \gamma_n, \end{aligned}$$

where $\delta_n = 2(k_n-1)+(k_n-1)^2$ and $\gamma_n = (k_n-1)s_n+2s_n$. Since $\sum_{n=1}^{\infty}(k_n-1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, it follows that $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Hence, we obtain the desired result.

(ii) : By (i) and Lemma 2.7, we obtain that $\lim_{n\to\infty} d(x_n, z)$ exists.

(iii) : By Lemma 1.2(ii) and (4.2), we have

$$\begin{aligned} d(x_{n+1}, z)^p &\le \alpha_n d(y_n, z)^p + (1-\alpha_n) d(T^n y_n, z)^p - \frac{\alpha_n(1-\alpha_n)}{2^{p-1}} d(T^n y_n, y_n)^p \\ &\le \alpha_n \left(\beta_n d(x_n, z)^p + (1-\beta_n) d(T^n x_n, z)^p - \frac{\beta_n(1-\beta_n)}{2^{p-1}} d(T^n x_n, x_n)^p \right) \\ &\quad + (1-\alpha_n)(k_n d(y_n, z) + s_n)^p - \frac{\alpha_n(1-\alpha_n)}{2^{p-1}} d(T^n y_n, y_n)^p \\ &\le \alpha_n\beta_n d(x_n, z)^p + \alpha_n(1-\beta_n)(k_n d(x_n, z) + s_n)^p \\ &\quad + (1-\alpha_n)(k_n^2 d(x_n, z) + k_n s_n + s_n)^p - \frac{\alpha_n\beta_n(1-\beta_n)}{2^{p-1}} d(T^n x_n, x_n)^p \\ &\quad - \frac{\alpha_n(1-\alpha_n)}{2^{p-1}} d(T^n y_n, y_n)^p. \end{aligned} \tag{4.3}$$

Since $k_n \ge 1$ and $s_n \ge 0$, we have

$$d(x_n, z) \le k_n d(x_n, z) + s_n \le k_n^2 d(x_n, z) + k_n s_n + s_n.$$

Then, by (4.3), we have

$$\begin{aligned} d(x_{n+1}, z)^p \le (k_n^2 d(x_n, z) + k_n s_n + s_n)^p &- \frac{\alpha_n\beta_n(1-\beta_n)}{2^{p-1}} d(T^n x_n, x_n)^p \\ &- \frac{\alpha_n(1-\alpha_n)}{2^{p-1}} d(T^n y_n, y_n)^p. \end{aligned}$$

This implies that

$$d(T^n x_n, x_n)^p \le \frac{2^{p-1}}{a^2(1-b)} \left((k_n^2 d(x_n, z) + k_n s_n + s_n)^p - d(x_{n+1}, z)^p \right),$$

and

$$d(T^n y_n, y_n)^p \le \frac{2^{p-1}}{a(1-b)} \left((k_n^2 d(x_n, z) + k_n s_n + s_n)^p - d(x_{n+1}, z)^p \right).$$

By $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} s_n = 0$ and $\lim_{n\to\infty} d(x_n, z)$ exists, we conclude that

$$\lim_{n\to\infty} d(T^n x_n, x_n) = 0, \tag{4.4}$$

and

$$\lim_{n\to\infty} d(T^n y_n, y_n) = 0. \tag{4.5}$$

By (4.4) and the uniform continuity of T, we have

$$\lim_{n\to\infty} d(T^{n+1}x_n, Tx_n) = 0. \tag{4.6}$$

By the definitions of x_{n+1} and y_n, we obtain

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n) \\ &\quad + d(T^{n+1}x_n, Tx_n) \\ &\le (1+k_{n+1})d(x_n, x_{n+1}) + s_{n+1} + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(T^{n+1}x_n, Tx_n) \\ &\le (1+k_{n+1})(\alpha_n d(x_n, y_n) + (1-\alpha_n)d(x_n, T^n y_n)) + s_{n+1} \\ &\quad + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_n, Tx_n) \\ &\le (1+k_{n+1})(\alpha_n d(x_n, y_n) + (1-\alpha_n)(d(x_n, y_n) + d(y_n, T^n y_n))) \\ &\quad + s_{n+1} + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_n, Tx_n) \\ &\le (1+k_{n+1})(d(x_n, y_n) + (1-\alpha_n)d(y_n, T^n y_n)) + s_{n+1} \\ &\quad + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_n, Tx_n) \\ &\le (1+k_{n+1})((1-\beta_n)d(x_n, T^n x_n) + (1-\alpha_n)d(y_n, T^n y_n)) \\ &\quad + s_{n+1} + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_n, Tx_n). \end{aligned}$$

Since $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} s_n = 0$, by (4.4), (4.5), and (4.6), we conclude that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Hence, we obtain the desired result. □

Now, we prove a strong convergence theorem for a generalized asymptotically nonexpansive semi-compact mapping in p-Hadamard spaces.

Theorem 4.2. *Suppose that X, C, T, $\{x_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are as in Lemma 4.1. If T^m is semi-compact for some $m \in \mathbb{N}$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 4.1(iii), $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Fix $m \in \mathbb{N}$, we have

$$d(x_n, T^m x_n) \le d(x_n, Tx_n) + d(Tx_n, T^2 x_n) + \cdots + d(T^{m-1}x_n, T^m x_n).$$

Since T is uniformly continuous, we have

$$\lim_{n\to\infty} d(x_n, T^m x_n) = 0.$$

That is, $\{x_n\}$ is an AFPS for T^m. By the semi-compactness of T^m, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in C$ such that $\lim_{k\to\infty} x_{n_k} = z$. Again, by the uniform continuity of T, we have

$$d(Tz, z) \le d(Tz, Tx_{n_k}) + d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, z) \to 0 \text{ as } k \to \infty.$$

Then $z \in F(T)$. By Lemma 4.1(ii), $\lim_{n\to\infty} d(x_n, z)$ exists, thus z is the strong limit of the sequence $\{x_n\}$ itself. This completes the proof. □

Finally, we prove a strong convergence theorem for a generalized asymptotically nonexpansive mapping which satisfies condition (I) in p-Hadamard spaces.

Theorem 4.3. *Suppose that X, C, T, $\{x_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are as in Lemma 4.1. If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By condition (I), there exists a nondecreasing function $\rho : [0, \infty) \to [0, \infty)$ with $\rho(0) = 0$ and $\rho(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x_n, Tx_n) = \rho(\text{dist}(x_n, F(T))).$$

It implies by Lemma 4.1(iii) that

$$\lim_{n\to\infty} \rho(\text{dist}(x_n, F(T))) = 0.$$

Then we have

$$\lim_{n\to\infty} \text{dist}(x_n, F(T)) = 0.$$

By Lemma 4.1(i), we obtain that the sequence $\{x_n\}$ is of monotone type (I) with respect to $F(T)$. This implies by Lemma 2.6 that the sequence $\{x_n\}$ converges strongly to a point $z \in F(T)$. This completes the proof. □

Remark 4.4. Any complete CAT(0) space is a 2-Hadamard space, therefore the results in this paper can be applied to any complete CAT(0) space.

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