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L^p-boundedness ($1 \leq p \leq \infty$) for Bergman Projection on a Class of Convex Domains of Infinite Type in $\mathbb{C}^2$

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Abstract. The main purpose of this paper is to show that over a large class of bounded domains $\Omega \subset \mathbb{C}^2$, for $1 < p < \infty$, the Bergman projection $\mathcal{P}$ is bounded from $L^p(\Omega, dV)$ to the Bergman space $A^p(\Omega)$; from $L^\infty(\Omega)$ to the holomorphic Bloch space $\mathrm{BlHol}(\Omega)$; and from $L^1(\Omega, P(z,z)dV)$ to the holomorphic Besov space $\mathrm{Besov}(\Omega)$, where $P(\zeta, z)$ is the Bergman kernel for Ω.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^2$ with smooth boundary $b\Omega$. Let ρ be a defining function for Ω so that $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ and $b\Omega = \{z \in \mathbb{C}^2 : \rho(z) = 0\}$, $\nabla\rho \neq 0$ on $b\Omega$. Let $\mathcal{O}(\Omega)$ be the space of functions that are holomorphic in Ω, with the topology of uniform convergence on compact subsets of Ω. For $1 < p < \infty$, let $L^p(\Omega, dV)$ be the standard Lebesgue space over Ω with respect to the Lebesgue volume measure dV of $\mathbb{R}^4$, and let the Bergman space $A^p(\Omega) = L^p(\Omega, dV) \cap \mathcal{O}(\Omega)$. The Bergman projection $\mathcal{P}$ is the orthogonal projection of $L^2(\Omega)$ onto the Bergman space $A^2(\Omega)$. The most important property of the Bergman projection is that there exists a function $P\colon \Omega \times \Omega \to \mathbb{C}$ such that

$$\mathcal{P}[u](z) = \int_\Omega u(\zeta)P(\zeta, z)\, dV(\zeta), \tag{1.1}$$

for all $u \in L^2(\Omega)$, $z \in \Omega$. Here, $P(\zeta, z)$ is the Bergman kernel on Ω, which is holomorphic with respect to $z \in \Omega$, and anti-holomorphic in ζ. In this paper, we

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investigate the $L^p(\Omega)$-boundedness of the projection $\mathcal{P}$. In the recent forty years, there have been many papers focused on studying $L^p(\Omega)$-boundedness (see for example [16, 1, 14, 15, 2]) and its applications in studying commutator operators (see for example [10]), composition operators (see for example [4, 9]). Although there are many results on the $L^p(\Omega)$-boundedness, the case $p = 1$ and the case $p = \infty$ are still open. In this paper, we provide an answer to solve these problems.

Definition 1.1. ([13, p. 478]) A differentiable function u on Ω is said to be a Bloch function if and only if

$$\|u\|_{\mathrm{Bl}(\Omega)} = \sup_{z\in\Omega} (|\rho(z)||u(z)| + |\rho(z)||\nabla u(z)|) < \infty.$$

The space of all Bloch functions defined on Ω is denoted by $\mathrm{Bl}(\Omega)$ and by $\mathrm{BlHol}(\Omega) = \mathrm{Bl}(\Omega) \cap \mathcal{O}(\Omega)$ the space of holomorphic Bloch functions on Ω. We also define $\|u\|_{\mathrm{BlHol}(\Omega)} = \|u\|_{\mathrm{Bl}(\Omega)}$ for all $u \in \mathrm{BlHol}(\Omega)$.

Since $P(z, z) > 0$ for all $z \in \Omega$, $P(z, z)dV(z)$ is a biholomorphically invariant measure of Ω.

Definition 1.2. A function $u \in A^2(\Omega, dV)$ is said to be a Besov function if and only if

$$\|u\|_{\mathrm{Besov}(\Omega)} = \left(\int_\Omega |\nabla^3 u(z)|(-\rho(z))^3 P(z, z)dV(z)\right) < \infty,$$

where $|\nabla^3 u(z)| = \sum_{1\le j+k\le 3} \left|\frac{\partial^{j+k}u}{\partial z_1^j \partial z_2^k}(z)\right|$. The space of all holomorphic Besov functions defined on Ω is denoted by $\mathrm{Besov}(\Omega)$. Here we have an explanation for this definition. Assume that Ω is a smoothly bounded, strongly pseudoconvex domains. The classical Besov space $\mathcal{B}(\Omega)$ is a subspace of $A^2(\Omega, dV)$ in which we equip the semi-norm

$$\|u\|_B = \int_\Omega |\nabla u(z)|(-\rho(z))P(z, z)dV(z) < \infty.$$

Since $\int_\Omega (-\rho(z))^{-1}dV(z) = \infty$, $(\mathcal{B}(\Omega), \|\cdot\|_B)$ consists only constant functions on Ω. In order to make more natural, we use the semi-norm $\|\cdot\|_{\mathrm{Besov}(\Omega)}$ instead of $\|\cdot\|_B$. This idea was used in [13] for strongly pseudoconvex domains.

The main result in this paper is following.

Main Theorem. *Let Ω be a smoothly bounded convex domain in $\mathbb{C}^2$ admitting a type F at all boundary points (see Definition 2.2) and satisfying the condition (B) (see Definition 2.4). Then the Bergman projection is bounded from:*

1. *$L^p(\Omega, dV)$ to $A^p(\Omega, dV)$ for all $1 < p < \infty$.*
2. *$L^\infty(\Omega)$ to BlHol(Ω).*

3. $L^1(\Omega, P(z,z))$ *to Besov*(Ω).

Phong and Stein in [16] established the $L^p \to A^p$ boundedness when Ω is a strongly pseudoconvex domain. Then, this result was generalized to a certain class of convex domains in $\mathbb{C}^2$ (see [1]) and to finite type convex domains in $\mathbb{C}^n$ (see [15]). Even when Ω is the unit ball in $\mathbb{C}^n$, for $n \geq 2$, the Bergman projection $\mathcal{P}$ can not be extended continuously from $L^p(\Omega)$ onto $A^p(\Omega)$ when $p = 1$ or $p = \infty$ (for example, see [20, Section 7.1]). In [14], using Cauchy-Fantappiè integral theory, Ligocka obtained the $L^\infty(\Omega) \to \mathrm{BlHol}(\Omega)$ boundedness on bounded strongly pseudoconvex domains. Recently, in studying Besov spaces on general domains in $\mathbb{C}^n$, Li and Luo (see [13]) have proved the $L^1(\Omega, P(z,z)) \to \mathrm{Besov}(\Omega)$ boundedness also on bounded strongly pseudoconvex domains or convex domains of finite type in $\mathbb{C}^2$.

The structure of the paper is as follows. Section 2 deals with preliminaries for the Bergman projection in terms of Cauchy-Fantappiè forms on convex domains admitting the F-type condition. Section 3 deals with the proof of the Main Theorem.

2. Preliminaries

Let $\Omega \subset \mathbb{C}^2$ be a bounded convex domain with smooth boundary $b\Omega$ with a defining function ρ. By the hypothesis that Ω is convex,

$$\sum_{i,j=1}^{4} \frac{\partial^2 \rho}{\partial x_i \partial x_j}(x) a_i a_j \geq 0,$$

in which $x \in b\Omega$, $z_j = x_{2j-1} + \sqrt{-1} x_{2j}$ and $a \in \mathbb{R}^4$ be a non-zero vector such that $\sum_{j=1}^4 a_j \frac{\partial \rho}{\partial x_j}(x) = 0$ on $b\Omega$. Let us define, for $(\zeta, z) \in b\Omega \times \Omega$:

$$\Phi(\zeta, z) = \sum_{j=1}^{2} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j). \tag{2.1}$$

The convexity of Ω gives

$$\mathrm{Re}\left(\sum_{j=1}^{2} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) \right) \neq 0,$$

so that $\Phi(\zeta, z) \neq 0$ for all $\zeta \in b\Omega, z \in \Omega$. Moreover, the following lemma proved in [17] is a consequence of the definition of $\Phi(\zeta, z)$.

Lemma 2.1. *For any* $P \in b\Omega$*, there are positive constants* δ, c *such that for all boundary points* $\zeta \in b\Omega \cap B(P, \delta)$*, we have*

1. $\Phi(\zeta, z)$ *is holomorphic in* $z \in B(\zeta, \delta)$*;*

2. *$\Phi(\zeta,\zeta)=0$, and $d_z\Phi|_{z=\zeta}\neq 0$;*
3. *There exists a constant $A>0$ such that $|\Phi(\zeta,z)|\geq A$ for all $z\in\Omega$ and $|z-\zeta|\geq c$;*
4. *$\rho(z)>0$ for all z with $\Phi(\zeta,z)=0$ and $0<|z-\zeta|<c$.*

Now we set

$$C(\zeta,z)=\frac{1}{2\pi i}\left[\sum_{j=1}^{2}\frac{\partial\rho}{\partial\zeta_j}(\zeta)d\zeta_j\right]\frac{1}{\Phi(\zeta,z)}\quad\text{for }(\zeta,z)\in b\Omega\times\Omega,$$

which is a $(1,0)$-form of ζ-variables. The Cauchy-Leray kernel for the convex domain Ω is

$$\Omega_0\left(C(\zeta,z)\right)=C(\zeta,z)\wedge(\bar{\partial}_\zeta C(\zeta,z)) \tag{2.2}$$

$$=\sum_{j_0\in\{1,2\}}\frac{A_{j_0}(\zeta)}{\Phi^2(\zeta,z)}d\zeta_1\wedge d\zeta_2\wedge d\bar{\zeta}_{j_0}, \tag{2.3}$$

which is a Cauchy-Fantappiè $(2,1)$-form on $b\Omega\times\Omega$, where $A_{j_0}(\zeta)$ is a polynomial involving first and second derivatives in ζ of ρ.

For each $z\in\Omega$ we extend $C(.,z)$ smoothly to the interior of Ω as follows

$$\widetilde{C}(\zeta,z)=\frac{1}{2\pi i}\left[\sum_{j=1}^{2}\frac{\partial\rho}{\partial\zeta_j}(\zeta)d\zeta_j\right]\frac{1}{\Phi(\zeta,z)-\rho(\zeta)}.$$

Definition 2.2. Let $F:[0,\infty)\to[0,\infty)$ be a smooth, strictly increasing function such that

1. $F(0)=0$,
2. $\displaystyle\int_0^\sigma\left|\ln F(r^2)\right|dr<\infty$ for some $\sigma>0$ which is small enough,
3. $\dfrac{F(t)}{t}$ is non-decreasing function.

Let $\Omega\subset\mathbb{C}^2$ be a smooth bounded, convex domain. We say that Ω admitting F-type at a point $P\in b\Omega$ if there are positive constants c,c' satisfy that for all $\zeta\in b\Omega\cap B(P,c')$:

$$\rho(z)\gtrsim F(|z-\zeta|^2),$$

for all $z\in B(\zeta,c)$ with $\Phi(\zeta,z)=0$.

If Ω admits the same F-type at every point on $b\Omega$, we simply call that Ω admitting F-type. In case $F(t) = t^m$, for $m = 1, 2, \ldots$, the F-type notion agrees with the finite type condition in the sense of Range in [17, 18]. Here the notation $B(\zeta, r)$ means the Euclidean ball centered at ζ of radius $r > 0$. Also the notations $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant, and $\approx$ means the combination of $\lesssim$ and $\gtrsim$.

Some examples to illustrate that the F-type condition consists a large class of convex domains of finite and infinite type in $\mathbb{C}^2$ can be found in [8, 9].

The following lemma provides the important lower estimate for the Cauchy-Fantappiè form. Its proof is rather similar to the proof of [5, Lemma 3.3] with a negligible modification and can be found in [7, before Corollary 2.6].

Lemma 2.3. *Let Ω be a smoothly bounded, convex domain in $\mathbb{C}^2$ admitting an F-type at $P \in b\Omega$. Then there is a positive constant c such that the support function $\Phi(\zeta, z)$ satisfies the following estimate*

$$|\Phi(\zeta, z) - \rho(\zeta)| \gtrsim |\rho(\zeta)| + |\rho(z)| + |\operatorname{Im} \Phi(\zeta, z)| + F(|z - \zeta|^2), \tag{2.4}$$

for every $\zeta \in \bar{\Omega} \cap B(P, c)$, and $z \in \Omega$, $|z - \zeta| < c$.

Definition 2.4. ([13, Definition 2.1]) We say that a smoothly bounded domain $\Omega \subset \mathbb{C}^2$ has B-property if there is a positive constant C_Ω such that the following holds:

$$(-\rho(\zeta))^3 \int_\Omega |\nabla_z^3 P(\zeta, z)| dV(z) + \frac{1}{P(\zeta, \zeta)} \int_\Omega |\nabla_z^3 P(\zeta, z)| (-\rho(z))^3 P(z, z) dV(z) \le C_\Omega$$

for all $\zeta \in \Omega$.

In $\mathbb{C}^2$, there are many bounded domains which admitting a type F at all boundary points and satisfying the condition (B). Firstly, all strictly convex domains in $\mathbb{C}^2$ admits type $F(t) = t$ at all boundary points. Secondly, let m_1, m_2 be positive integers, and let

$$\Omega_m = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m_1} + |z_2|^{2m_2} - 1 < 0\}$$

be convex domain in $\mathbb{C}^2$. The family $\{\Omega_m\}$ is the certain class of weakly convex domains in $\mathbb{C}^2$. Then, in [5], the author shows that Ω_m admits type $F(t) = t^m$ at all boundary points. In [13, p. 480-p. 481], it is proved that any strictly convex domain or any Ω_m satisfies B-property.

For $u \in C^1(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ and u is holomorphic on Ω, by the Stoke Theorem, we get

$$u(z) = \int_\Omega u(\zeta) \bar{\partial}_\zeta \Omega_0(\widetilde{C}(\zeta, z)), \qquad z \in \Omega. \tag{2.5}$$

By the smoothness of each component in $\Omega_0((\widetilde{C}(\zeta, z))$ then the form $\bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z))$ also is a smooth form on $(\bar{\Omega} \times \overline{\Omega}) \setminus \{(z, z), z \in b\Omega\}$.

For $0 < c < \delta$ (c is the constant in Lemma 2.3), let us define $\Omega_\delta = \{z \in \mathbb{C}^2 : \rho(z) < \delta\}$ and let P_z be the Hörmander solution operator to the $\bar{\partial}$-equation in the variables $z \in \Omega_\delta$ (the existence of P_z can be found in [11]).

Definition 2.5. For $(\zeta, z) \in \overline{\Omega} \times \overline{\Omega}_\delta$, let us define

$$Q(\zeta, z) = -P_z(\bar{\partial}_z \bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z))),$$
$$G(\zeta, z) = Q(\zeta, z) + \bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z)),$$

where $G(\zeta, z)$ is holomorphic in z.

The fact $Q(\zeta, z) \in C^\infty(\overline{\Omega}) \times C^1(\overline{\Omega})$ implies that

$$G(\zeta, z) = \frac{1}{\pi^2} \frac{1}{(\Phi(\zeta, z) - \rho(\zeta)^3} \left[\mathrm{O}(|\zeta - z|) + \det \begin{pmatrix} \rho(\zeta) & \frac{\partial \rho}{\partial \zeta_1}(\zeta) & \frac{\partial \rho}{\partial \zeta_2}(\zeta) \\ \frac{\partial \rho}{\partial \bar{\zeta}_1}(\zeta) & \frac{\partial^2 \rho}{\partial \zeta_1 \partial \bar{\zeta}_1}(\zeta) & \frac{\partial^2 \rho}{\partial \zeta_2 \partial \bar{\zeta}_1}(\zeta) \\ \frac{\partial \rho}{\partial \bar{\zeta}_2}(\zeta) & \frac{\partial^2 \rho}{\partial \zeta_1 \partial \bar{\zeta}_2}(\zeta) & \frac{\partial^2 \rho}{\partial \zeta_2 \partial \bar{\zeta}_2}(\zeta) \end{pmatrix} \right]$$
$$d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_2 + \text{non-singular terms.}$$

Let u be a holomorphic function defined on Ω_δ, since

$$\begin{aligned} \int_\Omega u(\zeta) P_z(\bar{\partial}_z \bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z))) &= \int_\Omega P_z(u(\zeta) \bar{\partial}_z \bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z))) \\ &= P_z(\int_\Omega u(\zeta) \bar{\partial}_z \bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z))) \\ &= P_z(\int_\Omega u(\zeta) \bar{\partial}_\zeta \bar{\partial}_z \Omega_0((\widetilde{C}(\zeta, z))) \\ &= P_z(\int_{b\Omega} u(\zeta) \bar{\partial}_z \Omega_0((\widetilde{C}(\zeta, z))) \\ &= 0 \quad \text{(see [12, 1.4.2])}, \end{aligned}$$

we have the reproductive property of $G(\zeta, z)$ that $u(z) = \int_\Omega u(\zeta) G(\zeta, z)$ for all $z \in \Omega$. More generally, let $u \in L^2(\Omega)$, and let us define

$$\mathcal{G}[u](z) = \int_\Omega u(\zeta) G(\zeta, z)$$

and its dual

$$\mathcal{G}^*[u](z) = \int_\Omega u(\zeta) \overline{G(\zeta, z)}.$$

Then $\mathcal{G} : L^2(\Omega) \to A^2(\Omega)$ is a well-defined, continuous operator. Moreover, we also have:

Theorem 2.6. ([6, Theorem 3.4][Ligocka's decomposition]) *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, convex domain. Assume that Ω admits a F-type at all boundary points for some function F. Then $\mathcal{P}[u](z) = \mathcal{G}(I-\mathcal{B})^{-1}[u](z) = (I+\mathcal{B})^{-1}\mathcal{G}^*[u](z)$, where*

$$\mathcal{B}[u](z) = \mathcal{G}^*[u](z) - \mathcal{G}[u](z).$$

3. Proof of the Main Theorem

3.1. Proof of the assertion (1)

This fact has been proved in [9]. For convenience, we briefly sketch its proof here. The $L^p(\Omega, dV)$-boundedness (for $p \in (1,\infty)$) is a consequence of the following lemma.

Lemma 3.1. *The operators $\mathcal{G}$ and $\mathcal{G}^*$ are bounded on $L^p(\Omega, dV)$. In particular, we have*

$$\|\mathcal{G}[u]\|_{L^p(\Omega,dV)} \lesssim \|u\|_{L^p(\Omega,dV)} \quad \text{for all } u \in L^p(\Omega, dV),\ 1 \le p \le \infty,$$

and

$$\|\mathcal{G}^*[u]\|_{L^p(\Omega,dV)} \lesssim \|u\|_{L^p(\Omega,dV)} \quad \text{for all } u \in L^p(\Omega, dV),\ 1 < p < \infty.$$

Due to the strong duality and the Marcinkiewicz Interpolation Theorem from harmonic analysis (see Theorem B.7, Appendix B in [3] for more details), it is sufficient to show that

$$\|\mathcal{G}[u]\|_{L^1(\Omega,dV)} \lesssim \|u\|_{L^1(\Omega,dV)} \quad \text{and} \quad \|\mathcal{G}[u]\|_{L^\infty(\Omega,dV)} \lesssim \|u\|_{L^\infty(\Omega,dV)}.$$

Firstly, we recall the change of variables $(\alpha, w) = (\alpha_1, \alpha_2, w_1, w_2) = (\zeta_1, \zeta_2, z_1 - \zeta_1, \rho(\zeta) + i\mathrm{Im}(\Phi(\zeta, z)))$ and let J be the Jacobian of this change. Then

$$\det(J) = \frac{\partial \mathrm{Im}(\Phi(\zeta,z))}{\partial x_4}\frac{\partial \rho(z)}{\partial x_2} - \frac{\partial \mathrm{Im}(\Phi(\zeta,z))}{\partial x_2}\frac{\partial \rho(z)}{\partial x_4}.$$

Since $\rho(z) \neq 0$, we can find a sufficiently small $0 < \delta < c$ so that $\dfrac{\partial \rho}{\partial x_4}$ dominates others partial derivatives of ρ and $|z-\zeta| \le \delta$. As a consequence, we have $\det(J) \neq 0$ on $|\zeta - z| \le \delta$.

Now let $\delta' > 0$ depend on Ω, c, δ and ρ, and $u \in L^1(\Omega, dV)$. Since $\{(z,\zeta) : |\zeta - z| < c\}$, the kernel $G(\zeta, z)$ is bounded from above by

$$\frac{|\zeta - z|}{|\Phi(\zeta,z) - \rho(\zeta)|^3} \lesssim \frac{1}{(|\rho(z)| + |\rho(\zeta)| + |\operatorname{Im}\Phi(\zeta,z)| + F(|\zeta - z|^2))^3}.$$

and by Lemma 2.3, we have

$$\begin{aligned}
&\iint_{(\zeta,z)\in(\Omega\cap B(0,c/2))^2} |G(\zeta,z)u(\zeta)|\,dV(\zeta,z)\\
&\qquad\lesssim \iint_{(\alpha,w)\in(\Omega\cap B(0,\delta'))\times B(0,\delta')} \frac{|u(\alpha)|}{(|w_2|^2+F^2(|w_1|^2))|w_1|}\,dV(\alpha,w)\\
&\qquad\lesssim \|u\|_{L^1(\Omega,dV)}\int_0^{\delta'}\int_0^{\delta'} \frac{r_1r_2}{(r_2^2+F^2(r_1^2))r_1}\,dr_2\,dr_1\\
&\qquad\lesssim \|u\|_{L^1(\Omega,dV)}\int_0^{\delta'} \ln F(r_1^2)\,dr_1 \lesssim \|u\|_{L^1(\Omega,dV)}\\
&\qquad\text{(by the property of } F\text{)}.
\end{aligned}$$

Therefore, we obtain the $L^1(\Omega, dV)$-boundedness.

Next, let $u \in L^\infty(\Omega)$. The Hölder's Inequality and Lemma 2.3 imply

$$\begin{aligned}
&\int_{\Omega\cap B(0,c/2)} |G(\zeta,z)\|u(\zeta)|dV(\zeta) \lesssim \|u\|_{L^\infty(\Omega)}\int_{\Omega\cap B(0,c/2)} \frac{|\zeta-z|}{|\Phi(\zeta,z)-\rho(\zeta)|^3}dV(\zeta)\\
&\lesssim \|u\|_{L^\infty(\Omega)}\int_{\Omega\cap B(0,\delta)} \frac{dV(w_1,w_2)}{(|\rho(z)|+|w_2|+F(|w_1|^2))^2|w_1|}\\
&\lesssim \|u\|_{L^\infty(\Omega)}\int_{|(t_1,t_2,t_3,t_4)|\le\delta} \frac{dt_1dt_2dt_3dt_4}{(|\rho(z)|+|t_3|+|t_4|+F(t_1^2+t_2^2))^2|(t_1,t_2)|}\\
&\text{(where } w_1=t_1+\sqrt{-1}t_2, w_2=t_3+\sqrt{-1}t_4\text{)}\\
&\lesssim \|u\|_{L^\infty(\Omega)}\int_{|(t_1,t_2,t_3)|\le\delta} \frac{dt_1dt_2dt_3}{(|\rho(z)|+|t_3|+F(t_1^2+t_2^2))|(t_1,t_2)|}\\
&\lesssim \|u\|_{L^\infty(\Omega)}\int_0^\delta |\ln F(r^2)|dr \lesssim \|u\|_{L^\infty(\Omega)} \quad \text{(by the property of } F\text{)}.
\end{aligned}$$

Hence the L^∞-boundedness is established and the proof of $L^p(\Omega, dV)$, for $p \in (1,\infty)$, is complete.

3.2. Proof of the assertion (2)

Since the continuity of $\mathcal{B}$ in Theorem 2.6 and the fact that $\mathrm{Ker}[I-\mathcal{B}] = \{0\}$, $I-\mathcal{B}$ is a Fredholm isomorphism of $L^\infty(\Omega)$. Thus, it is sufficient to prove that $\mathcal{G}$ maps continuously $L^\infty(\Omega)$ into $\mathrm{BlHol}(\Omega)$.

Let $u \in L^\infty(\Omega)$, we must show that

$$(|\rho(z)||\mathcal{G}u(z)| + |\rho(z)||\nabla_z\mathcal{G}u(z)|) \le \|u\|_\infty \tag{3.1}$$

for all $z \in \Omega$.

We consider the first term in (3.1). Since the integral $\int_\Omega |Q(\zeta,z)|\,dV(\zeta)$ is non-singular, we have

$$|\rho(z)||\mathcal{G}u(z)| \lesssim \|u\|_\infty \left(1 + |\rho(z)|\int_\Omega \left|\bar\partial_\zeta\Omega_0((\widetilde{C}(\zeta,z))\right| dV(\zeta)\right).$$

For $0 < c < \sigma$ (c is the constant in Lemma 2.3), let $h \in C^\infty(\mathbb{C}^2)$ be a cutoff function such that $h = 1$ on $\{\zeta \in \mathbb{C}^2 : |\rho(\zeta)| + |\rho(z)| + |\operatorname{Im}(\Phi(\zeta, z))| + F(|\zeta - z|^2) < \sigma/2\}$ and $h = 0$ on $\{\zeta \in \mathbb{C}^2 : |\rho(\zeta)| + |\rho(z)| + |\operatorname{Im}(\Phi(\zeta, z))| + F(|\zeta - z|^2) > \sigma\}$. Then,

$$\begin{aligned}
&\int_\Omega \left|\bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z))\right| dV(\zeta) \\
&\quad \lesssim 1 + \int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma} \left|\bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z))\right| dV(\zeta) \\
&\quad \lesssim \int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma} \left|\bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z))\right| dV(\zeta).
\end{aligned}$$

Since $|\bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z))|$ is dominated by $\dfrac{|\zeta - z|}{|\Phi(\zeta, z) - \rho(\zeta)|^3}$ when ζ near to z, we obtain

$$\begin{aligned}
&|\rho(z)| \int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma} \left|\bar{\partial}_\zeta \Omega_0((\widetilde{C}(\zeta, z))\right| dV(\zeta) \\
&\qquad \lesssim \int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma} \frac{|\zeta - z|}{|\Phi(\zeta, z) - \rho(\zeta)|^2} dV(\zeta).
\end{aligned}$$

To estimate the last integral in the above inequality, we use the following Henkin coordinates on Ω (see [19, Lemma V3.4]). These coordinates do exist since $\nabla\rho(\zeta)|_{\zeta=z}$ and $\nabla \operatorname{Im}\Phi(\zeta, z)|_{\zeta=z}$ are nonzero and are not proportial.

Lemma 3.2 (Henkin's coordinates)**.** *There exist positive constants M, a and $\eta \leq c$, and, for each z with $dist(z, b\Omega) \leq a$, there is a smooth local coordinate system $(t_1, t_2, t_3, t_4) = t = t(\zeta, z)$ on the ball $B(z, c)$ such that we have*

$$\begin{cases}
t(z, z) = 0, \\
t_1(\zeta) = \rho(\zeta) - \rho(z), \\
t_2(\zeta) = \operatorname{Im}(\Phi(\zeta, z)), \\
|t| < \delta \quad \text{for } \zeta \in B(z, c), \\
|J_\mathbb{R}(t)| \leq M \quad \text{and} \quad |det J_\mathbb{R}(t)| \geq \frac{1}{M},
\end{cases}$$

where $J_\mathbb{R}(t)$ is the Jacobian of the transformation t.

Therefore, for some $0 < \sigma' < \sigma$ small enough,

$$\begin{aligned}
&\int_{|\rho(\zeta)|+|\rho(z)|+|\operatorname{Im}(\Phi(\zeta,z))|+F(|\zeta-z|^2)<\sigma} \frac{1}{|\Phi(\zeta,z)-\rho(\zeta)|^2} dV(\zeta)\\
&\lesssim \int_{|(t_1,\ldots,t_4)|\le\sigma} \frac{1}{(|t_1|+|t_2|+F(|(t_3,t_4)|^2))^2} dt_1\ldots dt_4\\
&\lesssim \iint_{(r_1,r_2)\in(0,\sigma')^2} \frac{r_1 r_2}{r_1^2+F^2(r_2^2)} dr_1 dr_2\\
&\text{(using the polar coordinates } r_1 = |(t_1,t_2)| \text{ and } r_2 = |(t_3,t_4)|)\\
&\lesssim \int_0^{\sigma'} |\ln F(r^2)| dr < \infty.
\end{aligned}$$

Next, for the second term in (3.1), we have the note that $\left|\frac{\partial}{\partial z_j}\bar{\partial}_\zeta \Omega_0(\widetilde{C}(\zeta,z))\right|$ is dominated by $\frac{1}{|\Phi(\zeta,z)-\rho(\zeta)|^4}$. Thus, for all $z \in \Omega$, using the Henkin coordinates and the cutoff function h again, we have

$$\begin{aligned}
|\rho(z)|\,|\nabla_z \mathcal{G}u(z)| &\lesssim \|u\|_\infty \left(1 + \int_\Omega h(\zeta)\frac{dV(\zeta)}{|\Phi(\zeta,z)-\rho(\zeta)|^3}\right)\\
&\lesssim \|u\|_\infty \left(1 + \int_{|(t_1,\ldots,t_4)|\le\sigma'} \frac{dt_1\ldots dt_4}{(|t_1|+|t_2|+F(|(t_3,t_4)|^2))^2 |(t_1,\ldots,t_4)|}\right)\\
&\lesssim \|u\|_\infty \left(1 + \int_0^{\sigma'} |\ln F(r^2)| dr\right) < \infty.
\end{aligned}$$

Therefore, we conclude that for all $u \in L^\infty(\Omega)$, $\mathcal{G}[u] \in \mathrm{BlHol}(\Omega)$. So $\mathcal{P}$ is bounded from $L^\infty(\Omega)$ to $\mathrm{BlHol}(\Omega)$.

3.3. Proof of the assertion (3)

Let $u \in L^1(\Omega, P(\cdot,\cdot)dV)$. Firstly we have

$$\begin{aligned}
\|\mathcal{P}[u]\|_{\mathrm{Besov}(\Omega)} &= \int_\Omega |\nabla_z^3 \mathcal{P}[u](z)|(-\rho(z))^3 P(z,z)dV(z)\\
&= \int_\Omega \left|\nabla_z^3 \int_\Omega P(\zeta,z)u(\zeta)dV(\zeta)\right| (-\rho(z))^3 P(z,z)dV(z)\\
&= \int_\Omega \left|\int_\Omega \nabla_z^3 P(\zeta,z)u(\zeta)dV(\zeta)\right| (-\rho(z))^3 P(z,z)dV(z)\\
&\le \int_\Omega \int_\Omega |\nabla_z^3 P(\zeta,z)u(\zeta)|dV(\zeta)(-\rho(z))^3 P(z,z)dV(z)\\
&\le \int_\Omega \int_\Omega |\nabla_z^3 P(\zeta,z)|(-\rho(z))^3 P(z,z)dV(z)|u(\zeta)|dV(\zeta).
\end{aligned}$$

Secondly the (B)-property implies

$$\int_\Omega |\nabla_z^3 P(\zeta, z)|(-\rho(z))^3 P(z,z) dV(z) \le C_\Omega P(\zeta,\zeta)$$

for all $\zeta \in \Omega$. Hence, combining these facts yields that

$$\|\mathcal{P}[u]\|_{\mathrm{Besov}(\Omega)} \le C_\Omega \int_\Omega |u(\zeta)||P(\zeta,\zeta)| dV(\zeta) = C_\Omega \|u\|_{L^1(\Omega, P(\cdot,\cdot)dV)}.$$

Therefore the proof of the assertion (3) is complete.

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